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Generalization and randomization of some number-theoretic special functions

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1 Introduction

There are many special number-theoretic functions around the Riemann zeta function $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$, Re $s>1$, such as

$$\begin{aligned}
\zeta(s,x) &= \sum_{k=1}^{\infty}(k+x)^{-s}, \quad \mathrm{Re}\ s>1,\ \ x>-1, \quad \text{(Hurwitz zeta function)},^{1} \\
\Gamma(1+x)^{-1} &= \exp\left(\zeta'(0)-\zeta'(0,x)\right) \quad \left[\zeta'(0,x):=\frac{\partial}{\partial s}\zeta(0,x)\right]^{2} \qquad (1)\\
&= e^{\gamma x}\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)e^{-x/k} \quad \left[\text{where } \gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\right] \\
&= \lim_{n\to\infty}n^{x}\prod_{k=1}^{n}\left(1+\frac{x}{k}\right), \\
\psi(x+1) &= (\log\Gamma(x+1))'=\frac{\Gamma'(x+1)}{\Gamma(x+1)} \\
&= -\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+x}-\log n\right). \quad \text{(digamma function)} \qquad (2)
\end{aligned}$$

As we see in the above infinite sum or infinite product formulas, these special functions are related to the sequence of natural numbers $\{k\}_{k=1}^{\infty}$. In this article, we study what we get when $\{k\}_{k=1}^{\infty}$ is replaced with other positive increasing sequences, including random sequences.

The most popular method for generalization of number-theoretic special functions is the so-called *zeta regularization.*

Definition 1 ([4, 6]) Let a positive sequence $a=\{a_k\}_{k=1}^{\infty}$ satisfy $\sum_{k=1}^{\infty}a_k^{-\alpha}<\infty$ for some $\alpha>0$. Then we define the zeta function

$$z(s) := \sum_{k=1}^{\infty}a_k^{-s},$$

[2]Slightly different from the traditional definition.

[2]This notation will be used for any functions of two variables in this article.

which is holomorphic in Re $s > \alpha$. If $z(s)$ is analytically continued to a meromorphic function which is holomorphic at $s = 0$, a is said to be *zeta regularizable.* Then we write

$$\text{z-}\prod_{k=1}^{\infty} a_k := \exp(-z'(0))$$

and call it the *zeta regularized product* of $\prod_{k=1}^{\infty} a_k$.

But, for our purpose, this notion is too strong, indeed, it is quite unlikely that random sequences become zeta regularizable. We therefore assume a rather mild condition (Assumption 1 below) which random sequences can satisfy.

This work is somewhat an experimental one. We are not sure that it is a promising research. However, we think that some of results, such as Example 2, Theorem 6, Theorem 7,and their extensions in § 4.1 are fully interesting by themselves.

2 Deterministic generalization

2.1 Zeta regularized product

In this article, we consider real sequences which satisfy the following condition.

Assumption 1 (i) $a = \{a_k\}_{k=1}^{\infty}$ is a positive non-decreasing sequence diverging to ∞. (ii) a is uniformly distributed in the half line $(0, \infty)$ with the same density as $\mathbf{N}$ in the following sense: Setting

$$F(x) := \#\{k \in \mathbf{N}\,;\, a_k \le x\},$$

there exists some $\delta > 0$ such that

$$F(x)x^{-1} = 1 + O(x^{-\delta}), \quad x \to \infty. \tag{3}$$

Remark 1 As we will see later, Assumption 1 alone does not assure $a = \{a_k\}_{k=1}^{\infty}$ to be zeta regularizable.

Throughout this section § 2 (except Remark 4), we consider everything under Assumption 1.

Lemma 1 For any $\varepsilon > 0$, we have $\sum_{k=1}^{\infty} a_k^{-1-\varepsilon} < \infty$.

Proof. Since $k \le F(a_k)$, we see that $ka_k^{-1} \le F(a_k)a_k^{-1} \to 1$ as $k \to \infty$, which implies[3] $\limsup_{k\to\infty} ka_k^{-1} \le 1$. From this, the assertion of the lemma easily follows. Q.E.D.

Lemma 2 *The following limit exists:*

$$\lim_{x\to\infty}\left[\sum_{a_k\le x} a_k^{-1} - \log x\right] = \lim_{n\to\infty}\left[\sum_{k=1}^{n} a_k^{-1} - \log n\right] =: q. \tag{4}$$

[3]In fact, we have $\lim_{k\to\infty} ka_k^{-1} = 1$ ([5]).

Proof. Take $0 < \varepsilon < a_1$, and note that $F(\varepsilon) = 0$. By integration by parts formula,

$$\begin{aligned}
\int_\varepsilon^x (F(t)t^{-1} - 1)t^{-1}dt &= \int_\varepsilon^x F(t)t^{-2}dt - \int_\varepsilon^x t^{-1}dt \\
&= -F(x)x^{-1} + F(\varepsilon)\varepsilon^{-1} + \int_\varepsilon^x t^{-1}dF(t) - (\log x - \log \varepsilon) \\
&= -F(x)x^{-1} + \left(\sum_{a_k \le x} a_k^{-1} - \log x\right) + \log \varepsilon.
\end{aligned}$$

Since Assumption 1 implies $\int_\varepsilon^\infty |F(t)t^{-1} - 1|t^{-1}dt < \infty$ and that $\lim_{x\to\infty} F(x)x^{-1} = 1$, the term $\lim_{x\to\infty}[\sum_{a_k\le x} a_k^{-1} - \log x]$ of the last right-hand side of the above also has a limit as $x \to \infty$. We thus have

$$\int_\varepsilon^\infty (F(t)t^{-1} - 1)t^{-1}dt = -1 + \log \varepsilon + \lim_{x\to\infty}\left[\sum_{a_k\le x} a_k^{-1} - \log x\right]. \tag{5}$$

Since we have

$$\begin{aligned}
\liminf_{x\to\infty}\left[\sum_{a_k\le x} a_k^{-1} - \log F(x)\right] &\le \liminf_{n\to\infty}\left[\sum_{k=1}^n a_k^{-1} - \log n\right] \\
&\le \limsup_{n\to\infty}\left[\sum_{k=1}^n a_k^{-1} - \log n\right] \\
&\le \limsup_{x\to\infty}\left[\sum_{a_k\le x} a_k^{-1} - \log F(x)\right],
\end{aligned}$$

and since (3) implies

$$\begin{aligned}
\liminf_{x\to\infty}\left[\sum_{a_k\le x} a_k^{-1} - \log F(x)\right] &= \limsup_{x\to\infty}\left[\sum_{a_k\le x} a_k^{-1} - \log F(x)\right] \\
&= \lim_{x\to\infty}\left[\sum_{a_k\le x} a_k^{-1} - \log x\right],
\end{aligned}$$

we know (4) is valid. Q.E.D.

Proposition 1 (cf. [4] Theorem 2) *$z(s)$ is analytically continued to a meromorphic function in* Re $s > 1 - \delta$ *with a unique single pole at $s = 1$, whose residue is 1. In addition, the 'finite part' of $z(s)$ at the pole is equal to q, i.e.,*

$$\lim_{s\to 1}\left[z(s) - \frac{1}{s-1}\right] = q. \tag{6}$$

Proof. Let $\sigma :=$ Re $s > 1$ and let $0 < \varepsilon < a_1$. By integration by parts,

$$\begin{aligned}
\sum_{a_n\le x} a_n^{-s} &= \int_\varepsilon^x t^{-s}dF(t) = F(x)x^{-s} + s\int_\varepsilon^x F(t)t^{-s-1}dt \\
&= O(x^{1-\sigma}) + s\int_\varepsilon^x (F(t) - t)t^{-s-1}dt + s\int_\varepsilon^x t^{-s}dt \\
&= O(x^{1-\sigma}) + s\int_\varepsilon^x (F(t) - t)t^{-s-1}dt + \frac{s\varepsilon^{-s+1}}{s-1} - \frac{sx^{-s+1}}{s-1}.
\end{aligned}$$

Letting $x \to \infty$, we have

$$\begin{aligned} z(s) &= \frac{s\varepsilon^{-s+1}}{s-1} + s\int_\varepsilon^\infty (F(t)-t)t^{-s-1}dt \\ &= \frac{1}{s-1} + \frac{s\varepsilon^{-s+1}-1}{s-1} + s\int_\varepsilon^\infty (F(t)t^{-1}-1)t^{-s}dt. \end{aligned}$$

This expression and Assumption 1 implies that $z(s)$ is analytically continued to a meromorphic function in Re $s > 1-\delta$ with a unique single pole at $s=1$, whose residue is 1. Moreover

$$\lim_{s\to 1}\left[z(s) - \frac{1}{s-1}\right] = 1 - \log\varepsilon + \int_\varepsilon^\infty (F(t)t^{-1}-1)t^{-1}dt.$$

Then (5) shows that

$$\lim_{s\to 1}\left[z(s) - \frac{1}{s-1}\right] = \lim_{x\to\infty}\left[\sum_{a_k\le x} a_k^{-1} - \log x\right] = q.$$

Q.E.D.

It is easy to see that the corresponding Hurwitz zeta function

$$z(s,x) := \sum_{k=1}^\infty (a_k+x)^{-s}, \quad x > -a_1,$$

is also analytically continued to a meromorphic function in Re $s > 1-\delta$ with a unique single pole at $s=1$, whose residue is 1 (cf. [4] Theorem 1).

However, in general, $z(s)$ and $z(s,x)$ do not necessarily become holomorphic at $s=0$. Indeed, for the existence of $z'(0)$, the integral $\int_\varepsilon^\infty (F(t)-t)t^{-1}dt$ should be convergent, which Assumption 1 does not assure. Nevertheless their difference becomes holomorphic at $s=0$.

Proposition 2 *For each $x > -a_1$, the difference function $g(s,x) := z(s) - z(s,x)$ is analytically continued to a holomorphic function in* Re $s > -\delta$.

Proof. Since Proposition 1 implies that $sz(s+1)$ is holomorphic in Re $s > -\delta$, it is enough to show that

$$h(s) := g(s,x) - s\,z(s+1)x$$

is holomorphic in Re $s > -\delta$.

First, $h(s)$ is expressed in the following series in Re $s > 1$.

$$h(s) = \sum_{k=1}^\infty a_k^{-s} - \sum_{k=1}^\infty (a_k+x)^{-s} - s\sum_{k=1}^\infty a_k^{-s-1}x.$$

Suppose $|x| < a_{k_0}$. Then applying the Taylor expansion (negative binomial theorem)

$$\begin{aligned} (a_k+x)^{-s} &= a_k^{-s}\sum_{j=0}^\infty \binom{s+j-1}{j}\left(\frac{-x}{a_k}\right)^j \\ &= a_k^{-s} + \lambda s a_k^{-s-1} + a_k^{-s}\sum_{j=2}^\infty \binom{s+j-1}{j}\left(\frac{-x}{a_k}\right)^j, \quad k \ge k_0, \qquad (7) \end{aligned}$$

which converges absolutely, we see

$$h(s) = -s\sum_{k=1}^{k_0-1}(a_k+x)^{-s} - s\sum_{k=k_0}^{\infty} a_k^{-s}\sum_{j=2}^{\infty}\frac{(s+1)(s+2)\cdots(s+j-1)}{j!}\left(\frac{-x}{a_1}\right)^j\left(\frac{a_1}{a_k}\right)^j. \tag{8}$$

Since

$$\begin{aligned}
&\sum_{k=k_0}^{\infty}\left|a_k^{-s}\sum_{j=2}^{\infty}\frac{(s+1)(s+2)\cdots(s+j-1)}{j!}\left(\frac{-x}{a_1}\right)^j\left(\frac{a_1}{a_k}\right)^j\right| \\
&\quad\leq \sum_{k=k_0}^{\infty} a_k^{-\mathrm{Re}\,s}\left(\frac{a_1}{a_k}\right)^2\sum_{j=2}^{\infty}\left|\frac{(s+1)(s+2)\cdots(s+j-1)}{j!}\right|\left(\frac{-x}{a_1}\right)^j \\
&\quad= a_1^2\sum_{k=k_0}^{\infty} a_k^{-\mathrm{Re}\,s-2}\sum_{j=2}^{\infty}\left|\frac{(s+1)(s+2)\cdots(s+j-1)}{j!}\right|\left(\frac{-x}{a_1}\right)^j
\end{aligned}$$

is finite in $\mathrm{Re}\, s > -1$ by Lemma 1, $h(s)$ becomes holomorphic in $\mathrm{Re}\, s > -1$. Q.E.D.

Definition 2 We define the zeta regularized product of $\prod_{k=1}^{\infty}(1+\frac{x}{a_k})$ by

$$\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right) := \exp\left(g'(0,x)\right). \tag{9}$$

Remark 2 If $a = \{a_k\}_{k=1}^{\infty}$ is zeta regularizable, we have

$$\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right) = \frac{\text{z-}\prod_{k=1}^{\infty}(a_k+x)}{\text{z-}\prod_{k=1}^{\infty}a_k} = \exp\left(z'(0)-z'(0,x)\right).$$

2.2 Generalized Wallis formula

Proposition 3 (Weierstrass' infinite product formula, [4] Theorem 2, [6])

$$\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right) = e^{qx}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right)\exp\left(-\frac{x}{a_k}\right).$$

Proof. Noting $\lim_{s\to 0} sz(s+1) = 1$, we first calculate $h'(0)$.

$$\begin{aligned}
h'(0) &= g'(0,x) - \lim_{s\to 0}\frac{s\,z(s+1)-1}{x}\lambda \\
&= g'(0,x) - \lim_{s\to 0}\left[z(s+1)-\frac{1}{s}\right]x \\
&= g'(0,x) - qx \qquad \text{(cf. (6))}.
\end{aligned} \tag{10}$$

On the other hand, (8) implies $h(0) = 0$ and so that $h'(0) = \lim_{s\to 0} h(s)/s$. Therefore

$$\begin{aligned}
h'(0) &= -\sum_{k=1}^{\infty}\sum_{j=2}^{\infty}\frac{(j-1)!}{j!}\left(\frac{-x}{a_k}\right)^j = \sum_{k=1}^{\infty}\left[-\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{-x}{a_k}\right)^j - \frac{x}{a_k}\right] \\
&= \sum_{k=1}^{\infty}\left[\log\left(1+\frac{x}{a_k}\right) - \frac{x}{a_k}\right].
\end{aligned}$$

This and (10) imply that

$$g'(0,x) = qx + \sum_{k=1}^{\infty}\left[\log\left(1+\frac{x}{a_k}\right) - \frac{x}{a_k}\right]. \tag{11}$$

Plugging this into the exponential function, we finally obtain

$$\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right) = e^{qx}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right)e^{-\frac{x}{a_k}}.$$

Q.E.D.

Theorem 1 (Generalized Wallis formula)

$$\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right) = \lim_{n\to\infty} n^{-x}\prod_{k=1}^{n}\left(1+\frac{x}{a_k}\right). \tag{12}$$

Remark 3 *For the special case where $a_k = k$, $k = 1, 2, \ldots$, and $x = -1/2$, we have*

$$\begin{aligned}\text{z-}\prod_{k=1}^{\infty}\left(1-\frac{1}{2k}\right) &= \Gamma(1/2)^{-1} = \pi^{-1/2},\\ n^{1/2}\prod_{k=1}^{n}\left(1-\frac{1}{2k}\right) &= n^{1/2}\binom{2n}{n}2^{-2n}.\end{aligned}$$

So (12) implies now the classical Wallis formula.

Proof of Theorem 1. From (4) and Proposition 3, it follows that

$$\begin{aligned}\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{\lambda}{a_k}\right) &= \lim_{n\to\infty}\exp\left((a_1^{-1}+a_2^{-1}+\cdots+a_n^{-1}-\log n)x\right)\prod_{k=1}^{n}\left(1+\frac{x}{a_k}\right)e^{-\frac{x}{a_k}}\\ &= \lim_{n\to\infty} n^{-x}\prod_{k=1}^{n}\left(1+\frac{x}{a_k}\right).\end{aligned}$$

Q.E.D.

By definition, z-$\prod(1+\frac{x}{a_k})$ is neither 0 nor infinite. Consequently, Proposition 3 and Theorem 1 have substantial meaning.

Example 1 The square of the classical Wallis formula is in fact a zeta regularized product:

$$\pi^{-1} = \text{z-}\prod_{k=1}^{\infty}\left(1-\frac{1}{2k}\right)^2 = \text{z-}\prod_{k=1}^{\infty}\left(1-\frac{1}{a_k}\right),$$

where $\frac{1}{a_k} = \frac{1}{k} - \frac{1}{4k^2}$ or

$$a_k = \frac{1}{\frac{1}{k}-\frac{1}{4k^2}} = k + \frac{1}{4} + \frac{1}{4(4k-1)}, \quad k = 1, 2, \ldots,$$

which satisfies Assumption 1. Then let us show that

$$q = \lim_{s\to 1}\left(z(s) - \frac{1}{s-1}\right) = \gamma - \frac{\pi^2}{24}.$$

Since

$$\begin{aligned}\pi^{-1} &= \lim_{n\to\infty} n\prod_{k=1}^{n}\left(1-\frac{1}{2k}\right)^2 \\ &= \prod_{k=1}^{\infty}\left(1-\frac{1}{2k}\right)^2 \exp\left(\frac{1}{k}-\frac{1}{4k^2}\right)\cdot \lim_{n\to\infty} n\prod_{k=1}^{n}\exp\left(-\frac{1}{k}+\frac{1}{4k^2}\right),\end{aligned}$$

we must have

$$e^{-q} = \lim_{n\to\infty} n\prod_{k=1}^{n}\exp\left(-\frac{1}{k}+\frac{1}{4k^2}\right),$$

namely,

$$q = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{4k^2}\right)-\log n\right] = \gamma-\sum_{k=1}^{\infty}\frac{1}{4k^2} = \gamma-\frac{\pi^2}{24}.$$

Remark 4 In case $\sum_{k=1}^{\infty} a_k^{-1} < \infty$, $z(s)$ becomes finite at $s=1$, so that its 'finite part' q at $s=1$ is, of course, $\sum_{k=1}^{\infty} a_k^{-1}$. Then it holds that z-$\prod_{k=1}^{\infty}(1+\frac{x}{a_k}) = \prod_{k=1}^{\infty}(1+\frac{x}{a_k})$. Let us show it.
(i) For a finite sequence $a=\{a_k\}_{k=1}^{N}$,

$$z(s) := \sum_{k=1}^{N} a_k^{-s}, \quad z(s,x) := \sum_{k=1}^{N}(a_k+x)^{-s}, \quad 0\le\lambda<a_1,$$

which are entire functions, it is easy to see that $\exp(z'(0)-z'(0,x)) = \prod_{k=1}^{N}(1+\frac{x}{a_k})$.
(ii) For an infinite sequence $a=\{a_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_k^{-1}<\infty$,

$$z(s) := \sum_{k=1}^{\infty} a_k^{-s}, \quad z(s,x) := \sum_{k=1}^{\infty}(a_k+x)^{-s}, \quad 0\le\lambda<a_1,$$

are finite at $s=1$, but we do not know whether they are analytically continued beyond Re $s>1$. Nevertheless their difference $g(s,x) := z(s)-z(s,x)$ is analytically continued to a holomorphic function in Re $s>-1$, which is shown in a similar way as Proposition 3. Indeed, by (7),

$$g(s,x) = -\sum_{k=1}^{\infty} a_k^{-s}\sum_{j=1}^{\infty}\binom{s+j-1}{j}\left(\frac{-x}{a_k}\right)^j,$$

from which it follows that

$$g'(0,x) = -\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{(j-1)!}{j!}\left(\frac{-x}{a_k}\right)^j = -\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{-x}{a_k}\right)^j = \sum_{k=1}^{\infty}\log\left(1+\frac{x}{a_k}\right).$$

Thus

$$\exp(g'(0,x)) = \prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right).$$

2.3 Generalized digamma function

If $a = \{a_k\}_{k=1}^{\infty}$ satisfies Assumption 1, so does $\{a_k + x\}_{k=1}^{\infty}$ for each $x > 0$, and hence we can define

$$q(x) := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{x+a_k} - \log n\right), \quad x > 0.$$

Comparing with (2), we can say that $-q(x)$ is a generalized digamma function.

Suppose that unit electric charges are located at each point of $\{a_k\}_{k=1}^{\infty}$ on the real line $\mathbf{R}$. Then $q(x)$ can be regarded as the renormalized Coulomb potential at $-x$ caused by those electric charges. Indeed, we see

$$\begin{aligned} q'(x) &= \frac{d}{dx}\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{x+a_k} - \log n\right) \\ &= \frac{d}{dx}\sum_{k=1}^{\infty}\left(\frac{1}{x+a_k} - \log\frac{k+1}{k}\right) \\ &= \sum_{k=1}^{\infty}\frac{d}{dx}\left(\frac{1}{x+a_k} - \log\frac{k+1}{k}\right) \\ &= -\sum_{k=1}^{\infty}\frac{1}{(x+a_k)^2} = -z(2,x). \end{aligned}$$

By (11), we have

$$q(x) = -\frac{d}{ds}\left(z(s,x) - z(s,x-1)\right)\Big|_{s=0} + \sum_{k=1}^{\infty}\left[\frac{1}{x+a_k} + \log\left(1 - \frac{1}{x+a_k}\right)\right]. \tag{13}$$

Applying this formula to the sequence $\{a_k = k\}_{k=1}^{\infty}$, we have

$$-\psi(x+1) = -\log x + \sum_{k=1}^{\infty}\left[\frac{1}{x+k} + \log\left(1 - \frac{1}{x+k}\right)\right], \quad (x > 0), \tag{14}$$

because

$$\frac{d}{ds}\left(\zeta(s,x) - \zeta(s,x-1)\right)\Big|_{s=0} = \frac{d}{ds}\left(-x^{-s}\right)\Big|_{s=0} = \log x.$$

Theorem 2 *For any sequence $a = \{a_k\}_{k=1}^{\infty}$ satisfying Assumption 1 and*

$$a_k k^{-1} = 1 + O(k^{-\delta'}),\ k \to \infty, \quad \delta' > 0,$$

we have

$$q(x) = -\log x + O(x^{-1}), \quad x \to \infty.$$

Proof. From (14) it follows that

$$-\psi(x+1) = -\log x + O(x^{-\min(1,\delta')}), \quad x \to \infty.$$

On the other hand, for $x > 0$, we have

$$\begin{aligned} q(x) + \psi(x+1) &= \sum_{k=1}^{\infty}\left(\frac{1}{x+k} - \frac{1}{x+a_k}\right) \\ &= \sum_{k=1}^{\infty}\frac{a_k - k}{(x+k)(x+a_k)} \\ &= \sum_{k=1}^{\infty}\frac{O(k^{1-\delta})}{(x+k)(x+a_k)} = O(x^{-\delta'}), \quad x \to \infty. \end{aligned}$$

Q.E.D.

2.4 Generalized Gamma functions

The following lemma is easily derived from Theorem 1.

Lemma 3 *For each $n \in \mathbf{N}$,*

$$\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right) = \prod_{k=1}^{n}\left(1+\frac{x}{a_k}\right) \text{z-}\prod_{k=n+1}^{\infty}\left(1+\frac{x}{a_k}\right).$$

Now, recalling $\text{z-}\prod_{k=1}^{\infty}(1+\frac{x}{k}) = \Gamma(1+x)^{-1}$, Lemma 3 implies

$$\begin{aligned}\Gamma(n+1+x) &= \Gamma(1+x)\prod_{k=1}^{n}(k+x) = \frac{\prod_{k=1}^{n}(k+x)}{\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)} \\ &= \frac{n!\prod_{k=1}^{n}\left(1+\frac{x}{k}\right)}{\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)} = \frac{n!}{\text{z-}\prod_{k=n+1}^{\infty}\left(1+\frac{x}{k}\right)}.\end{aligned}$$

Therefore

$$\Gamma(x) = \frac{n!}{\text{z-}\prod_{k=n+1}^{\infty}\left(1+\frac{x-n-1}{k}\right)}. \tag{15}$$

We consider an analogy of this.

Definition 3 For each $n \in \mathbf{N}$, we define

$$G^{(n+1)}(x) := \frac{\prod_{k=1}^{n} a_k}{\text{z-}\prod_{k=n+1}^{\infty}\left(1+\frac{x-a_{n+1}}{a_k}\right)}. \tag{16}$$

Obviously, we have

$$G^{(n+1)}(a_{n+1}) = \prod_{k=1}^{n} a_k \tag{17}$$

$$\text{z-}\prod_{k=1}^{\infty}\left(1+\frac{x}{a_k}\right) = \frac{\prod_{k=1}^{n}(a_k+x)}{G^{(n+1)}(a_{n+1}+x)}, \quad n=1,2,\ldots \tag{18}$$

By (15), when $a_k = k$ for each $k \in \mathbf{N}$, $G^{(n+1)}(x) = \Gamma(x)$ holds for any $n \in \mathbf{N}$. In general, for $a = \{a_k\}_{k=1}^{\infty}$ satisfying the following assumption, the corresponding $G^{(n+1)}$ has a Gamma function-like property.

Assumption 2 There exists some $\alpha > 0$ such that $a_{k+1} - a_k = 1 + O(k^{-\alpha}), \quad k \to \infty$.

Theorem 3 *If $a = \{a_k\}_{k=1}^{\infty}$ satisfies Assumption 1 and Assumption 2, it holds for any $j \in \mathrm{N}$ that*

$$G^{(n+1)}(a_{n+1-j}) \sim \prod_{k=1}^{n-j} a_k, \quad n \to \infty. \tag{19}$$

Here "$\sim$" indicates that the ratio of the both hand sides tends to 1 in the specified limit.

Proof. For $j < n$,

$$\begin{aligned} G^{(n+1)}(a_{n+1-j}) &= \frac{\prod_{k=1}^{n} a_k}{\text{z-}\prod_{k=n+1}^{\infty}\left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)} \\ &= \frac{\prod_{k=1}^{n-j} a_k}{\prod_{k=n+1-j}^{n} a_k^{-1}\ \text{z-}\prod_{k=n+1}^{\infty}\left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)}. \end{aligned}$$

Therefore it is sufficient to show that

$$\lim_{n\to\infty} \prod_{k=n+1-j}^{n} a_k^{-1}\ \text{z-}\prod_{k=n+1}^{\infty}\left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) = 1. \tag{20}$$

By Proposition 3, we have

$$\begin{aligned} &\text{z-}\prod_{k=n+1}^{\infty}\left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \\ &\qquad = \exp\left(-q_{n+1}(a_{n+1} - a_{n+1-j})\right) \\ &\qquad\quad \times \prod_{k=n+1}^{\infty}\left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)\exp\left(\frac{a_{n+1} - a_{n+1-j}}{a_k}\right), \end{aligned}$$

where

$$\begin{aligned} q_{n+1} &:= \lim_{N\to\infty}\left[\sum_{k=n+1}^{N} a_k^{-1} - \log(N - n + 1)\right] \\ &= q - \sum_{k=1}^{n} a_k^{-1} = -\log n + o(1), \quad n \to \infty. \end{aligned}$$

Then Assumption 2 implies that

$$\begin{aligned} \exp\left(-q_{n+1}(a_{n+1} - a_{n+1-j})\right) &= n^{j - O(n^{-\alpha})} e^{o(1)(-j + O(n^{-\alpha}))} \\ &\sim n^j, \quad n \to \infty. \end{aligned}$$

The following is obvious.

$$\prod_{k=n+1}^{\infty}\left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)\exp\left(\frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \to 1, \quad n \to \infty.$$

From these, it follows that

$$\text{z-}\prod_{k=n+1}^{\infty}\left(1-\frac{a_{n+1}-a_{n+1-j}}{a_k}\right) \sim n^j, \quad n\to\infty.$$

And hence

$$\begin{aligned}&\lim_{n\to\infty}\prod_{k=n+1-j}^{n} a_k \;\text{z-}\prod_{k=n+1}^{\infty}\left(1-\frac{a_{n+1}-a_{n+1-j}}{a_k}\right)\\ &= \lim_{n\to\infty}\left(\prod_{k=n+1-j}^{n} a_k\right)\cdot n^j = \prod_{k=0}^{j-1}\left(\lim_{n\to\infty} a_{n-k}^{-1} n\right) = 1.\end{aligned}$$

Q.E.D.

If $a = \{a_k\}_{k=1}^{\infty}$ satisfies Assumption 1 and Assumption 2, the expression (18) and Theorem 3 can be used for numerical evaluation of z-$\prod_{k=1}^{\infty}(1+\frac{x}{a_k})$ in some cases. The method is as follows: First, for a suitably large n and $j_0 < n$, construct a Lagrange's polynomial $h_a^{(n,j_0)}(x)$ of degree (j_0-1) that interpolates the points

$$(x_j, y_j) = \left(a_{n+1-j}, \sum_{k=1}^{n-j}\log a_k\right), \quad j = 0,1,\ldots,j_0-1.$$

Substituting $h_a^{(n,j_0)}(x)$ for $\log G^{(n+1)}(x)$ in (18), we calculate

$$c^{(n,n',j_0)}(x) := \frac{\prod_{k=1}^{n'-1}(a_k+x)}{\exp\left(h_a^{(n,j_0)}(a_{n'}+x)\right)}$$

as an approximated value of z-$\prod_{k=1}^{\infty}(1+\frac{x}{a_k})$. (In doing this, to prevent overflow or underflow, all calculations should be done by taking logarithm, i.e., we calculate

$$\sum_{k=1}^{n'-1}\log(a_k+x) - h_a^{(n,j_0)}(a_{n'}+x)$$

then plug the result into the exponential function.) Here n' is a suitable integer between $n-j_0$ and n. Probably, it is better to pick up n' from the middle of the interval $[n+1-j_0, n]$.

Example 2 Let us consider the square of the Wallis formula again. The sequence dealt in Example 1, i.e., $a_k = k + \frac{1}{4} + \frac{1}{4(4k-1)}$ satisfies Assumption 2 for $\alpha = 2$, so that we can apply the above method to get an approximated value of z-$\prod_{k=1}^{\infty}(1-\frac{1}{a_k})$.

For $n = 30, 300, 3000$, we constructed Lagrange polynomials $h_a^{(n,5)}$, and calculated $c^{(n,n-2,5)}(1)$, which are listed in the table below. Since the true value is

$$1/\pi = 1/3.14159265\ldots,$$

roughly speaking, the error decreases at the rate of $O(n^{-2})$.

For comparison, we also calculated $w(n) := n\prod_{k=1}^{n}(1-\frac{1}{a_k})$ as approximated values due to the Wallis formula. This time, the error decreases at the rate of $O(n^{-1})$.

n	$c^{(n,n-2,5)}(1)$	$w(n)$
30	1/3.14059	1/3.16788
300	1/3.1415789	1/3.14421
3000	1/3.141592468	1/3.14185

In this way, $c^{(n,n-2,5)}(1)$ is much better than $w(n)$. But this example may be a special case, and since we have not established a precise error estimate, we do not know if our method is valid for general cases.

3 Randomized special functions

By randomizing the objects in the previous sections, we can find a new type of limit theorems in probability theory.

3.1 In the case of Poisson process

Let $\{\xi_i\}_{i=0}^{\infty}$ be a positive i.i.d. random variables whose common distribution is the exponential distribution with parameter 1, i.e.,[4]

$$P(\xi_i \le x) = \int_0^x e^{-t}dt = 1 - e^{-x}, \quad x \ge 0,$$

and set

$$X = \{X_k\}_{k=1}^{\infty} := \{\xi_1 + \cdots + \xi_k\}_{k=1}^{\infty}.$$

Then by virtue of the strong law of large numbers, the sequence $\{b + X_k\}_{k=1}^{\infty}$, $b > 0$, satisfies Assumption 1 almost surely. Note that

$$\eta(t) := F_X(t) = \#\{k | X_k \le t\}, \quad t \ge 0,$$

is a standard Poisson process.

3.1.1 Randomization of the Wallis formula

First let us calculate the distribution of the following random variable.

$$W(b,\lambda) := \text{z-}\prod_{k=1}^{\infty}\left(1 + \frac{\lambda}{b + X_k}\right), \quad \lambda \ge -b,\ b > 0.$$

Theorem 4 The n-th moment of $z(b,\lambda)$ is calculated as follows. [5]

$$\mathbf{E}\left[W(b,\lambda)^n\right] = \begin{cases} b^{-\lambda}, & n = 1, \\ b^{-n\lambda}\exp\left(\sum_{r=2}^{n}\binom{n}{r}\frac{\lambda^r}{(r-1)b^{r-1}}\right), & n = 2,3,\ldots \end{cases}$$

Lemma 4 (Durrett[1], (5.1) Theorem, Chapt.3.) *Under the conditional probability measure $P(\cdot \mid \eta(t) = N)$, $t > 0$, the distribution of $\{X_k\}_{k=1}^{N}$ coincides with that of the order statistics of N independent uniformly distributed random variables in $[0,t]$.*

[4] P stands for *probability*.

[5] $\mathbf{E}$ stands for *expectation*.

Sketch of Proof of Theorem 4.
Step 1. Let $\{X_{t,k}\}_{k=1}^{\infty}$ be independent uniformly distributed random variables in $[0,t]$. First, we define a random variable

$$W_{t,N,\lambda} := \prod_{k=1}^{N}\left(1+\frac{\lambda}{b+X_{t,k}}\right), \quad N\in\mathbf{N}.$$

and calculate its moments.

$$\begin{aligned}
&\mathbf{E}\left[(W_{t,N,\lambda})^n\right]\\
&= \prod_{k=1}^{N}\mathbf{E}\left[\left(1+\frac{\lambda}{b+X_{t,k}}\right)^n\right] = \left(\int_0^t\left(1+\frac{\lambda}{b+y}\right)^n\frac{dy}{t}\right)^N\\
&= \left(\sum_{r=0}^{n}\binom{n}{r}\lambda^r\frac{1}{t}\int_0^t\frac{dy}{(b+y)^r}\right)^N\\
&= \left(1+\frac{n\lambda}{t}\Big[\log(b+y)\Big]_0^t+\sum_{r=2}^{n}\binom{n}{r}\lambda^r\frac{1}{t}\Big[\frac{-1}{(r-1)(b+y)^{r-1}}\Big]_0^t\right)^N\\
&= \left(1+\frac{n\lambda}{t}(\log(b+t)-\log b)+\frac{1}{t}\sum_{r=2}^{n}\binom{n}{r}\frac{\lambda^r}{r-1}\Big[\frac{1}{b^{r-1}}-\frac{1}{(b+t)^{r-1}}\Big]\right)^N\\
&= \left(1+\frac{n\lambda}{t}(\log(b+t)-\log b)+\frac{1}{t}\sum_{r=2}^{n}\binom{n}{r}\frac{\lambda^r}{(r-1)b^{r-1}}+O(t^{-2})\right)^N\\
&= \left(1+\frac{n\lambda\log(b+t)}{t}-\frac{n\lambda\log b+C}{t}+O(t^{-2})\right)^N, \quad t\to\infty,
\end{aligned}$$

where

$$C := \sum_{r=2}^{n}\binom{n}{r}\frac{\lambda^r}{(r-1)b^{r-1}}.$$

Step 2. Since $P(\eta(t)=N)=t^Ne^{-t}/N!$, Lemma 4 implies that

$$\begin{aligned}
\mathbf{E}\left[\eta(t)^{-n\lambda}\prod_{k=1}^{\eta(t)}\left(1+\frac{\lambda}{1+X_k}\right)^n\right] &= \sum_{N=0}^{\infty}\mathbf{E}\left[N^{-n\lambda}\prod_{k=1}^{N}\left(1+\frac{\lambda}{1+X_k}\right)^n;\eta(t)=N\right]\\
&= \sum_{N=0}^{\infty}\mathbf{E}\left[N^{-n\lambda}\prod_{k=1}^{N}\left(1+\frac{\lambda}{1+X_k}\right)^n\,\middle|\,\eta(t)=N\right]\frac{t^Ne^{-t}}{N!}\\
&= \sum_{N=0}^{\infty}N^{-n\lambda}\mathbf{E}\left[(W_{t,N,\lambda})^n\right]\frac{t^Ne^{-t}}{N!}.
\end{aligned}$$

A change of variables $u := N/t$ shows

$$\mathbf{E}\left[\eta(t)^{-n\lambda}\prod_{k=1}^{\eta(t)}\left(1+\frac{\lambda}{1+X_k}\right)^n\right] = \sum_{u\in\frac{1}{t}\mathbf{N}}(tu)^{-n\lambda}\mathbf{E}\left[(W_{t,tu,\lambda})^n\right]\frac{t^{tu}e^{-t}}{(tu)!}.$$

Since the distribution of $\eta(t)/t$ is convergent to the Dirac measure $\delta_1(du)$, we see

$$\mathbf{E}\left[W(b,\lambda)^n\right] = \mathbf{E}\left[\lim_{t\to\infty}\eta(t)^{-n\lambda}\prod_{k=1}^{\eta(t)}\left(1+\frac{\lambda}{1+X_k}\right)^n\right]$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \mathbf{E}\left[\eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left(1+\frac{\lambda}{1+X_k}\right)^n\right] \\
&= \lim_{t\to\infty} \sum_{u\in\frac{1}{t}\mathbf{N}} (tu)^{-n\lambda}\mathbf{E}\left[(W_{t,tu,\lambda})^n\right] \frac{t^{tu}e^{-t}}{(tu)!} \\
&= \lim_{t\to\infty} t^{-n\lambda}\mathbf{E}\left[(W_{t,t,\lambda})^n\right] = b^{-n\lambda}e^{C}.
\end{aligned}
$$

The above calculation can be rigorously justified. Q.E.D.

In case $b=1$, if $|\lambda| \ll 1$, then the n-th moments of $W(1,\lambda)$ is approximately equal to $\exp(\frac{n(n-1)}{2}\lambda^2)$, which is nothing but the n-th moment of a log normal distribution, more precisely, the distribution of the random variable e^Y where Y is distributed as $\mathcal{N}(-\frac{\lambda^2}{2},\lambda^2)$. Hence when $|\lambda| \ll 1$, the distribution of $W(1,\lambda)$ is close to that of e^Y. [6]

3.1.2 Random digamma function

Proposition 3 implies that

$$\log W(1,\lambda) = Q\lambda + \sum_{k=1}^{\infty}\left[\log\left(1+\frac{\lambda}{1+X_k}\right) - \frac{\lambda}{1+X_k}\right],$$

where

$$Q = \lim_{N\to\infty}\left[\sum_{k=1}^{N}\frac{1}{1+X_k} - \log N\right].$$

This limit exists a.s.

Now, we have

$$\mathbf{E}[Q] = 0, \qquad \mathbf{E}[Q^2] = 1,$$

which is shown in the following way. First, it is easy to see that

$$\sum_{k=1}^{\infty}\left[\log\left(1+\frac{\lambda}{1+X_k}\right) - \frac{\lambda}{1+X_k}\right] = O(\lambda^2), \quad \lambda\to 0.$$

Hence we see

$$\mathbf{E}[\log W(1,\lambda)] = -\mathbf{E}[Q]\lambda + O(\lambda^2), \quad \lambda \to 0.$$

On the other hand, since the mean of $\log W(1,\lambda)$ is approximately equal to $-\lambda^2/2$, we see $\mathbf{E}[Q]=0$. And the 2-nd moment of $\log W(1,\lambda)$ is approximately equal to λ^2, so we see $\mathbf{E}[Q^2]=1$.

Suppose $|\lambda| \ll 1$. Since $-Q\lambda$ is the main part of $\log W(1,\lambda)$, and $\log W(1,\lambda)$ is approximately distributed as $\mathcal{N}(-\frac{\lambda^2}{2},\lambda^2)$, one may well expect that Q is distributed as $\mathcal{N}(0,1)$. But although its distribution is close to $\mathcal{N}(0,1)$, it is not exactly distribute as $\mathcal{N}(0,1)$. In deed we have $\mathbf{E}[Q^3]=1/2$.

Let us investigate a little bit more general case. Let

$$Q(x) := \lim_{N\to\infty}\left[\sum_{k=1}^{N}\frac{1}{x+X_k} - \log N\right], \quad x>0.$$

Comparing with (2), we can say that $-Q(x)$ is a random digamma function.

[6]This may hold in any case where $\{X_k\}_{k=1}^{\infty}$ is the partial sum of positive i.i.d. random variables with mean 1.

Theorem 5 The mean of $Q(x)$ is $\mathbf{E}[Q(x)] = -\log x$, and the centered moments of $Q(x)$ are

$$\mathbf{E}[(Q(x)+\log x)^n] = n! \sum_{p=1}^{\lfloor\sqrt{2n}\rfloor} \sum_{\substack{2\le n_1<\cdots<n_p \\ k_1n_1+\cdots+k_pn_p=n}} \frac{1}{x^{n-(k_1+\cdots+k_p)}} \times \prod_{j=1}^{p} \frac{1}{k_j!(n_j!)^{k_j}(n_j-1)^{k_j}}, \quad n=2,3,\ldots \tag{21}$$

More concretely, the centered moments of order $1,2,\ldots,6$ are

$$0,\quad \frac{1}{x},\quad \frac{1}{2x^2},\quad \frac{3}{x^2}+\frac{1}{3x^3},\quad \frac{5}{x^3}+\frac{1}{4x^4},\quad \frac{15}{x^3}+\frac{15}{2x^4}+\frac{1}{5x^5}.$$

Sketch of Proof. As in the previous section, let $\{X_{t,k}\}_{k=1}^N$ be i.i.d. uniform random variables in $[0,t]$. Define

$$Y_{t,N} := \sum_{k=1}^{N}\left(\frac{1}{x+X_{t,k}} - c(t)\right), \quad c(t) = \frac{1}{t}\left(\log(x+t) - \log x\right).$$

Then, when $t \to \infty$, we have

$$\mathbf{E}\left[\left(\frac{1}{x+X_{t,k}} - c(t)\right)^n\right] = \begin{cases} 0 & (n=1) \\ \dfrac{1}{x^{n-1}(n-1)t} + O(t^{-2+\varepsilon}) & (n\ge 2) \end{cases} \tag{22}$$

Here $\varepsilon > 0$ can be arbitrarily small. Indeed, if we write the L.H.S. of the above expression by integrals,

$$\begin{aligned}
&= \int_0^t \left(\frac{1}{x+y} - c(t)\right)^n \frac{dy}{t} \\
&= \sum_{r=0}^{n} \binom{n}{r} (-c(t))^{n-r} \frac{1}{t} \int_0^t (x+y)^{-r} dy \\
&= \sum_{r=0}^{n} \binom{n}{r} (-c(t))^{n-r} \times \begin{cases} 1 & (r=0) \\ c(t) & (r=1) \\ \frac{1}{(r-1)t}\left(x^{-r+1} - (x+t)^{-r+1}\right) & (r\ge 2) \end{cases} \\
&= c(t)^n - nc(t)^n + \sum_{r=2}^{n} \binom{n}{r} (-c(t))^{n-r} \frac{1}{(r-1)t}\left(\frac{1}{x^{r-1}} - \frac{1}{(x+t)^{r-1}}\right).
\end{aligned}$$

Here we have $c(t) = O(t^{-1+\varepsilon})$, $t \to \infty$, and hence we obtain (22).

Under these preparations, we see

$$\begin{aligned}
&\mathbf{E}\left[Y_{t,N}^n\right] \\
&= \sum_{n_1+n_2+\cdots+n_N=n} \mathbf{E}\left[\prod_{k=1}^{N}\left(\frac{1}{x+X_{t,k}} - c(t)\right)^{n_k}\right] \\
&= \sum_{n_1+n_2+\cdots+n_N=n} \prod_{k=1}^{N} \mathbf{E}\left[\left(\frac{1}{x+X_{t,k}} - c(t)\right)^{n_k}\right]
\end{aligned}$$

$$= \sum_{p=1}^{n} \frac{N!}{(N-p)!} \sum_{\substack{2\le n_1<\cdots<n_p \\ k_1n_1+k_2n_2+\cdots+k_pn_p=n}} \frac{1}{k_1!k_2!\cdots k_p!} \cdot \frac{n!}{(n_1!)^{k_1}(n_2!)^{k_2}\cdots(n_p!)^{k_p}} \times$$

$$\prod_{j=1}^{p} \mathbf{E}\left[\left(\frac{1}{x+X_{t,1}} - c(t)\right)^{n_j}\right]^{k_j}$$

$$= \sum_{p=1}^{n} N(N-1)\cdots(N-p+1) \times$$

$$\sum_{\substack{2\le n_1<\cdots<n_p \\ k_1n_1+k_2n_2+\cdots+k_pn_p=n}} \frac{1}{k_1!k_2!\cdots k_p!} \cdot \frac{n!}{(n_1!)^{k_1}(n_2!)^{k_2}\cdots(n_p!)^{k_p}} \times$$

$$\prod_{j=1}^{p} \left(\frac{1}{x^{n_j-1}(n_j-1)t} + O(t^{-2+\varepsilon})\right)^{k_j}$$

$$= \sum_{p=1}^{n} \left(1-\frac{1}{N}\right)\cdots\left(1-\frac{p-1}{N}\right)\cdot\left(\frac{N}{t}\right)^{p} \times$$

$$\sum_{\substack{2\le n_1<\cdots<n_p \\ k_1n_1+k_2n_2+\cdots+k_pn_p=n}} \frac{1}{x^{n-(k_1+k_2+\cdots+k_p)}} \cdot \frac{1}{k_1!k_2!\cdots k_p!} \cdot \frac{n!}{(n_1!)^{k_1}(n_2!)^{k_2}\cdots(n_p!)^{k_p}} \times$$

$$\prod_{j=1}^{p} \left(\frac{1}{n_j-1} + O(t^{-1+\varepsilon})\right)^{k_j}.$$

From this, it follows similarly to the previous theorem that

$$\begin{aligned}
\mathbf{E}[(Q(x)+\log x)^n] &= \mathbf{E}\left[\lim_{t\to\infty}\left(\sum_{k=1}^{\eta(t)}\left(\frac{1}{x+X_k} - c(t)\right)\right)^{n}\right] \\
&= \lim_{t\to\infty} \mathbf{E}\left[\left(\sum_{k=1}^{\eta(t)}\left(\frac{1}{x+X_k} - c(t)\right)\right)^{n}\right] \\
&= \lim_{t\to\infty} \sum_{N=0}^{\infty} \mathbf{E}\left[Y_{t,N}^{n}\right]\frac{t^N e^{-t}}{N!} \\
&= \lim_{t\to\infty} \sum_{u\in\frac{1}{t}\mathbf{N}} \mathbf{E}\left[Y_{t,tu}^{n}\right]\frac{t^{tu} e^{-t}}{(tu)!} \left(= \lim_{t\to\infty}\mathbf{E}\left[Y_{t,t}^{n}\right]\right) \\
&= \sum_{p=1}^{n} \sum_{\substack{2\le n_1<\cdots<n_p \\ k_1n_1+k_2n_2+\cdots+k_pn_p=n}} \frac{1}{x^{n-(k_1+k_2+\cdots+k_p)}} \times \\
&\qquad \frac{1}{k_1!k_2!\cdots k_p!} \cdot \frac{n!}{(n_1!)^{k_1}(n_2!)^{k_2}\cdots(n_p!)^{k_p}} \prod_{j=1}^{p} \frac{1}{(n_j-1)^{k_j}}.
\end{aligned}$$

Now it is easy to get (21). Q.E.D.

The following theorem is an easy consequence of the law of iterated logarithm and Theorem 2.

Theorem 6 *For any* $\varepsilon > 0$, $Q(x) = -\log x + O(x^{-(1/2)+\varepsilon})$, $x \to \infty$, *a.s.*

Regarding this theorem as a law of large numbers, the following corresponds to the central limit theorem.

Theorem 7 *The distribution of $\sqrt{x}(Q(x)+\log x)$ converges to $\mathcal{N}(0,1)$ as $x\to\infty$.*

Proof. The n-th moment of $\sqrt{x}(Q(x)+\log x)$ is

$$\mathbf{E}\left[(\sqrt{x}(Q(x)+\log x))^n\right] = n! \sum_{p=1}^{\lfloor\sqrt{2n}\rfloor} \sum_{\substack{2\le n_1<\cdots<n_p \\ k_1n_1+\cdots+k_pn_p=n}} \frac{1}{x^{n/2-(k_1+\cdots+k_p)}} \prod_{j=1}^{p} \frac{1}{k_j!(n_j!)^{k_j}(n_j-1)^{k_j}}.$$

In the sum $\sum_{p=1}^{\lfloor\sqrt{2n}\rfloor}$ of the R.H.S., the sum for p such that $n/2 > k_1+\cdots+k_p$ converge to 0 as $x\to\infty$. Therefore as $x\to\infty$, what survive are the terms for p such that $n/2 = k_1+\cdots+k_p$ (from this n must be even), that is, for $p=1$, $k_1=n/2$, $n_1=2$. Hence we see that

$$\lim_{x\to\infty}\mathbf{E}\left[(\sqrt{x}(Q(x)+\log x))^n\right] = \frac{n!}{2^{n/2}(n/2)!}, \quad (\ n\text{: even }).$$

This is nothing but the n-th moment of $\mathcal{N}(0,1)$. Q.E.D.

Suppose that unit electric charges are located at each random point of $\{X_k\}_{k=1}^{\infty}$, the renormalized Coulomb potential $Q(x)$ at the location $-x$ is distributed approximately as $\mathcal{N}(-\log x, 1/x)$ if x is large.

Another proof of Theorem 7. Since the distribution function $F_{\{x+X_k\}}(t)$ corresponding to the sequence $\{x+X_k\}_{k=1}^{\infty}$ is exactly $\eta(t-x)$, (5) implies that

$$\begin{aligned} Q(x) &= -\log x + 1 + \int_x^{\infty} (\eta(t-x)-t)\frac{dt}{t^2} \\ &= -\log x + \int_0^{\infty} (\eta(t)-t)\frac{dt}{(t+x)^2}. \end{aligned}$$

Since $\mathbf{E}[\eta(t)-t]=0$, we readily see $\mathbf{E}[Q(x)] = -\log x$. From the following expression

$$\sqrt{x}\,(Q(x)+\log x) = \int_0^{\infty} (\eta(t)-t)\frac{\sqrt{x}}{(t+x)^2}dt, \tag{23}$$

let us derive Theorem 7 by using the Lindeberg - Feller theorem ([2] Chapt.2 (4.5) Theorem).

First, note that $\tilde{\eta}(t) := \eta(t)-t$ is a martingale with mean 0. The Fubini theorem (or integration by parts formula) implies

$$\begin{aligned} \int_S^T \tilde{\eta}(t)\frac{\sqrt{x}}{(t+x)^2}dt &= \int_S^T \left(\tilde{\eta}(S) + \int_S^t d\tilde{\eta}(s)\right)\frac{\sqrt{x}}{(t+x)^2}dt \\ &= \int_S^T \left(\int_s^T \frac{\sqrt{x}}{(t+x)^2}dt\right)d\tilde{\eta}(s) + \tilde{\eta}(S)\int_n^T \frac{\sqrt{x}}{(t+x)^2}dt \\ &= \int_S^T \left(\frac{\sqrt{x}}{s+x} - \frac{\sqrt{x}}{T+x}\right)d\tilde{\eta}(s) + \left(\frac{\sqrt{x}}{S+x} - \frac{\sqrt{x}}{T+x}\right)\tilde{\eta}(S) \\ &= \int_S^T \frac{\sqrt{x}}{s+x}d\tilde{\eta}(s) - \frac{\sqrt{x}}{T+x}\tilde{\eta}(T) + \frac{\sqrt{x}}{S+x}\tilde{\eta}(S). \end{aligned}$$

Letting $S \to 0$, $T \to \infty$, we have the fallowing expression.

$$\sqrt{x}\,(Q(x) + \log x) = \int_0^\infty \tilde{\eta}(t)\frac{\sqrt{x}}{(t+x)^2}dt = \int_0^\infty \frac{\sqrt{x}}{t+x}d\tilde{\eta}(t), \quad \text{a.s.}$$

Now put

$$\begin{cases} U_{n,m} := \displaystyle\int_{(m-1)^2}^{m^2} \frac{\sqrt{n}}{t+n}d\tilde{\eta}(t), \quad m = 1, 2, \dots, n-1, \\ U_{n,n} := \displaystyle\int_{(n-1)^2}^{\infty} \frac{\sqrt{n}}{t+n}d\tilde{\eta}(t). \end{cases}$$

Let us the triangular array $\{U_{n,m}\}_{1\le m\le n, 1\le n}$ satisfies the Lindeberg - Feller's conditions.

Step 1. Since $\{\eta(t)\}_{t\ge 0}$ is an independent increment process, $\{U_{n,m}\}_{m=1}^n$ is independent sequence for each n.

Step 2. It holds (without letting $n \to \infty$) that

$$\begin{aligned} \sum_{m=1}^{n} \mathbf{E}\left[U_{n,m}^2\right] &= \sum_{m=1}^{n-1} \int_{(m-1)^2}^{m^2} \left(\frac{\sqrt{n}}{t+n}\right)^2 dt + \int_{(n-1)^2}^{\infty} \left(\frac{\sqrt{n}}{t+n}\right)^2 dt \\ &= \int_0^\infty \frac{n}{(t+n)^2}dt = 1. \end{aligned}$$

Step 3. Now, to prove the theorem, it is sufficient to show the last Lindeberg - Feller's condition: for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \sum_{m=1}^{n} \mathbf{E}\left[U_{n,m}^2\,;\, |U_{n,m}| > \varepsilon\right] = 0. \tag{24}$$

Lemma 5 *If a random variable U has the 4-th moment, then*

$$\mathbf{E}\left[U^2; |U| > \varepsilon\right] \le \varepsilon^{-2}\mathbf{E}[U^4].$$

Proof.

$$\mathbf{E}\left[U^2; |U| > \varepsilon\right] \le \mathbf{E}\left[U^2 \cdot \frac{U^2}{\varepsilon^2}; |U| > \varepsilon\right] \le \frac{1}{\varepsilon^2}\mathbf{E}\left[U^4\right].$$

Q.E.D.

Since

$$\mathbf{E}[U_{n,n}^2] = \int_{(n-1)^2}^{\infty} \left(\frac{\sqrt{n}}{t+n}\right)^2 dt = \frac{n}{(n-1)^2 + n} \longrightarrow 0, \quad n \to \infty,$$

it is sufficient to prove

$$\lim_{n\to\infty} \sum_{m=1}^{n-1} \mathbf{E}[U_{n,m}^4] = 0 \tag{25}$$

to show (24) by Lemma 5.

Let us estimate each of $\mathbf{E}[U_{n,m}^4]$. Putting

$$U_m(t) := \int_{(m-1)^2}^{t} \frac{\sqrt{n}}{s+n}d\tilde{\eta}(s), \quad m \le t,$$

and applying the Itô formula, we have

$$\begin{aligned}
\mathbf{E}\left[U_{n,m}^4\right] &= \mathbf{E}\left[\int_{(m-1)^2}^{m^2}\left(\left(U_m(t)+\frac{\sqrt{n}}{t+n}\right)^4 - U_m(t)^4 - 4U_m(t)^3\frac{\sqrt{n}}{t+n}\right)dt\right] \\
&= \int_{(m-1)^2}^{m^2}\left(6\,\mathbf{E}\left[U_m(t)^2\right]\left(\frac{\sqrt{n}}{t+n}\right)^2 + \left(\frac{\sqrt{n}}{t+n}\right)^4\right)dt \\
&= \int_{(m-1)^2}^{m^2}\left(6\left(\frac{n}{(m-1)^2+n} - \frac{n}{t+n}\right)\frac{n}{(t+n)^2} + \frac{n^2}{(t+n)^4}\right)dt \\
&= n^2\int_{(m-1)^2}^{m^2}\left(6\cdot\frac{t-(m-1)^2}{((m-1)^2+n)(t+n)^3} + \frac{1}{(t+n)^4}\right)dt \\
&< n^2\int_{(m-1)^2}^{m^2}\left(6\cdot\frac{2m-1}{((m-1)^2+n)^4} + \frac{1}{((m-1)^2+n)^4}\right)dt \\
&= 6n^2\cdot\frac{(2m-1)^2}{((m-1)^2+n)^4} + n^2\cdot\frac{2m-1}{((m-1)^2+n)^4} \\
&= n^2\cdot\frac{6(2m-1)^2+2m-1}{((m-1)^2+n)^4}.
\end{aligned}$$

From this, we derive

$$\sum_{m=1}^{n-1}\mathbf{E}\left[U_{n,m}^4\right] < n^2\sum_{m=1}^{n-1}\frac{6(2m-1)^2+2m-1}{((m-1)^2+n)^4} = O(n^{-1/2}), \quad n\to\infty,^7$$

thus (25) holds. Now the proof of Theorem 7 is complete. Q.E.D.

3.2 In the case of random walk

We next consider the case where $\{\xi_i\}_{i=1}^\infty$ is a Bernoulli sequence with $P(\xi_i = 0) = P(\xi_i = 2) = 1/2$, and $X_n := \sum_{i=1}^n \xi_i$. Again by the strong law of large numbers, $\{x + X_k\}_{k=1}^\infty$, $x > 0$, satisfies Assumption 1 almost surely.

3.2.1 Random digamma function

Defining G_k by

$$G_k := \#\{n\in\mathbf{N}\,;\, X_n = 2k\}, \quad k = 1,2,\ldots.$$

$\{G_k\}_{k=1}^\infty$ is an i.i.d. sequence with a geometric distribution $P(G_k = n) = 2^{-n}$, $n \in \mathbf{N}$, and we have

$$\sum_{k=1}^{G_1+\cdots+G_N}\frac{1}{x+X_k} = \sum_{k=0}^{N}\frac{G_k}{x+2k}.$$

Let us first look at the law of large numbers. Since $\mathbf{E}[G_k] = 2$, for sufficiently large N,

$$\log(G_1+G_2+\cdots+G_N) = \log\left(\frac{1}{N}\sum_{k=1}^{N}G_k\right) + \log N \approx \log 2 + \log N.$$

[7]For this estimate, use $n^2\int_0^\infty x^2/(x^2+n)^4dx = \pi/(32\sqrt{n})$.

Therefore, with probability 1,

$$Q(x) = \lim_{N\to\infty}\left[\sum_{k=0}^{N}\frac{G_k}{x+2k} - (\log 2 + \log N)\right]$$

converges. Theorem 2 implies that with probability 1,

$$Q(x) = -\log x + O(x^{-1}), \quad x \to \infty.$$

The mean of $Q(x)$ is computed as follows.

$$\begin{aligned}
\mathbf{E}[Q(x)] &= \lim_{N\to\infty}\mathbf{E}\left[\sum_{k=0}^{N}\frac{G_k}{x+2k} - (\log 2 + \log N)\right]\\
&= \lim_{N\to\infty}\left[\sum_{k=0}^{N}\frac{2}{x+2k} - (\log 2 + \log N)\right]\\
&= \lim_{N\to\infty}\left[\sum_{k=0}^{N}\frac{1}{(x/2)+k} - (\log 2 + \log N)\right]\\
&= -\psi((x/2)+1) - \log 2\\
&= -\log x + \sum_{k=0}^{\infty}\left[\frac{1}{(x/2)+k} + \log\left(1 - \frac{1}{(x/2)+k}\right)\right].
\end{aligned}$$

Next, let us look at the central limit theorem. We put

$$Q_N(x) := \sum_{k=0}^{N}\frac{G_k}{x+2k} - (\log 2 + \log N),$$

and calculate its characteristic function (Fourier transform). Noting that

$$\mathbf{E}\left[\exp(itG_k)\right] = \frac{\frac{1}{2}e^{it}}{1-\frac{1}{2}e^{it}},$$

we have

$$\begin{aligned}
\mathbf{E}\left[e^{itQ_N(x)}\right] &= \prod_{k=0}^{N}\mathbf{E}\left[\exp\left(it\cdot\frac{G_k}{x+2k}\right)\right]\exp\left(-it(\log 2 + \log N)\right)\\
&= \prod_{k=0}^{N}\left[\frac{\frac{1}{2}\exp\left(it\cdot\frac{1}{x+2k}\right)}{1-\frac{1}{2}\exp\left(it\cdot\frac{1}{x+2k}\right)}\right]\exp\left(-it(\log 2 + \log N)\right)\\
&= \prod_{k=0}^{N}\left[\frac{\frac{1}{2}\exp\left(-it\cdot\frac{1}{x+2k}\right)}{1-\frac{1}{2}\exp\left(it\cdot\frac{1}{x+2k}\right)}\right]\exp\left[-it\left(-\sum_{k=0}^{N}\frac{2}{x+2k} + \log 2 + \log N\right)\right].
\end{aligned}$$

Thus

$$\mathbf{E}\left[e^{itQ(x)}\right] = \prod_{k=0}^{\infty}\left[\frac{\exp\left(-it\cdot\frac{1}{x+2k}\right)}{2-\exp\left(it\cdot\frac{1}{x+2k}\right)}\right]\exp\left[-it\left(-\psi((x/2)+1) + \log 2\right)\right].$$

Let us take the limit $x \to \infty$ of the infinite product

$$\prod_{k=0}^{\infty}\left[\frac{\exp\left(-it\cdot\frac{1}{x+2k}\right)}{2-\exp\left(it\cdot\frac{1}{x+2k}\right)}\right] = \prod_{k=0}^{\infty}\left[\frac{1}{2\exp\left(it\cdot\frac{1}{x+2k}\right)-\exp\left(it\cdot\frac{2}{x+2k}\right)}\right].$$

Developing the denominator, we see

$$\begin{aligned}
&2\exp\left(it\cdot\frac{1}{x+2k}\right)-\exp\left(it\cdot\frac{2}{x+2k}\right)\\
&= 2\left(1+\frac{it}{x+2k}-\frac{1}{2}\cdot\frac{t^2}{(x+2k)^2}+\cdots\right)-\left(1+\frac{2it}{x+2k}-\frac{1}{2}\cdot\frac{4t^2}{(x+2k)^2}+\cdots\right)\\
&= 1+\frac{t^2}{(x+2k)^2}+\cdots,
\end{aligned}$$

and hence

$$\begin{aligned}
\prod_{k=0}^{\infty}\left[\frac{\exp\left(-it\cdot\frac{1}{x+2k}\right)}{2-\exp\left(it\cdot\frac{1}{x+2k}\right)}\right] &= \prod_{k=0}^{\infty}\left(1+\frac{t^2}{(x+2k)^2}+\cdots\right)^{-1}\\
&\sim \prod_{k=0}^{\infty}\exp\left(-\frac{t^2}{(x+2k)^2}\right)\\
&= \exp\left(-\frac{t^2}{2}\cdot\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{((x/2)+k)^2}\right), \quad x\to\infty.
\end{aligned}$$

From the above, when $x \gg 1$, the distribution of $Q(x)$ is close to the normal distribution with mean $-\psi((x/2)+1)+\log 2$ and variance $\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{((x/2)+k)^2}$. Since we have

$$\begin{aligned}
-\psi((x/2)+1)+\log 2 &= -\log x + O(x^{-1}),\\
\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{((x/2)+k)^2} &= x^{-1}+O(x^{-2}),
\end{aligned}$$

as $x \to \infty$, in the long run, we proved the following convergence in distribution.

$$\sqrt{x}(Q(x)+\log x) \to \mathcal{N}(1,0), \quad x\to\infty.$$

That is, the assertion of Theorem 7 holds in this case, too.

4 Further discussions

4.1 Extension of Theorem 6 and Theorem 7

Recently, Theorem 6 and Theorem 7 have been much more generalized by S.Takanobu.

Theorem 8 ([5]) *Let $\{\xi_i\}_{i=1}^{\infty}$ be an i.i.d. sequence with $\xi_i > 0$, $\mathbf{E}[\xi_i]=1$, and $\mathbf{E}[\xi_i^{\beta}] < \infty$ for some $\beta > 1$. Then we have $Q(x) = -\log x + O(x^{-1})$, $x\to\infty$, a.s..*

Theorem 9 ([5]) *Let $\{\xi_i\}_{i=1}^{\infty}$ be an i.i.d. sequence with $\xi_i > 0$, $\mathbf{E}[\xi_i]=1$, and $v := \mathbf{V}[\xi_i^2] < \infty$. Then, for $X = \{X_k\}_{k=1}^{\infty} = \{\xi_1+\cdots+\xi_k\}_{k=1}^{\infty}$, it holds that the distribution of $\sqrt{x}(Q_X(x)+\log x)$ converges to $\mathcal{N}(0,v)$ as $x\to\infty$.*

For i.i.d. sequences $\{\xi_i\}_{i=1}^{\infty}$ with $\xi_i > 0$, $\mathbf{E}[\xi_i] = 1$, but $\mathbf{V}[\xi_i^2] = \infty$, we have following limit theorem.

Theorem 10 ([5]) *(i) If* $[0, \infty) \ni s \mapsto \mathbf{E}[\xi_1^2; \xi_1 \le s] \in [0, \infty)$ *is slowly varying at* ∞, *there exists a positive sequence* $\{B_n\}_{n=1}^{\infty}$ *such that*

$$\frac{x}{B_{\lfloor x \rfloor}}(Q_X(x) + \log x) \longrightarrow \mathcal{N}(0,1), \quad x \to \infty, \quad \textit{in distribution.}$$

(ii) If there exist a $\beta \in (1,2)$ *and an* $L(\cdot)$ *which is slowly varying at* ∞ *such that*

$$P(\xi_1 > x) \sim L(x) x^{-\beta}, \quad x \to \infty,$$

then there exists $\{B_n\}_{n=1}^{\infty}$ *such that*

$$\begin{aligned}
&\lim_{x\to\infty} \mathbf{E}\left[\exp\left(\sqrt{-1}\,t(\beta-1)^{1/\beta}\frac{x}{B_{\lfloor x \rfloor}}(Q_X(x) + \log x)\right)\right] \\
&= \exp\left(\beta \int_0^{\infty}\left(e^{\sqrt{-1}by} - 1 - \sqrt{-1}by\right)\frac{dy}{y^{\beta+1}}\right).
\end{aligned}$$

These results with proofs will be written in a paper in near future.

4.2 The case of two dimensional random array of electric charges

We mentioned about the electro-static interpretation of random digamma function in § 3.1.2. In this context, a natural question arises: *Suppose that unit electrical charges are located at random in an unbouded domain of* $\mathbf{R}^2$. *Then, can we define a renormalized Coulomb potential as a random variable?*

Example 3 Suppose that the distribution of the unit electrical charges are described by a Poisson random measure on the out side of centered circle $B(O,x)^c$ with the Lebesgue measure as the intensity. Then the Coulomb potential at O will be expressed as

$$\int_x^{\infty} \frac{dN(\pi t^2)}{t} = \sqrt{\pi}\int_{x^2}^{\infty} \frac{dN(t)}{\sqrt{t}}$$

by a standard Poisson process $N(t)$, which is of course divergent. The renormalized potential would be

$$\sqrt{\pi}\int_{x^2}^{\infty} \frac{d\tilde{N}(t)}{\sqrt{t}}, \quad \tilde{N}(t) := N(t) - t,$$

but it is not well-defined because

$$\mathbf{E}\left[\left(\int_{x^2}^{\infty} \frac{d\tilde{N}(t)}{\sqrt{t}}\right)^2\right] = \int_{x^2}^{\infty} \frac{dt}{t} = \infty.$$

To look at the situation closely, let us observe the following deterministic case.: The sequence $a = \{\sqrt{k}\}_{k=1}^{\infty}$ is zeta regularizable, because the corresponding zeta function is $z(s) = \zeta(s/2)$. Hence by Theorem 2 in [4] and Theorem 1.8 in [3], we have

$$\text{z-}\prod_{k=1}^{\infty}\left(1 + \frac{x}{\sqrt{k}}\right) = \exp\left(\zeta\left(\frac{1}{2}\right)x - \frac{\gamma x^2}{2}\right)\prod_{k=1}^{\infty}\left(1 + \frac{x}{\sqrt{k}}\right)\exp\left(-\frac{x}{\sqrt{k}} + \frac{x^2}{2k}\right),$$

where

$$\zeta\left(\frac{1}{2}\right) = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{\sqrt{k}} - 2\sqrt{n}\right] = -1.46035\ldots$$

For $X = \{X_k\}_{k=1}^{\infty} := \{\xi_1 + \cdots + \xi_k\}_{k=1}^{\infty}$, partial sums of i.i.d. random variables, Assumption 1 is satisfied with $\delta < 1/2$, and then the corresponding zeta function

$$Z(s) = \sum_{k=1}^{\infty} X_k^{-s}$$

will become meromorphic in Re $s > 1/2$, but $Z(1/2)$ may not be defined. This fact has something to do with the non-existence of the limit

$$\int_{x^2}^{\infty}\frac{d\tilde{N}(t)}{\sqrt{t}} = \lim_{y\to\infty}\left[\sum_{x^2\le X_k\le y}\frac{1}{\sqrt{X_k}} - 2\sqrt{y} + 2x\right].$$

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