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EXISTENCE AND NONEXISTENCE OF NONTRIVIAL STEADY STATES OF AN ACTIVATOR-INHIBITOR SYSTEM

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1. Introduction

Following A. Turing's celebrated idea of ***diffusion-driven instability*** [10], A. Gierer and H. Meinhardt [2] proposed a mathematical model for pattern formations of spatial tissue structures of ***hydra*** in morphogenesis, a biological phenomenon discovered by A. Trembley in 1744 [9]. It is a system of reaction-diffusion equations of the form

$$\text{(1.1)}\qquad \begin{cases} u_t = d_1\Delta u - u + \frac{u^p}{v^q} + \sigma & \text{in} \quad \Omega\times[0,T)\,, \\ \tau v_t = d_2\Delta v - v + \frac{u^r}{v^s} & \text{in} \quad \Omega\times[0,T)\,, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 & \text{on} \quad \partial\Omega\times[0,T) \end{cases}$$

where Δ is the Laplace operator, Ω is a bounded smooth domain in $\mathbb{R}^n$, $n\geq 1$ and ν is the unit outer normal to $\partial\Omega$. Here u, v represent respectively the concentrations of two substances, activator and inhibitor, with diffusion rates d_1, d_2, and are therefore always assumed to be positive. The source term σ is a nonnegative constant representing the production of the activator, $\tau > 0$ is the response rate of v to the change of u, and the exponents p, q, r, s are nonnegative numbers satisfying the condition

$$\text{(1.2)}\qquad 0 < \frac{p-1}{r} < \frac{q}{s+1}.$$

We remark that the response rate τ was introduced mathematically and is an important parameter on the stability of the system.

In this expository paper, we will explain our methods and results in [4] where various existence and nonexistence results on nontrivial steady states of (1.1) were obtained using newly established a priori estimates. For earlier results on this system, we refer the readers to [1][3][5][6][7][8] and the references therein.

2. Nonexistence of nontrivial solutions

For any $\sigma\geq 0$, the corresponding stationary equation

$$\text{(2.1)}\qquad \begin{cases} d_1\Delta u - u + \frac{u^p}{v^q} + \sigma = 0 & \text{in} \quad \Omega, \\ d_2\Delta v - v + \frac{u^r}{v^s} = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

has a unique constant solution (u^*, v^*) such that

$$\text{(2.2)}\qquad \begin{cases} -u^* + (u^*)^{p-\frac{qr}{s+1}} + \sigma = 0, \\ v^* = (u^*)^{\frac{s+1}{r}}. \end{cases}$$

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When $\sigma = 0$, we have $(u^*, v^*) \equiv (1,1)$.

One of the main theorems in [4] is the following nonexistence result

Theorem 2.1. *Assume* $\max\{q,r\} < s+1$. *There exists a constant* $k > 0$ *such that whenever* $\frac{d_2}{d_1} \leq k$, *we have* $(u,v) \equiv (u^*, v^*)$.

Remark 2.2. *The constants k can be calculated explicitly. For example, when $\sigma = 0$ and $(p,q,r,s) = (2,4,2,4)$, the "common source" case, we have $k = 1$.*

Theorem 2.1 is new even when $n = 1$ and it indicates that the ratio of two diffusion rates alone can prevent the existence of nontrivial patterns while all previously known nonexistence results for this system require that at least one of the diffusion rates d_1, d_2 be suitably large.

Theorem 2.1 is a consequence of the following optimal bounds:

Lemma 2.3. *(i). If $q < s+1$, then there exists $k_1 > 0$ depending on p,q,r,s,σ, such that whenever $\frac{d_2}{d_1} \leq k_1$, we have $u \leq u^*, v \leq v^*$.*
(ii). If $r < s+1$, then there exists $k_2 > 0$ depending on p,q,r,s,σ, such that whenever $\frac{d_2}{d_1} \leq k_2$, we have $u \geq u^*, v \geq v^*$.

The main idea in proving Lemma 2.3 is to use quantities of the form $\frac{u}{v^\lambda}$ to bound both u and v. Here $\lambda > 0$ has to be chosen carefully. To get a feeling of the technique we used, we sketch here the proof of the first part when $\sigma = 0$:

Step 1: Let $0 < \lambda < \frac{s+1}{r}$. Then $\frac{u}{v^\lambda}$ controls both u and v, more precisely,

$$\inf_\Omega \frac{u}{v^\lambda} \leq \left(\inf_\Omega u\right)^{1-\frac{r}{s+1}\lambda} \leq \left(\inf_\Omega v\right)^{\frac{s+1}{r}-\lambda} \leq \left(\sup_\Omega v\right)^{\frac{s+1}{r}-\lambda} \leq \left(\sup_\Omega u\right)^{1-\frac{r}{s+1}\lambda} \leq \sup_\Omega \frac{u}{v^\lambda}.$$

Step 2: Let $0 < \lambda \leq 1$. Then at any point $x^* \in \overline{\Omega}$ where $\frac{u}{v^\lambda}$ achieves its maximum, we have

$$1 - \frac{\lambda d_1}{d_2} \leq \frac{u^{p-1}}{v^q} - \frac{\lambda d_1}{d_2}\frac{u^r}{v^{s+1}}. \tag{2.3}$$

Step 3: Let $\lambda < \frac{s+1-q}{r-(p-1)}$ and

$$\frac{d_2}{d_1} = \frac{\lambda(s+1-\lambda r)}{q - \lambda(p-1)}. \tag{2.4}$$

Then (2.3) implies $\frac{u}{v^\lambda}(x^*) \leq 1$ which follows from Young's inequality. Hence $\frac{u}{v^\lambda} \leq 1$ in Ω and step 1 implies $u \leq 1$ and $v \leq 1$ in Ω.

Step 4: The number k_1 is chosen so that whenever $\frac{d_2}{d_1} < k_1$, one can find λ satisfies (2.4) and $0 < \lambda < \min\left\{1, \frac{s+1}{r}, \frac{s+1-q}{r-(p-1)}\right\}$. The case when $\frac{d_2}{d_1} = k_1$ can be proved by a limiting process.

On the other hand, standard energy estimate could also yield nonexistence results if we have certain uniform a priori estimates. Actually, if we have positive lower and upper a priori bounds for solutions to (2.1) which are uniform, then we have energy estimate

$$d_1 \|\nabla u\|^2_{L^2(\Omega)} \leq C \|\nabla u\|^2_{L^2(\Omega)}$$

where C is a constant depending on the uniform bounds of u, v. Hence, large d_1 implies nonexistence of nontrivial steady states. For example, the following two theorems were proved in [4]:

Theorem 2.4. *Assume $\sigma > 0$ and $\frac{p-1}{r} < \min\left\{1, \frac{2}{n}\right\}$. Then there exists constant $c > 0$, such that whenever $d_1 \geq c$, (1.1) has no nontrivial steady states.*

Theorem 2.5. *Assume $\sigma = 0$, $n = 2$ and $\frac{p-1}{r} < 1$. Then for any $d^* > 0$, there exists constant $c > 0$, such that whenever $d_2 \geq d^*$, and $d_1 \geq c$, (1.1) has no nontrivial steady states.*

3. Existence of nontrivial solutions

Nontrivial solutions do exist under certain situations when d_1 is small.

We consider the linearization of (2.1) around (u^*, v^*),

$$\begin{cases} d_1 \Delta h + f_u(u^*, v^*) h + f_v(u^*, v^*) k = 0 & \text{in } \Omega, \\ d_2 \Delta k + g_u(u^*, v^*) h + g_v(u^*, v^*) k = 0 & \text{in } \Omega, \\ \frac{\partial h}{\partial \nu} = \frac{\partial k}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where

$$f(u,v) = -u + \frac{u^p}{v^q} + \sigma, g(u,v) = -v + \frac{u^r}{v^s}.$$

Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of $-\Delta$ under Neumann boundary conditions in Ω. We also use m_i to denote the multiplicity of eigenvalue λ_i, $i = 0, 1, 2, \cdots$. For simplicity, we use f_u to denote $f_u(u^*, v^*)$, the same applies to f_v, g_u and g_v. The linear system (3.1) will possess a nontrivial solution if and only if the matrix

$$\begin{pmatrix} f_u - d_1\lambda_i & f_v \\ g_u & g_v - d_2\lambda_i \end{pmatrix} = \begin{pmatrix} p(u^*)^{p-1-\frac{qr}{s+1}} - 1 - d_1\lambda_i & -q(u^*)^{p-\frac{(q+1)r}{s+1}} \\ r(u^*)^{\frac{r}{s+1}-1} & -s-1-d_2\lambda_i \end{pmatrix}$$

is singular for some i. Hence, given $d_2 > 0$, for each $i \geq 1$, the linear system (3.1) will possess a nontrivial solution if and only if

$$d_1 = d_{1i} \equiv \frac{1}{\lambda_i}\left[p(u^*)^{p-1-\frac{qr}{s+1}} - 1 - \frac{qr(u^*)^{p-1-\frac{qr}{s+1}}}{s+1+d_2\lambda_i}\right].$$

We also define for any $d > 0$,

$$A_d = \{i \geq 1 : d < d_{1i}\}, N_d = \sum_{i \in A_d} m_i.$$

Now we can state the existence result in [4].

Theorem 3.1. *Assume that $\sigma > 0$ or*

$$\sigma = 0 \text{ and } n = 2.$$

Assume in addition that $\frac{p-1}{r} < 1$. If $d_1 \neq d_{1i}$, $i = 1, 2, 3, \cdots$, and N_{d_1} is odd, then there exists at least one nontrivial solution to (2.1).

Remark 3.2. *When $\sigma = 0$, condition $n = 2$ can be replaced by the following complicated but more general assumption: $r < \frac{n}{n-2}$ and there exists $\delta \in (0,1]$ such that*

$$0 < \frac{1-\delta}{r} + \frac{\delta}{p} < 1,$$

and

$$\frac{\frac{(1-\delta)s}{r} + \frac{\delta q}{p}}{\frac{r-1+\delta}{r} - \frac{\delta}{p}} < \frac{n}{n-2} \text{ or } \frac{\frac{(1-\delta)s}{r} + \frac{\delta q}{p}}{\frac{r-1+\delta}{r} - \frac{\delta}{p}} \leq s+1.$$

Theorem 3.1 is proved using Leray-Schauder topological degree theory. Let $X = C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ be the Banach space with norm

$$\|(u,v)\|_X = \max\left\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\right\}$$

and X^+ be the positive cone in X, i.e.,

$$X^+ = \{(u,v) \in X : u > 0, v > 0 \text{ in } \overline{\Omega}\}.$$

We define solution operators $S = (I - d_2\Delta)^{-1}$ and $R = (\varpi I - d_1\Delta)^{-1}$ under Neumann boundary conditions. Here $\varpi > 0$ is a large constant to be determined later. Let

$$T(u,v) = (R(f(u,v) + \varpi u), S(g(u,v) + v)).$$

Then T is an operator defined on X^+ and it is easy to check that (u,v) is a positive solution to (2.1) if and only if it is a fixed point of T in X^+, i.e.,

$$T(u,v) = (u,v).$$

Lemma 3.3. *If $d_1 \neq d_{1i}$, $i = 1,2,3,\cdots$, then for every sufficiently small neighborhood V of (u^*,v^*), T has no fixed point on ∂V and*

$$\deg(I - T, V, (0,0)) = (-1)^{N_{d_1}}$$

provided that ϖ is sufficiently large.

Lemma 3.3 is proved by counting the number of negative eigenvalues of $I - L$ where L is the Fréchet derivative of T at (u^*,v^*).

Next, we consider a one-parameter family of elliptic systems

$$\left\{\begin{array}{lll} d_1\Delta u - u + \tau\left(\frac{u^p}{v^q} + \sigma\right) + (1-\tau)\rho = 0 & \text{in} & \Omega, \\ d_2\Delta v - v + \tau\frac{u^r}{v^s} + (1-\chi_\tau)\rho = 0 & \text{in} & \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} & \partial\Omega \end{array}\right. \tag{3.2}$$

with parameter $\tau \in [0,1]$. (We have abused the notation here since the parameter τ has nothing to do with the response rate in (1.1).) In (3.2), ρ is a given positive constant and

$$\chi_\tau = \left\{\begin{array}{lll} 2\tau & \text{if} & \tau \in \left[0, \frac{1}{2}\right], \\ 1 & \text{if} & \tau \in \left[\frac{1}{2}, 1\right]. \end{array}\right.$$

When τ changes from 0 to 1, (3.2) serves as a deformation from a trivial system which has a unique solution $(u,v) \equiv (\rho,\rho)$ to (2.1).

Lemma 3.4. *Under the assumptions of Theorem 3.1, positive solutions to* (3.2) *satisfies a priori bound*

$$0 < \alpha \leq u, v \leq \beta \tag{3.3}$$

for some positive constants α, β independent of τ.

Once we have a priori bounds uniform in τ, we can use deformation argument and conclude

Lemma 3.5. *There exists $\eta > 0$ such that T has no fixed point on $\partial\Lambda_\eta$ and*

$$\deg(I - T, \Lambda_\eta, (0,0)) = 1.$$

Here

$$\Lambda_\eta = \left\{(u,v) \in X : \eta < u, v < \frac{1}{\eta} \text{ in } \overline{\Omega}\right\}.$$

From the properties of topological degree, we have

$$\deg\left(I - T, \Lambda_\eta \backslash \overline{V}, (0,0)\right) = 1 - (-1)^{N_{d_1}} = 2 \neq 0,$$

hence T has at least one fixed point in $\Lambda_\eta \backslash \overline{V}$, which is a nontrivial solution to (2.1). Hence Theorem 3.1 is proved.

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