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Degenerate parabolic equation derived from kinetic theory

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Abstract

We study a degenerate parabolic equation derived from the kinetic theory using Rényi-Tsallis' entropy. If the exponent is critical, we have the formation of collapse for the blowup solution in finite time. This result is regarded as a higher-dimensional version of our previous work on the non-stationary Smoluchowski-Poisson equation associated with the Boltzmann entropy in two-space dimensions, and actually, we use the mass quantization of the blowup family of stationary solutions in the proof.

1 Introduction

The purpose of the present paper is to show the formation of collapse for the blowup in finite time solution to a degenerate parabolic equation with the space dimension greater than 2. This equation describes the motion of the mean field of many self-interacting particles, and is derived from the kinetic theory [2].

In fact, this theory induces the parabolic-elliptic system

$$\mu_t = \nabla[D_* \cdot (\nabla p + \mu \nabla \varphi)] \qquad \Delta\varphi = \mu \qquad \text{in } \Omega \times (0,T) \qquad (1)$$

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as the hydrodynamical limit of self-gravitating particles. Here, $\mu = \mu(x,t) \geq 0$ is the function describing particle density at $(x,t) \in \Omega \times (0,T)$, $\Omega \subset \mathbf{R}^n$ a domain, $\varphi = \varphi(x,t)$ is the Newton potential generated by μ, and $p \geq 0$ is the pressure determined by the density-pressure relation

$$p = p(\mu, \theta). \tag{2}$$

If Ω has the boundary $\partial\Omega$, the null-flux boundary condition

$$(\nabla p + \mu \nabla \varphi) \cdot \nu = 0$$

is imposed with ν denoting the outer unit normal vector so that the total mass

$$\lambda = \int_\Omega \mu(x,t) dx$$

is conserved during the evolution.

In more details, if $0 \leq f = f(x,v,t)$ is the density of particles at $(x,t) \in \Omega \times (0,T)$ moving at the velocity v, then it satisfies the kinetic equation

$$f_t + v \cdot \nabla_x f - \nabla\varphi \cdot \nabla_v f = -\nabla_v \cdot j$$

with the general dissipation flux term $-\nabla_v \cdot j$. This flux term is determined by the maximum entropy production principle, that is, f maximize the local entropy $S = \int_{\mathbf{R}^n} s(f(x,v,t))dv$ under the constraint

$$\mu(x,t) = \int_{\mathbf{R}^n} f(x,v,t)dv, \qquad p(x,t) = \frac{1}{n}\int_{\mathbf{R}^n} |v|^2 f(x,v,t)dv.$$

Averaging f over the velocities $v \in \mathbf{R}^n$, and then the passage to the limit of large friction or large times leads to (1) in the (x,t) space, see [2]. We have, thus, several mean field equations according to the entropy function $s(f)$ subject to the law of partition of particles into mezoscopic states; e.g., the entropies of Boltzmann, Fermi-Dirac, Bose-Einstein, and so forth.

System (1) with (2) is still under-determined, and there are two main theories to prescribe the temperature θ. First, the cannonical statistics takes iso-thermal setting, and hence the temperature $\theta > 0$ is a constant. Second, $\theta = \theta(t) > 0$ is the function of t in the micro-cannonical statistics, where

$$E = \frac{n}{2}\int_\Omega p dx + \frac{1}{2}\int_\Omega \mu\varphi dx$$

is the prescribed total energy independent of t.

If Rényi-Tsallis' entropy

$$S = \frac{-1}{q-1}\int_{\mathbf{R}^n}(f^q - f)dv$$

is adopted, then (2) becomes

$$p = \kappa\theta^{1-\frac{\gamma n}{2}}\mu^{1+\gamma},$$

where $\kappa > 0$ is a constant and $\frac{1}{\gamma} = \frac{1}{q-1} + \frac{n}{2}$, see [3, 1]. By normalizing constants and assuming $\Omega = \mathbf{R}^n$, then we can reduce (1) to the degenerate parabolic equation

$$u_t = \frac{m-1}{m}\Delta u^m - \nabla\cdot(u\nabla\Gamma * u), \quad u \geq 0 \qquad \text{in } \mathbf{R}^n \times (0,T) \tag{3}$$

in the iso-thermal setting, where the new unknown u is a positive constant times μ, $\frac{1}{m-1} = \frac{1}{q-1} + \frac{n}{2}$, and

$$\Gamma(x) = \frac{1}{\omega_{n-1}(n-2)\,|x|^{n-2}}$$

with ω_{n-1} denoting the area of the boundary of the unit ball in $\mathbf{R}^n$.

When $n = 3$ and $q = \frac{5}{3}$, the case $m = 2 - \frac{2}{n} = \frac{4}{3}$ actually arises to (3). Equation (3) of this exponent m is, mathematically, a higher-dimensional version of the Smoluchowski-Poisson equation associated with the Boltzmann entropy in two-space dimension. This two-dimensional equation is given by

$$u_t = \Delta u - \nabla\cdot(u\nabla\Gamma * u), \quad u \geq 0 \qquad \text{in } \mathbf{R}^2 \times (0,T) \tag{4}$$

defined for $\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$, and thus, is a relative to the simplified system of chemotaxis,

$$\begin{aligned} &u_t = \nabla\cdot(\nabla u - u\nabla v), \quad -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u dx \quad \text{in } \Omega\times(0,T) \\ &\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 \qquad\qquad \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{5}$$

associated with the total mass conservation $\|u(t)\|_1 = \|u_0\|_1$ and the decrease of the free energy,

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)dx - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u\otimes u dx dx',$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary, ν the outer unit normal vector, $u \otimes u = u(x,t)u(x',t)$, and $G = G(x,x')$ is the Green's function associated with

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u dx \quad \text{in } \Omega \times (0,T), \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T).$$

We have the formation of collapse for the blowup solution in finite time to (5), i.e.,

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_*(x_0)\delta_{x_0}(dx) + f(x)dx \tag{6}$$

as $t \uparrow T$ in $\mathcal{M}(\overline{\Omega})$, where $T < +\infty$ is the blowup time,

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \text{there exists } x_k \to x_0,\ t_k \uparrow T \text{ such that } u(x_k,t_k) \to +\infty\}$$

denotes the blowup set,

$$m(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega) \end{cases}$$

is the quantized mass, and $0 \le f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$.

Similarly to (5), there is a collapse formation with quantized mass of the blowup solution in finite time to (3), provided that $u_0 = u|_{t=0} \in X = L^1(\mathbf{R}^2, (1+|x|^2)dx) \cap L^\infty(\mathbf{R}^2) \cap H^1(\mathbf{R}^2)$. In fact, (3) is well-posed in this function space X locally in time, and it follows that $\limsup_{t\uparrow T} \int_{\mathbf{R}^2} |x|^2 u(x,t)dx < +\infty$. This guarantees the boundedness of the blowup set in $\mathbf{R}^2$, and then we obtain an analogous result of (6), see later arguments of this paper. We study the question whether or not this is also the case of (3) with $m = 2 - \frac{2}{n}$, $n \ge 3$.

The solution to (3) which we handle with is the weak solution obtained similarly to [7, 9, 8]. First, given the initial value

$$0 \le u_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \qquad \text{with} \qquad u_0^m \in H^1(\mathbf{R}^n), \tag{7}$$

we take the approximate solution $u_\varepsilon = u_\varepsilon(x,t)$ satisfying

$$\begin{aligned} u_{\varepsilon t} &= \frac{m-1}{m}\Delta(u_\varepsilon + \varepsilon)^m - \nabla \cdot (u_\varepsilon \nabla\Gamma * u_\varepsilon) && \text{in } \mathbf{R}^n \times (0,T) \\ u\big|_{t=0} &= u_{0\varepsilon} && \text{in } \mathbf{R}^n \end{aligned}$$

for $0 < \varepsilon \ll 1$, where

$$\begin{array}{ll} 0 \le u_{0\varepsilon} \in L^1 \cap W^{2,p}(\mathbf{R}^n) & \text{for any } p \in [\frac{n}{n-1}, n+3] \\ \|u_{0\varepsilon}\|_p \le \|u_0\|_p, & \text{for any } p \in [1,\infty] \\ \|\nabla u_{0\varepsilon}^m\|_2 \le \|u_0^m\|_2 & \\ u_{0\varepsilon} \to u_0 & \text{strongly in } L^p(\mathbf{R}^n) \text{ as } \varepsilon \downarrow 0 \text{ for some } p \in [\frac{n}{n-1}, \infty). \end{array}$$

The construction of this approximate solution assures its several uniform estimates with respect to $0 < \varepsilon \ll 1$ locally in time, and then, passing to a subsequence, we obtain their convergence to $u = u(x,t)$ satisfying

$$\begin{array}{l} u \in L^\infty([0,T]; L^1(\mathbf{R}^n)) \cap L^\infty_{loc}([0,T); L^\infty(\mathbf{R}^n)) \\ \nabla u^m \in L^\infty([0,T]; L^2(\mathbf{R}^n)) \\ \Gamma * u \in L^\infty_{loc}([0,T); H^1(\mathbf{R}^n)) \end{array}$$

and

$$\int_0^T \int_{\mathbf{R}^n} (\nabla u^m \cdot \nabla \xi - u \nabla \Gamma * u \cdot \nabla \xi - u \xi_t) dx dt = \int_{\mathbf{R}^n} u_0 \xi dx$$

for $0 < T \ll 1$, where $\xi \in H^1(0,T; L^2(\mathbf{R}^n)) \cap L^2(0,T; H^1(\mathbf{R}^n))$ is the test function such that $\xi(\cdot, t) = 0$ for $0 < T - t \ll 1$.

Remark 1 *The potential $\Gamma(x)$ used in [7, 9, 8] decays exponentially at ∞, and is different from ours. In our case, however, the Calderón-Zygmund estimate is applicable and it holds that $\Gamma * u \in L^\infty_{loc}([0,T); W^{2,p}(\mathbf{R}^n))$ for any $1 < p < \infty$ by $u \in L^\infty([0,T]; L^1(\mathbf{R}^n)) \cap L^\infty_{loc}([0,T); L^\infty(\mathbf{R}^n))$.*

Henceforth, $u = u(x,t)$ and $T = T_{\max} \in (0, +\infty]$ denote this weak solution and its existence time, respectively. The first theorem shows that there is a threshold of $\|u_0\|_1$ for the blowup of the solution in finite time, and this value λ_* is associated with the Sobolev constant $S = S(n)$, that is, $\lambda_* = \left(\frac{2}{mS}\right)^{n/2}$ and

$$S = \inf\{\|\nabla \xi\|_2^2 \mid \xi \in C_0^\infty(\mathbf{R}^n), \ \|\xi\|_{\frac{2n}{n-2}} = 1\}. \tag{8}$$

An analogous fact is shown by [9, 8] for the equation which they studied, see the above Remark 1, while a different argument using the Trudinger-Moser inequality will be provided here.

Theorem 1 *If $u_0 = u_0(x)$ is the initial value satisfying (7) and $\|u_0\|_1 < \lambda_*$, then $T = +\infty$ holds in (3) for $m = 2 - \frac{2}{n}$. There is, on the other hand, $u_0 = u_0(x)$ with (7) such that $\|u_0\|_1 > \lambda_*$ and $T < +\infty$.*

The blowup solution constructed in the above theorem is constructed for the case of

$$\int_{\mathbf{R}^n} |x|^2 u_0(x)dx < +\infty. \tag{9}$$

Actually, formation of collapse of the blowup solution to (3) is associated with this class.

This paper is composed of four sections. In the next section, we describe the scaling property to (3) and explain why the exponent $m = 2 - \frac{2}{n}$ and the value λ_* are selected for the L^1-threshold of the blowup in finite time to arise, and then prove Theorem 1. In section 3 we show that the formation of collapse arises when the free energy does not decay so fast. Section 4 deals with the related questions on the blowup rate, finiteness of the isolated blowup points, mass quantization, and so forth.

2 Preliminaries

For the moment, we take a formal argument concerning the scaling property of (3). The first observation is that it is a *model B equation*, see [12], associated with the *free energy*

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} dx - \frac{1}{2}\langle \Gamma * u, u\rangle .$$

In fact, we have

$$\delta\mathcal{F}(u)[v] = \frac{d}{ds}\mathcal{F}(u+sv)\Big|_{s=0} = \left\langle v, u^{m-1} - \Gamma * u\right\rangle ,$$

where $\langle\ ,\ \rangle$ denotes the L^2-inner product, and identifyiing $\mathcal{F}(u)$ with $u^{m-1} - \Gamma * u$, we can write (3) as

$$u_t = \nabla \cdot \left\{ \frac{m-1}{m} \nabla u^m - u\nabla\Gamma * u \right\} = \nabla \cdot u\nabla\delta\mathcal{F}(u) \qquad \text{in } \mathbf{R}^n \times (0,T). \tag{10}$$

From this form of (10), it is easy to infer, at least formally, the total mass conservation

$$\|u(t)\|_1 = \|u_0\|_1 = \lambda \tag{11}$$

and the decrease of the free energy

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(u) &= -\int_{\mathbf{R}^n} u\left|\nabla\delta\mathcal{F}(u)\right|^2 dx \\ &= -\int_{\mathbf{R}^n} u\left|\nabla(u^{m-1}-\Gamma*u)\right|^2 dx \leq 0. \end{aligned} \tag{12}$$

In fact, justifying (11) for the weak solution is rather easy. As for (12), on the other hand, we write, again formally, its right-hand side as

$$-\int_{\mathbf{R}^n}\left|\frac{m-1}{m-1/2}\nabla u^{m-1/2}-u^{1/2}\nabla\Gamma*u\right|^2 dx,$$

noting that $u^{1/2}\nabla\Gamma*u\in L^\infty_{loc}([0,T);L^2(\mathbf{R}^n))$ holds for the weak solution $u=u(x,t)$. Then, the above described construction of approximate solutions and the process of passing to the limit guarantee $u^{m-1/2}\in L^2_{loc}([0,T);H^1(\mathbf{R}^n))$, and furthermore, equality (12) is justified as

$$\frac{d}{dt}\mathcal{F}(u)=-\int_{\mathbf{R}^n}\left|\frac{m-1}{m-1/2}\nabla u^{m-1/2}-u^{1/2}\nabla\Gamma*u\right|^2 dx\leq 0 \tag{13}$$

for a.e. t.

We go back to the formal argument again. Regarding (11)-(12), we formulate the stationary state by

$$u^{m-1}-\Gamma*u=\text{constant in } \{u>0\}, \qquad \int_{\mathbf{R}^n} u dx=\lambda. \tag{14}$$

If the above constant is denoted by c, then $v=\Gamma*u+c$ satisfies

$$-\Delta v=v_+^q \quad \text{in } \mathbf{R}^n, \qquad \int_{\mathbf{R}^n} v_+^q dx=\lambda, \tag{15}$$

where $m=1+\frac{1}{q}$. (This constant may depend on the connected compent of $\{u>0\}$ at this moment, which eventually becomes unique by the following result.) Problem (15) is invariant under the scaling transformation

$$v(x)\mapsto v_\mu(x)=\mu^\gamma v(\mu x) \tag{16}$$

if and only if $\gamma=n-2$ and $q=\frac{1}{m-1}=\frac{n}{n-2}$, i.e., $m=2-\frac{2}{n}$, where $\mu>0$ is a constant. If this is the case, conversely, problem (16) admits a family of solutions, each of which is necessarily radially symmetric and has compact support, see [15]. Then, we define the normalized solution $v_* = v_*(x)$ to (15) and the quantized mass $\lambda_* > 0$ by

$$-\Delta v_* = v_{*+}^q, \ v_* \leq v_*(0) = 0 \quad \text{in } \mathbf{R}^n \qquad \text{and} \qquad \lambda_* = \int_{\mathbf{R}^n} v_{*+}^q dx,$$

respectively.

This profile of mass quantization of the stationary state on the whole space $\mathbf{R}^n$ is the origin of the quantized blowup mechanism for the family of solutions to

$$-\Delta v = v_+^q \ \text{ in } \Omega, \quad v = \text{constant} \ \text{ on } \partial\Omega, \quad \int_\Omega v_+^q dx = \lambda$$

with $q = \frac{n}{n-2}$, where $\Omega \subset \mathbf{R}^n$ is a bounded domain, $n \geq 3$. An analogous result to $n = 2$ arises to

$$-\Delta v = e^v \ \text{ in } \Omega, \quad v = \text{constant}, \ \text{ on } \partial\Omega, \quad \int_\Omega e^v dx = \lambda. \tag{17}$$

The free boundary problem (17) is, actually, regarded as a stationary state of (5), and its quantized blowup mechanism induces (6) similarly, see [13] and the references therein.

Remark 2 *The non-stationary problem (3) for $m = 2 - \frac{2}{n}$ has also the scaling property; if $u = u(x,t)$ is a solution, then $u_\mu(x,t) = \mu^n u(\mu x, \mu^n t)$ satisfies*

$$u_{\mu t} = \frac{m-1}{m}\Delta u_\mu^m - \nabla \cdot (u_\mu \nabla \Gamma * u_\mu), \quad u_\mu \geq 0 \qquad \textit{in } \mathbf{R}^n \times (0, T_\mu)$$
$$\int_{\mathbf{R}^n} u_\mu dx = \int_{\mathbf{R}^n} u dx \qquad \textit{for } t \in [0, T_\mu),$$

where $\mu > 0$ is a constant and $T_\mu = \mu^{-n} T$. This scaling is of course compatible to (16) for the stationary solution.

Lemma 1 *It holds that*

$$j_* = \inf\{\mathcal{F}(u) \mid 0 \leq u \in L^m(\mathbf{R}^n), \ \int_{\mathbf{R}^n} u = \lambda_*\} = 0$$

if $m = 2 - \frac{2}{n}$.

Proof: Higher-dimensional Trudinger-Moser inequality is given by

$$j_R = \inf\{\mathcal{F}(u) \mid u \geq 0, \text{ supp } u \subset B_R, \int_{\mathbf{R}^n} u = \lambda_*\} > -\infty, \tag{18}$$

in the dual form, see [16, 15], where $B_R = B(0,R)$. Here, it follows that

$$\mathcal{F}(u_\mu) = \mu^{n-2}\mathcal{F}(u)$$

for $u_\mu(x) = \mu^n u(\mu x)$ by $m = 2 - \frac{2}{n}$. Since supp $u_\mu \subset B_{\mu^{-1}R}$ if and only if supp $u \subset B_R$, therefore, we obtain

$$j_{\mu^{-1}R} = \mu^{n-2} j_R \geq j_R$$

for $\mu > 1$. This implies $j_R \geq 0$ and hence

$$j_* \geq 0$$

because $R > 0$ is arbitrary. We have

$$j_* = \mu^{n-2} j_*$$

again by the above scaling. This implies $j_* = 0$. ▮

Lemma 2 *It holds that*

$$\lambda_* = \left(\frac{2}{mS}\right)^{n/2} \tag{19}$$

Proof: Using Sobolev's constant (8), we obtain

$$0 \leq \langle \Gamma * u, u \rangle = \|\nabla\Gamma * u\|_2^2 \leq S\|u\|_{\frac{2n}{n+2}}^2 \leq S\|u\|_1^{2\theta}\|u\|_m^{2(1-\theta)}$$

for $\frac{\theta}{1} + \frac{1-\theta}{m} = \frac{n+2}{2n}$. Since $m = 2 - \frac{2}{n}$, it follows that $2(1-\theta) = m$, and this implies the relation

$$\mathcal{F}(u) \geq \left(\frac{1}{m} - \frac{S}{2}\lambda_*^{2\theta}\right)\|u\|_m^m$$

for $0 \leq u \in L^m(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} u = \lambda_*$. Regarding the Talenti family [14], we see that the above estimate is optimal, and therefore, it holds that

$$\frac{1}{m} - \frac{S}{2}\lambda_*^{2-m} = 0,$$

by Lemma 1. This means (19). ▮

Lemma 3 *If $\lambda < \lambda_*$, then we have*

$$\|u(t)\|_m + \langle \Gamma * u(t), u(t) \rangle \le C_1 \tag{20}$$

with a constant $C_1 > 0$ independent of $t \in [0, T)$.

Proof: We have $\|v\|_1 = \lambda_*$ for $v = \frac{\lambda_*}{\lambda} u$, and this implies

$$\begin{aligned} \mathcal{F}(u_0) &\ge \mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} dx - \frac{1}{2} \langle \Gamma * u, u \rangle \\ &= \left(\frac{\lambda}{\lambda_*}\right)^m \int_{\mathbf{R}^n} \frac{v^m}{m} dx - \frac{1}{2} \left(\frac{\lambda}{\lambda_*}\right)^2 \langle \Gamma * v, v \rangle \\ &\ge \begin{cases} \frac{1}{2}\left\{\left(\frac{\lambda}{\lambda_*}\right)^m - \left(\frac{\lambda}{\lambda_*}\right)^2\right\} \langle \Gamma * v, v \rangle \\ \left\{\left(\frac{\lambda}{\lambda_*}\right)^m - \left(\frac{\lambda}{\lambda_*}\right)^2\right\} \int_{\mathbf{R}^n} \frac{v^m}{m} dx \end{cases} \end{aligned}$$

by Lemma 3. Then, (20) follows from $0 < \lambda < \lambda_*$ and $0 < m < 2$. ∎

Lemma 4 *If the initial value u_0 satisfies $\mathcal{F}(u_0) < 0$ and (9), then $T < +\infty$ arises.*

Proof: Using the approximate solution, we can show that

$$t \in [0, T) \mapsto \int_{\mathbf{R}^n} \varphi(x) u(x, t) dx$$

is locally absolutely continuous for $\varphi \in C_0^\infty(\mathbf{R}^n)$, and it holds that

$$\frac{d}{dt} \int_{\mathbf{R}^n} \varphi u dx = \frac{m-1}{m} \int_{\mathbf{R}^n} u^m \Delta \varphi dx + \frac{1}{2} \int \int_{\mathbf{R}^n \times \mathbf{R}^n} \rho_\varphi u \otimes u dx dx'$$

for a.e. t, where $u \otimes u = u(x, t) u(x', t)$ and

$$\rho_\varphi = \rho_\varphi(x, x') = (\nabla \varphi(x) - \nabla \varphi(x')) \cdot \nabla \Gamma(x - x').$$

Here, the inequality

$$|\rho_\varphi(x, x')| \le (n-2) \|\nabla \varphi\|_\infty \Gamma(x - x')$$

is made use of for this purpose.

Under the assumption of (9), taking $\varphi = |x|^2$ is justified again, see [7]. Since

$$\Delta\varphi = 2n, \quad \rho_\varphi(x,x') = -2(n-2)\mathcal{F}(u) \tag{21}$$

holds for this $\varphi = |x|^2$, we can show that the function

$$t \in [0,T) \mapsto \int_{\mathbf{R}^n} |x|^2 u(x,t)dx \in [0,+\infty)$$

is locally absolutely continuous, and satisfies

$$\begin{aligned} \frac{d}{dt}\int_{\mathbf{R}^n} |x|^2 udx &= \frac{m-1}{m}\cdot 2n\int_{\mathbf{R}^n} u^m dx - (n-2)\langle \Gamma * u, u\rangle \\ &= 2(n-2)\mathcal{F}(u) \end{aligned} \tag{22}$$

for a.e. t. Since $\mathcal{F}(u(t)) \leq \mathcal{F}(u_0)$, it follows that

$$\int_{\mathbf{R}^n} |x|^2 u(x,t)dx < 0 \qquad \text{for } t \gg 1$$

if both $\mathcal{F}(u_0) < 0$ and $T = +\infty$ occur, a contradiction. Thus, $\mathcal{F}(u_0) < 0$ implies $T < +\infty$. ∎

Proof of Theorem 1: We can apply Moser's iteration scheme for the weak solution to (3) with $m = 2 - \frac{2}{n}$, see [8]. Thus, if there are $p > 1$ and $C_2 > 0$ such that $\sup_{t\in[0,T)} \|u(t)\|_p \leq C_2$, then it holds that $\sup_{t\in[0,T)} \|u(t)\|_\infty \leq C_3$ with a constant $C_3 > 0$ independent of T. This implies $T = +\infty$, see [9]. The first part of Theorem 1 is thus a consequence of Lemma 3.

Wang-Ye's Trudinger-Moser inequality (18), on the other hand, is sharp, and it holds that

$$\inf\{\mathcal{F}(u) \mid u \geq 0, \ \text{supp}\ u \subset B_R, \ \int_{\mathbf{R}^n} u = \lambda\} = -\infty$$

for any $R > 0$ and $\lambda > \lambda_*$. Each $\lambda > \lambda_*$, in particular, admits an admissible initial value $u_0 = u_0(x)$ with compact support such that $\|u_0\|_1 = \lambda$ and $\mathcal{F}(u_0) < 0$. For this u_0, it follows that $T < +\infty$ from Lemma 4, and the proof is complete. ∎

Remark 3 *The first difference between (3) with $m = 2 - \frac{2}{n}$, $n \geq 3$, and (4) with $n = 2$ is the linearity of the diffusion, while the recursive property (21) is the second difference. In fact, we have*

$$(\nabla\varphi(x) - \nabla\varphi(x'))\cdot\nabla\Gamma(x-x') = -\frac{1}{\pi}$$

for $\varphi(x) = |x|^2$ *and* $\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$.

3 Collapse Formation

In the following theorem,

$$\mathcal{S} = \{x_0 \in \mathbf{R}^n \mid \text{there exists } x_k \to x_0,\ t_k \uparrow T \text{ such that } u(x_k, t_k) \to +\infty\}$$

denotes the blowup set. Here, we emphasize that $T < +\infty$ implies

$$\limsup_{t\uparrow T} \|u(t)\|_\infty = +\infty,$$

see [7, 9, 8], while $\|u(t)\|_{L^\infty(|x|>R)}$ is bounded for $R \gg 1$ as we shall show below and therefore, the blowup set is always non-void in the case of $T < +\infty$.

To see this, first, ε-regularity is obtained by localized Moser's iteration scheme, i.e., localization of Lemma 1, see [8].

Lemma 5 *We have* $\varepsilon_0 > 0$ *and* $C_7 > 0$ *independent of* $x_0 \in \mathbf{R}^n$ *such that*

$$\limsup_{t\uparrow T} \int_{B(x_0,R)} u(x,t)dx < \varepsilon_0$$

implies

$$\limsup_{t\uparrow T} \|u(t)\|_{L^\infty(B(x_0,R/2))} \leq C_7$$

for $0 < R \ll 1$.

Next, we have

$$\int_{\mathbf{R}^n} |x|^2 u(x,t)dx \leq 2(n-2)T\mathcal{F}(u_0) + \int_{\mathbf{R}^n} |x|^2 u_0 dx \equiv C_4(T, u_0)$$

by (22), and hence

$$\sup_{t\in[0,T)} \int_{|x|>R} u(x,t)dx \leq \frac{1}{R^2} C_4(T, u_0). \tag{23}$$

This implies

$$\limsup_{t\uparrow T} \|u(t)\|_{L^\infty(|x|>R)} \leq C_5 \qquad \text{for } R \gg 1 \tag{24}$$

by Lemma 5 with a constant $C_5 > 0$ independent of $t \in [0, T)$. Then, it follows that $\mathcal{S} \subset \overline{B(0, R)}$.

Here, we shall show the formation of collapse to (3), prescribing the behavior of the free energy.

Theorem 2 *Given the initial value $u_0 = u_0(x)$ satisfying (7) and (9), assume $T < +\infty$ for the weak solution $u = u(x,t)$ to (3) with $m = 2 - \frac{2}{n}$, $n \geq 3$ and also*

$$\int_0^T (T-t)^{-\gamma} \mathcal{F}(u(t))dx > -\infty \tag{25}$$

for some $\gamma > 0$. Then the blowup set $\mathcal{S}$ of this $u = u(\cdot, t)$ is finite and it holds that

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \tag{26}$$

in $\mathcal{M}(\mathbf{R}^n) = C'(\mathbf{R}^n \cup \{\infty\})$ as $t \uparrow T$, where $\mathbf{R}^n \cup \{\infty\}$ is the one-point compactification of $\mathbf{R}^n$, $m(x_0) > 0$, and

$$0 \leq f = f(x) \in L^1(\mathbf{R}^n; (1+|x|^2)dx)) \cap L^\infty_{loc}((\mathbf{R}^n \cup \{\infty\}) \setminus \mathcal{S}). \tag{27}$$

Remark 4 *Inequality (25) may be replaced by*

$$\int_0^T a(t)\mathcal{F}(u(t))dt > -\infty,$$

where $a = a(t) > 0$ is a measurable function in $[0, T)$ satisfying

$$\int_0^T \frac{ds}{\int_s^T a(t)dt} < +\infty.$$

Remark 5 *We have always $\int_0^T \mathcal{F}(u(t))dt > -\infty$ and*

$$\mathcal{F}(u(t)) \geq -C_6(T-t)^{-1} \tag{28}$$

with a constant $C_6 > 0$ independent of $t \in [0, T)$.

Proof: The above relations are obvious if

$$\lim_{t\uparrow T} \mathcal{F}(u(t)) > -\infty. \tag{29}$$

In the other case,

$$\lim_{t\uparrow T} \mathcal{F}(u(t)) = -\infty, \tag{30}$$

we have $\mathcal{F}(u(t_0)) < 0$ for some $t_0 \in [0,T)$. We may assume $t_0 = 0$ without loss of generality.

First, (22) implies

$$\frac{dH}{dt} < 0 \quad \text{for} \quad H(t) = \int_{\mathbf{R}^n} |x|^2 u(x,t)dx \tag{31}$$

and therefore, there is $H(T) = \lim_{t\uparrow T} H(t) \geq 0$. Thus, we obtain

$$\int_0^T \mathcal{F}(u(t))dt = H(T) - H(0) > -\infty.$$

Next, equality (22) reads;

$$\begin{aligned} 2(n-2)\mathcal{F}(u) &= \frac{d}{dt}\int_{\mathbf{R}^n} |x|^2 udx = -\int_{\mathbf{R}^n} u\nabla(u^{m-1} - \Gamma * u) \cdot \nabla |x|^2 \\ &= -2\int_{\mathbf{R}^n} u\nabla(u^{m-1} - \Gamma * u) \cdot xdx, \end{aligned}$$

formally again, and then it holds that

$$\begin{aligned} \left|\frac{d}{dt}\int_{\mathbf{R}^n} |x|^2 udx\right|^2 &\leq 4\int_{\mathbf{R}^n} u\left|\nabla(u^{m-1} - \Gamma * u)\right|^2 dx \int_{\mathbf{R}^n} |x|^2 udx \\ &= -4\frac{d}{dt}\mathcal{F}(u) \cdot \int_{\mathbf{R}^n} |x|^2 udx. \end{aligned}$$

The above inequality is again justified through the approximate solution, and we obtain

$$\left(\frac{dg}{dt}\right)^2 \leq -\frac{d}{dt}\mathcal{F}(u) = -\frac{1}{2(n-2)}\frac{d^2}{dt^2}g^2$$

for $g = g(t) > 0$ defined by

$$g(t) = \left\{\int_{\mathbf{R}^n} |x|^2 u(x,t)dx\right\}^{1/2},$$

or equivalently,

$$gg'' + (n-1)(g')^2 \le 0$$

for a.e. t. This inequality is written as

$$\frac{d^2}{dt^2}\log g = \frac{gg'' - (g')^2}{g^2} \le -n\left(\frac{g'}{g}\right)^2 = -n\left(\frac{d}{dt}\log g\right)^2,$$

or

$$-\frac{d}{dt}h \le -nh^2$$

for $h = -\frac{d}{dt}\log g > 0$, recall (31). Thus, we obtain

$$\frac{d}{dt}h^{-1} \le -n < 0,$$

and there exists $h(T) = \lim_{t\uparrow T} h(t) \in (0, +\infty]$ satisfying

$$h^{-1}(T) - h^{-1}(t) \le -n(T-t)$$

for $t \in [0, T)$.

Neglecting this term, we obtain

$$h^{-1}(t) \ge n(T-t) \qquad \text{for } t \in [0, T),$$

and then it holds that

$$h(t) \le \frac{1}{n(T-t)} = -\frac{1}{n}\frac{d}{dt}\log\{T-t\}$$

or

$$\frac{d}{dt}\log\left\{\frac{H(t)}{(T-t)^{2/n}}\right\} \ge 0 \tag{32}$$

for a.e. t. Then (28) follows from (22). ∎

We shall follow the argument developed for Smoluchowski-Poisson equation (4) in two-space dimension [5, 10] to prove Theorem 2. The key lemma is the following.

Lemma 6 *If (25) holds with $T < +\infty$, then*

$$\lim_{t\uparrow T}\int_{\mathbf{R}^n}\varphi(x)u(x,t)dx \tag{33}$$

exists for any $\varphi \in C_0^1(\mathbf{R}^n)$.

Proof: The formal calculation

$$\begin{aligned}
\left|\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u dx\right|^2 &= \left|\int_{\mathbf{R}^n} u\nabla(u^{m-1}-\Gamma * u)\cdot\nabla\varphi dx\right|^2 \\
&\leq \int_{\mathbf{R}^n} u\left|\nabla(u^{m-1}-\Gamma * u)\right|^2 dx\cdot\int_{\mathbf{R}^n} u\left|\nabla\varphi\right|^2 dx \\
&\leq -\left\|\nabla\varphi\right\|_\infty^2 \lambda\frac{d}{dt}\mathcal{F}(u), \qquad (34)
\end{aligned}$$

is justified by taking the approximate solution, i.e.,

$$(A')^2 \leq -\frac{\|\nabla\varphi\|_\infty^2 \lambda}{2(n-2)}H'' \qquad (35)$$

for a.e. t for $A(t)=\int_{\mathbf{R}^n}\varphi u dx$. In the case of (29), we obtain

$$\int_0^T \left|\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u dx\right| dt \leq T^{1/2}\left\{\int_0^T\left|\frac{d}{dt}\int_{\mathbf{R}^n}\varphi u dx\right|^2 dt\right\}^{1/2} < +\infty$$

and then the existence of (33).

Thus, we may assume $\mathcal{F}(u_0)<0$ without loss of generality, and then it holds that

$$\begin{aligned}
\int_0^T\left(\int_s^T a(t)dt\right)A'(s)^2 ds &= \int_0^T a(t)dt\int_0^t A'(s)^2 ds \\
&\leq -C_7\int_0^T a(t)H'(t)dt < +\infty
\end{aligned}$$

by (25), where for $a(t)=(T-t)^{-\gamma}$ and $C_7=\frac{\|\nabla\varphi\|_\infty^2\lambda}{2(n-2)}$. We obtain

$$\begin{aligned}
|A(t_2)-A(t_1)|^2 &= \left|\int_{t_1}^{t_2} A'(s)ds\right|^2 \\
&\leq \int_0^T\frac{ds}{\int_s^T a(t)dt}\cdot\int_{t_1}^{t_2}\left(\int_s^T a(t)dt\right)A'(s)^2 ds
\end{aligned}$$

for $0\leq t_1\leq t_2<T$, and hence the existence of (33). ∎

Remark 6 *We have the scaling invariant inequality*

$$\sup_{t'\in[t,\theta t+(1-\theta)T]} A(t') \leq A(t) + \left\{(1-\theta)\log\frac{1}{\theta}\cdot\frac{(H(t)-H(T))\|\nabla\varphi\|_\infty^2\lambda}{n(n-2)}\right\}^{1/2} \tag{36}$$

in the case of $\mathcal{F}(u_0)<0$, *where* $0<\theta<1$.

Proof: Inequality (35) implies

$$\int_t^{t'}(t'-s)A'(s)^2ds \leq \frac{\|\nabla\varphi\|_\infty^2\lambda}{2(n-2)}\{H(t)-H(t')\}$$

for $0\leq t\leq t'<T$. Then, it holds that

$$|A(\theta t+(1-\theta)t')-A(t)|^2 = \left|\int_t^{\theta t+(1-\theta t'} A'(s)ds\right|^2$$

$$\leq (1-\theta)\cdot\int_t^{\theta t+(1-\theta)t'}(t'-s)^{-1}ds\cdot\int_t^{t'}(t'-s)A'(s)^2ds$$

$$\leq (1-\theta)\log\frac{1}{\theta}\cdot\frac{\|\nabla\varphi\|_\infty^2\lambda}{n(n-2)}\cdot(H(t)-H(T)).$$

Varying $t'\in[t,T)$, we get (36). ∎

Remark 7 *Inequality (36) combined with the argument [6] will be applicable to the study of the blowup in infinite time. Namely, we expect that*

$$\liminf_{t\uparrow+\infty}\|u(t)\|_{L^\infty(B(x_0,R/2))}<+\infty$$

holds if $T=+\infty$ *and* $\liminf_{t\uparrow+\infty}\|u(t)\|_{L^1(B(x_0,R))}<\varepsilon_0$.

Remark 8 *By Remark 5, we have*

$$0\leq -H'(t)\leq\frac{2}{n}(T-t)^{-1}H(0)$$

in the case of $\mathcal{F}(u_0)<0$. *If the above inequality is improved slightly, i.e.,*

$$0\leq -H'(t)\leq K(T-t)^{-1+\gamma} \tag{37}$$

with $K > 0$ and $0 < \gamma < 1$, the assumption on the free energy of Theorem 2 is valid. This implies also

$$0 \leq H(T) - H(t) \leq C_9(T-t)^\alpha. \tag{38}$$

Here, we note that if (38) holds, then there is ε_1 independent of $x_0 \in \mathcal{S}$, $0 < R \ll 1$, and $t \in [0,T)$ such that

$$\liminf_{t \uparrow T} \int_{\mathbf{R}^n} \varphi_{x_0,R}(x)u(x,t) \geq \varepsilon_1 \tag{39}$$

and therefore, the finiteness of $\mathcal{S}$.

Proof: Inequality (37) implies with $C_9 > 0$ and $0 < \alpha < 1$. Applying (36) for $\theta = 1/2$, we obtain $C_{10} > 0$ such that

$$\sup_{t' \in [t, \frac{t+T}{2}]} A(t') \leq A(t) + C_{10}(T-t)^\alpha.$$

Now, we define $t_k \uparrow T$ and a_k by

$$T - t_{k+1} = \frac{1}{2}(T - t_k) \qquad \text{and} \qquad a_k = \sup_{t' \in [t_k, t_{k+1}]} A(t'),$$

to obtain

$$a_{k+1} < a_k + C_{10}(T-t_1)^{k\alpha/2}$$

for $k = 1, 2, \ldots$. Then, we obtain $a_k < \varepsilon_0$ for $k = 1, 2, \cdots$ by assuming $a_1 < \varepsilon_1$ for some $0 < \varepsilon_1 \ll 1$. This is a contradiction, and we obtain (39). ∎

Proof of Theorem 2: Given $x_0 \in \mathcal{S}$, we take $\varphi = \varphi_{x_0,R} \in C_0^\infty(\mathbf{R}^n)$ satisfying $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B(x_0, R)$, and $\varphi = 0$ on $\mathbf{R}^n \setminus B(x_0, 2R)$. First, $\mathcal{S}$ is a bounded set in $\mathbf{R}^n$ by (24). Next, Lemma 5 guarantees

$$\limsup_{t \uparrow T} \int_{\mathbf{R}^n} \varphi_{x_0,R}(x)u(x,t)dx \geq \varepsilon_0 \tag{40}$$

for each $x_0 \in \mathcal{S}$, where $0 < R \ll 1$. Then, relation (40) is improved by

$$\liminf_{t \uparrow T} \int_{\mathbf{R}^n} \varphi_{x_0,R}(x)u(x,t)dx \geq \varepsilon_0$$

by Lemma 6. Then, the finiteness of $\mathcal{S}$ follows from (11).

We have the convergence of $u(x,t)dx \rightharpoonup \mu(dx,T)$ in $\mathcal{M}(\mathbf{R}^n)$ as $t \uparrow T$ by (11), (23), and the existence of (33) for $\varphi \in C_0^1(\mathbf{R}^n)$. There arises that

$$\text{supp } \mu_s(dx,T) = \mathcal{S}$$

and (27) if $\mu(dx,T) = \mu_s(dx,T) + f(x)dx$ denotes the Radon-Nikodym-Lebesgue decomposition. Then, we obtain

$$\mu_s(dx,T) = \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx)$$

with $m(x_0) \geq \varepsilon_0$ and the proof is complete. ∎

4 Further Discussions

This section is concerned with the mass quantization, $m(x_0) = \lambda_*$ in (26). First, we shall show the estimate of collapse mass from below. A blowup point x_0 is called isolated if $\mathcal{S} \cap B(x_0,R) = \{x_0\}$ and non-degenerate if

$$\liminf_{t \uparrow T} \inf_{x \in B(x_0,R)} u(x,t) > 0,$$

where $0 < R \ll 1$.

Theorem 3 *If $T < +\infty$ occurs to (3) and $x_0 \in \mathcal{S}$ is a non-degenerate isolated blowup point, then it holds that*

$$\limsup_{t \uparrow T} \mathcal{F}(\varphi^{1/m} u(t)) < +\infty, \tag{41}$$

where $\varphi = \varphi_{x_0,R}$ with $0 < R \ll 1$.

Proof: Given such $x_0 \in \mathcal{S}$, we apply the local elliptic-parabolic regularity. We may assume

$$\sup_{t \in [0,T)} \|u(t)\|_{L^\infty(B(x_0,2R) \setminus B(x_0,R/4))} < +\infty \tag{42}$$

and

$$\sup_{t \in [0,T)} \|\Gamma * u(t)\|_{W^{2,p}(B(x_0,R) \setminus B(x_0,R/2))} < +\infty \tag{43}$$

for $0 < R \ll 1$. Taking $\varphi = \varphi_{x_0,R}$, now we define the local free energy by

$$\mathcal{F}_\varphi(t) = \int_{\mathbf{R}^n} \frac{u^m}{m}\varphi - \frac{1}{2}\varphi u \Gamma * \varphi u \, dx \geq \mathcal{F}(\varphi^{1/m}u).$$

Using $\hat{\varphi} = \varphi_{x_0,R/2}$, thus we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_\varphi(t) &= \int_{\mathbf{R}^n} (u^{m-1} - \Gamma * \varphi u)\varphi u_t dx \\
&= -\int_{\mathbf{R}^n} u\nabla\varphi(u^{m-1} - \Gamma * \varphi u) \cdot \nabla(u^{m-1} - \Gamma * u)dx \\
&= -\int_{\mathbf{R}^n} u\varphi\nabla(u^{m-1} - \Gamma * \varphi u) \cdot \nabla(u^{m-1} - \Gamma * u)dx + O(1) \\
&= -\int_{\mathbf{R}^n} u\hat{\varphi}\nabla(u^{m-1} - \Gamma * \varphi u) \cdot \nabla(u^{m-1} - \Gamma * u)dx + O(1)
\end{aligned}$$

because $\Gamma * u(\cdot, t)$ is bounded in $W^{1,q}_{loc}(\mathbf{R}^n)$ for $1 \leq q < \frac{n}{n-1}$. Here, equality (11) implies

$$\begin{aligned}
&\left| \int_{\mathbf{R}^n} u\hat{\varphi}\nabla\Gamma * (1-\varphi)u \cdot \nabla(u^{m-1} - \Gamma * \varphi u)dx \right| \\
&\leq C_{11}\lambda \int_{\mathbf{R}^n} u\hat{\varphi}|\nabla(u^{m-1} - \Gamma * \varphi u)|dx
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_\varphi(t) &\leq -\int_{\mathbf{R}^n} u\hat{\varphi}\left|\nabla(u^{m-1} - \Gamma * \varphi u)\right|^2 dx \\
&\quad + C_{11}\lambda \int_{\mathbf{R}^n} u\hat{\varphi}|\nabla(u^{m-1} - \Gamma * \varphi u)|dx + O(1) \\
&\leq -\frac{1}{2}\int_{\mathbf{R}^n} u\hat{\varphi}\left|\nabla(u^{m-1} - \Gamma * \varphi u)\right|^2 dx + O(1).
\end{aligned}$$

Thus, we obtain $\mathcal{F}(\varphi^{1/m}u(t)) \leq C_{12}$ with a constant C_{12} independent of $t \in [0, T)$ as is desired. The proof is complete. ∎

Remark 9 *If $x_0 \in \mathcal{S}$ is isolated and non-degenerate, we have $0 < R \ll 1$ and $0 \leq f = f(x) \in L^1(B(x_0, 2R)) \cap C(B(x_0, 2R) \setminus \{x_0\})$ such that any $t_k \uparrow T$ admits $\{t'_k\} \subset \{t_k\}$ and $m(x_0) \geq 0$ satisfying*

$$u(x, t'_k)dx \rightharpoonup m(x_0)\delta_{x_0}(dx) + f(x)dx.$$

If $m(x_0) < \lambda_*$ *is the case, we obtain* $\|u(t'_k)\|_{L^m(B(x_0,R)} \le C'_{12}$*, which, however, does not imply* $\liminf_{t\uparrow T} \|u(t)\|_{L^\infty(B(x_0,R/2))} < +\infty$. *If (26) holds, then we can follow the argument of [5]. Thus, we obtain* $m(x_0) \ge \lambda_*$ *by the above theorem.*

We proceed to the blowup rate, regarding the scaling described in Remark 2. In fact, the backward self-similar transformation is defined by

$$v(y,s) = (T-t)u(x,t), \quad y = (x-x_0)/(T-t)^{1/n}, \ s = -\log(T-t) \quad (44)$$

from this property of scaling, where $x_0 \in \mathcal{S}$. Then, we say that the blowup point x_0 is type I if

$$\limsup_{t\uparrow T}(T-t)\,\|u(t)\|_{L^\infty(B(x_0,b(T-t)^{1/n}))} < +\infty$$

for each $b > 0$, and type II for the other case. The next theorem shows that any blowup point is type II if the free energy is bounded. A similar fact is shown to the semilinear parabolic equation with critical Sobolev growth, see [11]. We mention also that the Herrero-Velázquez solution [4] for the two-dimensional Smoluchowski-Poisson equation (4) has the same profile, boundedness of the free energy and type II blowup rate.

Theorem 4 *If (29) holds, then each* $x_0 \in \mathcal{S}$ *is type II. We have, more precisely,*

$$\lim_{t\uparrow T}(T-t)\,\|u(t)\|_{L^\infty(B(x_0,b(T-t)^{1/n})} = +\infty \quad (45)$$

for any $b > 0$.

Proof: By the proof of Lemma 6, it holds that

$$\int_0^T \left|\frac{d}{dt}\int_\Omega \varphi u dx\right| dx \le C_{13}\lambda\|\nabla\varphi\|_\infty \quad (46)$$

in the case of (29). Putting $\varphi = \varphi_{x_0,R}$, therefore, we obtain

$$\int_0^T \left|\frac{d}{dt}\int_{\mathbf{R}^n} \varphi u dx\right| dt \le C_{14}\lambda^{1/2}R^{-1}$$

with $C_{14} > 0$ independent of $0 < R \ll 1$. This implies

$$|\langle\varphi_{x_0,R}, u(t)\rangle - \langle\varphi_{x_0,R}, u(t')\rangle| \le C_{14}\lambda^{1/2}R^{-1}(t'-t)$$

for $0 \le t \le t' < T$, and hence

$$|\langle \varphi_{x_0,R}, u(t)\rangle - \langle \varphi_{x_0,R}, \mu(dx,T)\rangle| \le C_{14}\lambda^{1/2}R^{-1}(T-t) \tag{47}$$

for

$$\mu(dx,T) = \sum_{x_0\in S} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

by (26). Given $b > 0$, we can take $R = b(T-t)$ for $0 < T - t \ll 1$ in (47), and then it follows that

$$\limsup_{t\uparrow T} \left|\langle \varphi_{x_0,b(T-t)}, u(t)\rangle - m(x_0)\right| \le C_{14}\lambda^{1/2}b^{-1}.$$

Since $b > 0$ is arbitrary, this implies

$$\lim_{b\uparrow+\infty}\limsup_{t\uparrow T}\left|\int_{B(x_0,b(T-t))} u(x,t)dx - m(x_0)\right| = 0, \tag{48}$$

again for any $b > 0$. Under the transformation (44), inequality (48) reads;

$$\lim_{b\uparrow+\infty}\limsup_{s\uparrow+\infty}\left|\int_{B(0,be^{-\frac{n-1}{n}s})} v(s,y)dy - m(x_0)\right| = 0. \tag{49}$$

We have

$$\int_{\mathbf{R}^n} v(y,s)dy = \lambda \qquad \text{for } s > -\log T, \tag{50}$$

and therefore, any $t_k \uparrow T$ admits $\{s'_k\} \subset \{s_k\}$ for $s_k = -\log(T-t_k)$, such that

$$v(y,s'_k)dy \rightharpoonup \zeta(dy) \qquad \text{in } \mathcal{M}_0(\mathbf{R}^n) = C'_0(\mathbf{R}^n), \tag{51}$$

and this $\zeta(dy)$ satisfies

$$\zeta(dy) \ge m(x_0)\delta_0(dy) \tag{52}$$

by (49), where $C_0(\mathbf{R}^n) = \{\varphi \in C(\mathbf{R}^n \cup \{\infty\}) \mid \varphi(\infty) = 0\}$. Relations (51)-(52) imply

$$\lim_{k\to\infty} \|v(s'_k)\|_{L^\infty(B(0,b))} = +\infty$$

for any $b > 0$, and hence (45). The proof is complete. ∎

We finally examine the possibility of mass quantization, $m(x_0) \le \lambda_*$ for the isolated $x_0 \in \mathcal{S}$. In fact, using the backward self-similar transformation (44), we obtain

$$v_t = \frac{m-1}{m}\Delta v^m - \nabla \cdot \left(v\nabla\Gamma * v + \frac{|y|^2}{2n} \right)$$
$$v \ge 0 \qquad \text{in } \mathbf{R}^n \times (-\log T, +\infty), \tag{53}$$

and then it holds that the decrease of the free energy and its recursive relation between the second moment. They are, formally, given by

$$\frac{d}{ds}\hat{\mathcal{F}}(v) = -\int_{\mathbf{R}^n} v \left| \nabla \left(v^{m-1} - \Gamma * v - \frac{|y|^2}{2n} \right) \right|^2 dy \le 0 \,.$$
$$\frac{d}{ds}\int_{\mathbf{R}^n} |y|^2 v dy = 2(n-2)\hat{\mathcal{F}}(v) + \int_{\mathbf{R}^n} |y|^2 v dy, \tag{54}$$

where

$$\hat{\mathcal{F}}(v) = \left\{ \int_{\mathbf{R}^n} \left(\frac{v^m}{m} - \frac{|y|^2}{2n} v \right) dy - \frac{1}{2}\langle \Gamma * v, v \rangle \right\}.$$

Equation (53) is actually written as

$$v_t = \nabla \cdot v\nabla\delta\hat{\mathcal{F}}(v) \qquad \text{in } \mathbf{R}^n \times (-\log T, +\infty)$$

and hence the first equality of (54) reads;

$$\frac{d}{ds}\hat{\mathcal{F}}(v) = -\int_{\mathbf{R}^n} v \left| \nabla\delta\hat{\mathcal{F}}(v) \right|^2 dy.$$

Relation (54) now implies

$$\frac{d}{ds}\int_{\mathbf{R}^n} |y|^2 v dy \le 2(n-2)\hat{\mathcal{F}}(v_0) + \int_{\mathbf{R}^n} |y|^2 v dy$$

and therefore, the assumption

$$2(n-2)\hat{\mathcal{F}}(v_0) + \int_{\mathbf{R}^n} |y|^2 v_0 dy < 0$$

induces the contradiction, $\int_{\mathbf{R}^n} |y|^2 v dy < 0$ for $s \gg 1$. Thus, it holds that

$$2(n-2)\hat{\mathcal{F}}(v_0) + \int_{\mathbf{R}^n} |y|^2 v_0 dy \ge 0,$$

which must be translated in s:

$$2(n-2)\hat{\mathcal{F}}(v) + \int_{\mathbf{R}^n} |y|^2 v dy \geq 0 \quad \text{for any } s > -\log T. \tag{55}$$

Thus, we obtain some unusual relation (51)-(52) with $m(x_0) > \lambda_*$ and (55), which may suggest the possibility of $m(x_0) = \lambda_*$ for all $x_0 \in \mathcal{S}$ in the case of (29). The other interesting question is the construction of this type solution with radially symmetry, provided with a sharp blowup profile.

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