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A positive solution of a nonlinear scalar field equation

早稲田大学理工学研究科　　平田 潤 (Jun Hirata)

0. Introduction

This is a joint work with Kazunaga Tanaka. In this note we consider the following nonlinear Schrödinger equation:

$$\text{(NLS)} \begin{cases} -\Delta u + V(x)u = f(u) & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N). \end{cases}$$

Here $N \geq 3$, $V(x) \in C(\mathbf{R}^N, \mathbf{R})$ and $f(u) \in C(\mathbf{R}, \mathbf{R})$. Our main purpose is to show the existence of a positive solution of (NLS) with the nonlinearity

$$f(u) = |u|^{p-1}u - |u|^{q-1}u, \qquad 1 < p < q.$$

When $V(x) \equiv V_\infty$ is a constant, Berestycki-Lions **[BL]** obtain almost necessary and sufficient condition for the existence of a positive solution of (NLS). However, when $V(x)$ depends on x, this existence problem becomes delicate. For example, let us consider

$$-\Delta u + (1 + \varepsilon \arctan x_1)u = |u|^{p-1}u,$$

where $1 < p < \frac{N+2}{N-2}$. If $\varepsilon = 0$, this equation has a positive solution. However, for any $\varepsilon > 0$, this equation has only trivial solution. This example shows the existence of nontrivial solutions depends on $V(x)$ in a very delicate way. This difficulty comes from the lack of the Palais-Smale condition.

To overcome this difficulty, we usually assume $V(x) \to V_\infty > 0$ as $|x| \to \infty$ and $V(x) \leq V_\infty$ for all $x \in \mathbf{R}^N$. Rabinowitz **[R]** also assumes that $f(u)$ satisfies the global Ambrosetti-Rabinowitz condition and the monotonicity of $\frac{f(u)}{u}$ and he shows the existence of a positive solution of (NLS). Jeanjean-Tanaka **[JT2]** extends his result and they show that if $V(x) \to V_\infty$ suitably fast, (NLS) has a positive solution under the condition only $\frac{f(u)}{u} \to \infty$. However, when $f(u) \to -\infty$ as $u \to \infty$, the existence of problem seems not well-studied.

Our first result is the following:

Theorem 1. *We assume that* $N \geq 3$ *and* $V(x)$ *satisfies* $\inf_{x \in \mathbf{R}^N} V(x) > 0$ *and*

(v1) $\lim_{|x| \to \infty} V(x) = V_\infty$ *and*

$$0 < V_\infty < 2(q-p)\left(\frac{1}{(p+1)(q-1)}\right)^{\frac{q-1}{q-p}} (p-1)^{\frac{p-1}{q-p}} (q+1)^{\frac{p-1}{q-p}} \tag{0.1}$$

(v2) $x \cdot \nabla V(x) \in L^1(\mathbf{R}^N)$.

(v3) $V(x) \leq V_\infty$ *for all* $x \in \mathbf{R}^N$.

Then,

$$(*)\begin{cases} -\Delta u + V(x)u = |u|^{p-1}u - |u|^{q-1}u & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N). \end{cases}$$

has a positive solution.

As another approach to show the existence of a positive solution of (NLS), we use the symmetry of $V(x)$. Indeed, Hirata [**H2**] assumes that $V(x)$ is invariant under a finite group action, for example, $V(-x) = V(x)$ for all $x \in \mathbf{R}^N$. He also assume $V(x)$ converges to $V_\infty > 0$ suitably fast and $\frac{f(u)}{u} \gg 1$ as $u \to \infty$. Under above conditions, he shows the existence of a positive solution of (NLS) even without condition like (v3). (See also Adachi [**A**], Hirata [**H1**]). Our second result is in spirit of [**A,H1,H2**].

Theorem 2. *We assume that* $N \geq 3$ *and* $V(x)$ *satisfies* $\inf_{x \in \mathbf{R}^N} V(x) > 0$, *(v1)-(v2) and*

(v4) $V(-x) = V(x)$ *for all* $x \in \mathbf{R}^N$,

(v5) there exist $\alpha > 2$ *and* $C > 0$ *such that*

$$V_\infty - V(x) \geq -Ce^{-\alpha|x|} \qquad \text{for all } x \in \mathbf{R}^N.$$

Then, $(*)$ *has an even positive solution.*

Remark. (i) Theorem 2 does not need a condition like (v3). Thus we can apply Theorem 2 even if $V(x) > V_\infty$.

(ii) Conditions (v2) and (v5) mean $V(x) \to V_\infty$ suitably fast. In particular, (v2) and (v5) hold if $V(x) - V_\infty$ has compact support.

We also remark that if $V(x)$ is radially symmetric, Bartsch-Willem [**BWi**] show that the functional corresponding to (NLS) satisfies the Palais-Smale condition in radially symmetric functions space. In particular, Kikuchi [**K**] shows that $(*)$ has a positive solution if $V(|x|) = V(x)$ for all $x \in \mathbf{R}^N$ and $V(x) \to \infty$ as $|x| \to \infty$. See also Bartsch-Wang [**BWa**] and Hirata [**H2**] for study of (NLS) under more wide classes of symmetries.

In sections 1–2, we give outline of proofs of Theorems 1 and 2. In section 3, we deal with more general nonlinear scalar field equations.

1. Outline of the proof of Theorem 1

In this section, we find the nontrivial critical point of the following functional which corresponds to $(*)$:

$$I(u) := \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + V(x)u^2 dx - \int_{\mathbf{R}^N} \left(\frac{1}{p+1}|u|^{p+1} - \frac{1}{q+1}|u|^{q+1}\right) dx.$$

We remark that $I(u)$ has the mountain pass structure. However, since $I(u)$ does not satisfy the Palais-Smale condition, we cannot apply the mountain pass theorem to $I(u)$ directly. To overcome this difficulty, fist we use so-called the monotonicity method which originated by Struwe [**S**] (see also Jeanjean [**J**] and Rabier [**Ra**]) to find bounded Palais-Smale sequences.

1.1. Monotonicity method

For $\lambda \in [0, \frac{1}{2}]$, we consider the following perturbed equation:

$$(*)_\lambda \begin{cases} -\Delta u + V(x)u = (1+\lambda)|u|^{p-1}u - |u|^{q-1}u & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N). \end{cases}$$

The corresponding functional is

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + V(x)u^2 dx - \int_{\mathbf{R}^N} \left(\frac{1+\lambda}{p+1}|u|^{p+1} - \frac{1}{q+1}|u|^{q+1}\right) dx.$$

Since $I_\lambda(u)$ has a mountain pass structure, there is a function $v_\lambda \in H^1(\mathbf{R}^N)$ such that $I(v_\lambda) < 0$. We define the mountain pass level b_λ for

$$b_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbf{R}^N)) \mid \gamma(0) = 0, \gamma(1) = v_\lambda\}.$$

Using ideas in Struwe [**S**], Jeanjean [**J**], and Rabier [**Ra**], we have

Lemma 1.1. *(c.f. [**S,J,Ra**]) For almost every $\lambda \in [0, \frac{1}{2}]$, $I_\lambda(u)$ has a bounded Palais-Smale sequence.*

We remark that since $I_\lambda(u)$ has a mountain pass structure, we can see that $I_\lambda(u)$ has a Palais-Smale sequence by Ekeland's principle. However, since the nonlinearities $|u|^{p-1}u - |u|^{q-1}u$ does not satisfy the global Ambrosetti-Rabinowitz condition, that Palais-Smale sequence may not be bounded. On the other hand, Lemma 1.1 says that there is a sequence $(\lambda_j)_{j=1}^\infty \subset [0, \frac{1}{2}]$, $\lambda_j \searrow 0$ such that $I_{\lambda_j}(u)$ has a bounded Palais-Smale sequence $(u_n^{\lambda_j})_{n=1}^\infty \subset H^1(\mathbf{R}^N)$. Taking a subsequence if necessary, we may assume that $u_n^{\lambda_j}$ converges to a weak limit u_j. Next, we show that u_j is a nontrivial critical point of $I_{\lambda_j}(u)$.

1.2 Weak convergence of $I_\lambda(u)$

To show that u_j is a nontrivial critical point of $I_{\lambda_j}(u)$, the following limit equation and corresponding functional play important roles:

$$(**)_\lambda \begin{cases} -\Delta u + V_\infty u = (1+\lambda)|u|^{p-1}u - |u|^{q-1}u & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), \end{cases}$$

$$I_\lambda^\infty(u) := \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + V_\infty u^2 dx - \int_{\mathbf{R}^N} \left(\frac{1+\lambda}{p+1}|u|^{p+1} - \frac{1}{q+1}|u|^{q+1}\right) dx.$$

Since (0.1) holds, $(**)_\lambda$ has a ground-state solution ω (see Berestycki-Lions [**BL**]). Moreover, since $V(x) \le V_\infty$ and $V(x) \not\equiv V_\infty$, we have $b_{\lambda_j} < I^\infty(\omega)$. Thus, by usual concentration compactness argument, we can see that u_j is a nontrivial critical point of $I_{\lambda_j}(u)$ with

$I_{\lambda_j}(u_j) \le b_{\lambda_j}$. In next section, we show that $(u_j)_{j=1}^{\infty} \subset H^1(\mathbf{R}^N)$ is a bounded Palais-Smale sequence for the functional corresponding to the original problem $(*)$.

1.3 A priori estimate

In this section we show that (u_j) is a bounded Palais-Smale sequence. A similar result is shown in Jeanjean-Tanaka [**JT2**] for an equation (NLS) with a property $\frac{f(u)}{u} \to \infty$. For our problem, we argue as follows:

Since u_j is a critical point of $I_{\lambda_j}(u)$, we have the Pohozaev's identity:

$$\int_{\mathbf{R}^N} |\nabla u_j|^2 dx = N I_{\lambda_j}(u_j) + \frac{1}{2}\int_{\mathbf{R}^N} x \cdot \nabla V(x) u_j^2 dx. \tag{1.1}$$

On the other hand, by maximum principle, it is not difficult to find that (u_j) is bounded in $L^\infty(\mathbf{R}^N)$. Thus, the boundedness of $\|\nabla u_j\|_{L^2(\mathbf{R}^N)}$ follows from (v2) and (1.1). Since $\|\nabla u_j\|_{L^2(\mathbf{R}^N)}$ is bounded, we can see that (u_j) is a bounded Palais-Smale sequence of $I(u)$ by a similar way to [**JT2**].

1.4 Conclusion

Since (u_j) is bounded Palais-Smale sequence, we use concentration compactness argument again and we get a weak limit u_0 of (u_j) is a nontrivial critical point of $I(u)$. Thus, we have Theorem 1.

2. Outline of the proof of Theorem 2.

In this section, we give an outline of the proof of Theorem 2. We define the space of even functions

$$E := \{u(x) \in H^1(\mathbf{R}^N) \mid u(-x) = u(x) \quad \text{for all } x \in \mathbf{R}^N\}$$

and we consider the functional $I(u)$ corresponding to $(*)$ in E. We remark that $I(u)$ has a mountain pass structure. The following Lemma 2.1 is the key of this proof.

Lemma 2.1. *We assume (v1), (v4) and (v5). Let $v_0 \in E$ such that $I(v_0) < 0$ and we define the mountain pass level $b_E = b_E(v_0)$ by*

$$b_E = \inf_{\gamma \in \Gamma_E} \max_{t \in [0,1]} I(\gamma(t)),$$

$$\Gamma_E = \{\gamma(t) \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) = v_0\}.$$

Then, we have

$$b_E < 2 I^\infty(\omega).$$

Here $I^\infty(u)$ is the functional corresponding to the limit equation

$$(**)\begin{cases} -\Delta u + V_\infty u = |u|^{p-1}u - |u|^{q-1}u & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), \end{cases}$$

and $\omega(x)$ is its ground-state solution.

For a proof of Lemma 2.1, we need

$$I(\omega(x-s)+\omega(x+s)) < 2I^\infty(\omega) \qquad \text{for } s \in \mathbf{R}^N, |s| \gg 1. \tag{2.1}$$

We remark that this type estimates are so-called interaction estimates which are studied by many authors in various situation (see Taubes [**T**], Bahri-Li [**BaLi**], ...). We aslo remark that (2.1) follows from the fact that $\omega(x)$ has an exponential decay and $V(x)$ satisfies (v5). To estimate b_E, we use the following sample path:

$$\gamma(t) = \begin{cases} \omega(\frac{x}{t}-s)+\omega(\frac{x}{t}+s) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

where $s \in \mathbf{R}^N$ and $|s| \gg 1$. We remark that the path $t \mapsto \omega(\frac{x}{t})$ is used in Jeanjean-Tanaka [**JT1**] to show that for the autonomous equation $(**)$, the mountain pass solution is the ground state solution. Indeed, they show that $\omega(\frac{x}{t}) \to 0$ as $t \to 0$, $I^\infty(\omega(\frac{x}{t})) < I^\infty(\omega(x))$ for all $t \neq 1$, and $I^\infty(\omega(\frac{x}{t})) \to -\infty$ as $t \to \infty$. Our path $\gamma(t)$ is the even symmetry version of their path. From (2.1), we have

$$\gamma(0) = 0, \qquad I(\gamma(t)) \to -\infty \text{ as } t \to \infty,$$

$$\max_{t\in[0,\infty)} I(\gamma(t)) < 2I^\infty(\omega).$$

This implies Lemma 2.1.

Now, we prove Theorem 2. We consider the perturbed equation $(*)_\lambda$ and the corresponding functional $I_\lambda(u)$. By Lemma 2.1 and continuity of $\lambda \mapsto I_\lambda(u)$, there exists $v_0 \in E$ and $\lambda_0 \in (0, \frac{1}{2}]$ such that

$$I_\lambda(v_0) < 0 \qquad \text{for all } \lambda \in [0, \lambda_0],$$

$$b_\lambda = \inf_{\gamma\in\Gamma_E} \max_{t\in[0,1]} I_\lambda(\gamma(t)) < 2I^\infty(\omega) \qquad \text{for all } \lambda \in [0, \lambda_0].$$

Arguing as in section 1.1, we have a sequence $(\lambda_j)_{j=1}^\infty \subset [0, \lambda_0]$, $\lambda_j \to 0$ such that $I_{\lambda_j}(u)$ has a bounded Palais-Smale sequence $(u_n^{\lambda_j})_{n=1}^\infty \subset E$ at mountain pass level b_{λ_j}. The following Lemma 2.2 ensures that the weak limit u_j of $(u_n^{\lambda_j})$ is a nontirivial critical point of $I_{\lambda_j}(u)$.

Lemma 2.2. *We assume (v1) and (v4). Let $\lambda \in [0, \frac{1}{2}]$ and $(u_n) \subset E$ be a bounded Palais-Smale sequence of $I_\lambda(u)$ at level c. Moreover if $c < 2I^\infty(\omega)$, then a weak limit $u_0 \in E$ of (u_n) is a critical point of $I_\lambda(u)$ with $I_\lambda(u_0) \leq c$.*

We remark that Lemma 2.2 follows from the concentration compactness argument under symmentry assumption (see [**A,H1,H2**]). By Lemmas 2.1 and 2.2, we have that $u_j \in E$ is a nontrivial critical point of $I_{\lambda_j}(u)$. Thus, in a similar way to sections 1.3 and

1.4, we have that (u_j) is a bounded Palais-Smale sequence of $I(u)$ and it converges weakly to a nontrivial solution u_0 for $(*)$. Thus, we have Theorem 2.

3. Nonlinear scalar field equations

With the same idea to deal with Theorem 1, we can study more general equations. Here we give just a result for x-dependent nonlinear scalar field equations, which can be regarded as an x-dependent version of results of [**BGK,BL**]. More precisely we study the following nonlinear elliptic equation:

$$(\sharp)\begin{cases} -\Delta u = g(x,u) & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N). \end{cases}$$

Here $N \geq 2$ and $g(x,\xi) \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$. We remark that when $g(x,\xi) = -V(x)\xi + f(\xi)$ with $V(x) \in C(\mathbf{R}^N, \mathbf{R})$ and $f(\xi) \in C(\mathbf{R},\mathbf{R})$, $(\sharp)$ is a nonlinear Schrödinger equation (NLS). To state our main result, we set $G(x,\xi) = \int_0^\xi g(x,\tau)\,d\tau$ and assume

(g0) $G(x,\xi) : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is of class C^1.

(g1) When $N \geq 3$,

$$\limsup_{\xi\to\infty} \frac{g(x,\xi)}{\xi^{\frac{N+2}{N-2}}} = 0 \quad \text{uniformly in } x \in \mathbf{R}^N.$$

When $N = 2$, for any $\alpha > 0$ there exists $C_\alpha > 0$ such that

$$g(x,\xi) \leq C_\alpha e^{\alpha\xi^2} \quad \text{for all } x \in \mathbf{R}^N \text{ and } \xi \in \mathbf{R}.$$

(g2) $g(x,0) \equiv 0$ for all $x \in \mathbf{R}^N$ and there exists $m > 0$ such that

$$-\infty < \liminf_{\xi\to 0} \frac{g(x,\xi)}{\xi} \leq \limsup_{\xi\to 0} \frac{g(x,\xi)}{\xi} \leq -m < 0$$

uniformly in $x \in \mathbf{R}^N$.

(g3) There exists a function $g_\infty(\xi) \in C(\mathbf{R},\mathbf{R})$ such that

$$\lim_{|x|\to\infty} g(x,\xi) = g_\infty(\xi) \quad \text{uniformly on } \xi \text{ bounded.}$$

(g4) There exists $\zeta_0 > 0$ such that $G_\infty(\zeta_0) > 0$, where $G_\infty(\xi)$ is defined by

$$G_\infty(\xi) = \int_0^\xi g_\infty(\tau)d\tau.$$

(g5) $G(x,\xi) \geq G_\infty(\xi)$ for all $x \in \mathbf{R}^N$ and $\xi \in \mathbf{R}$.

(g6) There exists a continuous function $\nu : [0,\infty) \to [0,\infty)$ such that

$$\left| \int_{\mathbf{R}^N} x \cdot \nabla_x G(x,u)\, dx \right| \leq \nu(\|u\|_{L^\infty(\mathbf{R}^N)})$$

for $u \in H^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$.

(g7) $g(x,\xi)$ satisfies one of the following conditions:

(g7-a) There exists a uniformly continuous function $h(x) : \mathbf{R}^N \to (0,\infty)$ such that

(i) there exist c_1, $c_2 > 0$ such that

$$c_1 \le h(x) \le c_2 \qquad \text{for all } x \in \mathbf{R}^N .$$

(ii) There exists $p \in (1, \frac{N+2}{N-2})$ when $N \ge 3$, $p \in (1,\infty)$ when $N = 2$ such that

$$\lim_{\xi\to\infty} \frac{g(x,\xi)}{\xi^p} = h(x) \qquad \text{uniformly in } x \in \mathbf{R}^N.$$

(g7-b) There exists $\zeta_1 > \zeta_0$ such that

$$g(x,\zeta_1) \le 0 \quad \text{for all } x \in \mathbf{R}^N .$$

Our main result is as follows

Theorem 3. *We assume $N \ge 2$ and $g(x,\xi)$ satisfies (g0)–(g7). Then (♯) has a positive solution.*

For a proof of Theorem 3 we refer to [**HT**] and we give some remarks on conditions (g0)–(g7).

(i) When $N \ge 3$ and $g(x,\xi)$ is independent of the space variable x, that is, $g(x,\xi) = g(\xi) = g_\infty(\xi)$, the conditions (g1), (g2), (g4) are given in [**BL**] for the existence of a positive solution of x-independent problem:

$$-\Delta u = g(u) \qquad \text{in } \mathbf{R}^N .$$

Conditions (g5), (g6) hold if $g(x,\xi)$ is independent of x and Theorem 3 can be regarded as an extension of the result of [**BL**] to x-dependent equations.

(ii) When $N = 2$ and $g(x,\xi)$ is independent of x, [**BGK**] assumes (g1), (g2) and the following condition

$$\lim_{\xi\to 0} \frac{g(\xi)}{\xi} = -m < 0 \text{ exists},$$

which is slightly stronger than (g4). We remark that with our method we can extend the result of [**BGK**] slightly and we can show the existence of a positive solution for x-independent problem under conditions (g1), (g2), (g4) when $N = 2$.

(iii) The condition (g7) is a condition that ensures an a priori L^∞-bound for positive solutions and which covers many applications; (g7-a) covers nonlinear Schrödinger equations of type

$$-\Delta u + V(x)u = \pm u^p + u^q \qquad \text{in } \mathbf{R}^N$$

with $1 < p < q < \frac{N+2}{N-2}$ $(N \geq 3)$ and $1 < p < q < \infty$ $(N = 2)$. (g7-b) covers

$$-\Delta u + V(x)u = u^p - u^q \qquad \text{in } \mathbf{R}^N$$

with $1 < p < q$. In particular, Theorem 1 is the special case of Theorem 3.

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