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Spectrum and Energy of Waves in Mean Shear Flows

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Waves in inviscid shear flows exhibit temporally and spatially complicated behavior due to the presence of the continuous spectrum and the nonorthogonal property of eigenmodes of non-selfadjoint operators. The non-selfadjointness of the linearized systems is associated with the noncanonical Hamiltonian structure of ideal fluids and plasmas. By imposing the kinematical constraint on disturbances, the adjoint equation establishes a formal definition of the wave energy for both point and continuous spectra. This observation will serve for the bifurcation theory of various shear flows in fluids and plasmas.

I. INTRODUCTION

The effect of flow in plasmas has recently attracted considerable attention in fusion research and astrophysics. For the purpose of the magnetic confinement, the stability of plasmas described by ideal magnetohydrodynamics (MHD) is the most fundamental problem. However, theoretical work on flowing plasmas is quit limited in comparison with static plasmas. This fact is essentially due to the non-selfadjoint property of the linearized hydrodynamic equations. The fluid equation linearized about an equilibrium state can be written in the form of the evolution equation $i\partial_t\tilde{u} = \mathcal{L}\tilde{u}$ ($\tilde{u}$: perturbation), where the linear operator $\mathcal{L}$ is non-selfadjoint due to the inhomogeneity (i.e., gradient or shear) of the mean fields. In the ideal limit (no dissipation), it is known that the spectrum of $\mathcal{L}$ includes the continuous spectrum as well as the point spectrum. In contrast with the Schrödinger equation, however, the spectral theory for non-selfadjoint operator is generally unknown especially for the continuous spectrum. Even for an inviscid parallel shear flow, the Rayleigh equation [1] has the non-selfadjoint property, and its necessary and sufficient condition for stability is still a nontrivial problem [2]. Obviously, plasma inherits this difficulty as a fluid model.

Instead of the spectral analysis, one may adopt the variational principle, which provides *a priori* estimate of stability without knowing explicit solutions. If one finds a conserved quantity whose first variation vanishes at an equilibrium state and whose second variation is positive (or negative) definite there, such a state turns out to be (linearly) stable. Since the conserved quantity is usually a combination of the Hamiltonian and the Casimir invariants, this is called the energy-Casimir method. If the conserved quantity was furthermore convex for any finite variation, one could insist the Lyapunov stability [3]. However, it must be remarked that this estimate gives only sufficient condition for stability, and we do not have so many Casimir invariants for three-dimensional fluid motions.

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According to the Krein's theorem [4] (see also Moser [5]), the signature of the energy of each eigenmode is essential for understanding the loss of stability, namely, the bifurcation of steady state. In canonical systems of finite degree of freedom, linear instability is only possible when pairs of eigenvalues of positive and negative energy modes collide. This bifurcation theory is, however, under development for continuous spectrum of infinite dimensional systems, because the corresponding singular eigenmodes require more mathematical techniques than the non-singular modes of point spectrum. In Sec. II and Sec. III, we will associate the non-selfadjoint property of the linearized MHD equation with the noncanonical Hamiltonian structure [6], which will enable us to define the *wave energy* (= energy of eigenmode) including the continuous spectrum in a general manner.

In Sec. IV and Sec. V, we will formulate the wave energy as well as wave action for both point and continuous spectra in the MHD case. The hydrodynamic case can be derived by dropping the magnetic field and will be discussed elsewhere. Generally speaking, the wave energy may be negative in the presence of mean shear flow. In analogy with the Krein's theory, the negative energy mode might cause instability in cooperation with other positive energy mode. While the spectral resolution is still nontrivial for the continuous spectrum, our formulation avoids this difficulty by invoking the hyperfunction theory [7] to deal formally with the singularity.

II. LINEARIZED LIE-POISSON SYSTEMS

The noncanonical Hamiltonian structure of the MHD equation (including the hydrodynamic equation) was uncovered by Morrison & Greene [6]. It was also derived by the Lie-Poisson reduction from a canonical system for the Lagrangian variables [8]. The MHD equation is, thus, regarded as a Lie-Poisson system on the dual $\mathfrak{g}^*$ of the Lie algebra, $\mathfrak{g} = \mathfrak{X}\circledS (\Lambda^1 \oplus \Lambda^0 \oplus \Lambda^3)$, where $\mathfrak{X}$ and Λ^n denote, respectively, the spaces of vector fields and n-forms on the domain $D \subset \mathbb{R}^3$, and $\circledS$ denotes the semidirect product. Denote an element of the dual space by $u = (\boldsymbol{M}, \boldsymbol{B}, \rho, s)^T \in \mathfrak{g}^*$, where u is composed of fluid momentum density $\boldsymbol{M}$, magnetic field $\boldsymbol{B}$, mass density ρ and specific entropy s. The velocity field $\boldsymbol{v}$ is given by $\boldsymbol{M} = \rho \boldsymbol{v}$.

For any functionals $F, G : \mathfrak{g}^* \to \mathbb{R}$, the Lie-Poisson bracket is generically written in the form of

$$\{F, G\} = \left\langle u, \left[\frac{\delta F}{\delta u}, \frac{\delta G}{\delta u} \right] \right\rangle, \tag{1}$$

using the standard pairing $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and the Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. The Hamiltonian equation $\partial_t F = \{F, H\}$ is posed for a prescribed Hamiltonian function $H : \mathfrak{g}^* \to \mathbb{R}$. The Lie-Poisson bracket for the MHD equation is explicitly given in Ref. 6 or Ref. 18.

Let us introduce a notation following the *adjoint representation* of the Lie group theory [9]. A linear operator $\mathrm{ad}(\zeta) : \mathfrak{g} \to \mathfrak{g}$ for any $\zeta \in \mathfrak{g}$ is defined by

$$[\zeta_1, \zeta_2] = -[\zeta_2, \zeta_1] = \mathrm{ad}(\zeta_1)\zeta_2 = -\mathrm{ad}(\zeta_2)\zeta_1 \quad \text{for } \forall \zeta_1, \zeta_2 \in \mathfrak{g}. \tag{2}$$

The dual operator of $\mathrm{ad}(\zeta)$ with respect to the inner bracket $\langle , \rangle$ is denoted by $\mathrm{ad}^*(\zeta) : \mathfrak{g}^* \to \mathfrak{g}^*$. Using this notation, the Hamiltonian equation allows another expression;

$$\partial_t u = -\mathrm{ad}^*\left(\frac{\delta H}{\delta u}\right) u. \tag{3}$$

Any Lie-Poisson system is commonly written in this form of nonlinear evolution equation. Actually, one can reproduce the ideal MHD equation by substituting the corresponding Lie-Poisson bracket and the Hamiltonian.

It is important to notice that the operator $\mathrm{ad}^*(\circ)$ determines, to some extent, how $u(t)$ evolves infinitesimally at each time. Suppose that the Hamiltonian in (3) is replaced by arbitrary linear functional, i.e., $H = \langle u, \zeta \rangle$ for some $\zeta \in \mathfrak{g}$. The corresponding *virtual* variation $\delta u = -\mathrm{ad}^*(\zeta)u$ is said to be kinematically accessible (or dynamically accessible, according to Morrison [10]). Such variations δu generated by every $\zeta \in \mathfrak{g}$ do not span the whole space $\mathfrak{g}^*$, which implies that there is some constraint on the dynamics. The existence of such kinematical constraints is peculiar to the noncanonical systems and, physically, it is related to the conservation laws.

Any equilibrium state $u_e \in \mathfrak{g}^*$ is characterized by an extremum point ($\delta H = 0$) of H *with respect to* the kenamatically accessible variations;

$$\delta H = \left\langle \mathrm{ad}^*(\zeta)u_e, \left.\frac{\delta H}{\delta u}\right|_e \right\rangle = 0 \quad \text{for all } \zeta \in \mathfrak{g}, \tag{4}$$

where $\delta H/\delta u|_e$ denotes the value of the functional derivative $\delta H/\delta u \in \mathfrak{g}$ at $u = u_e$. The linearization of (3) about this u_e leads to a linear evolution equation for perturbation $\tilde{u}(t) \in \mathfrak{g}^*$,

$$\partial_t \tilde{u} = (\mathcal{A}\mathcal{H} + \mathcal{B})\tilde{u}, \quad \tilde{u}(0) = \tilde{u}_0, \tag{E}$$

where we defined some linear operators, $\mathcal{A} : \mathfrak{g} \to \mathfrak{g}^*$, $\mathcal{H} : \mathfrak{g}^* \to \mathfrak{g}$ and $\mathcal{B} : \mathfrak{g}^* \to \mathfrak{g}^*$ by

$$\mathcal{A} := -\mathrm{ad}^*(\circ)u_e, \quad \mathcal{H} := \left.\frac{\delta^2 H}{\delta u^2}\right|_e \quad \text{and} \quad \mathcal{B} := -\mathrm{ad}^*\left(\left.\frac{\delta H}{\delta u}\right|_e\right). \tag{5}$$

Note that $\mathcal{A}$ is an anti-symmetric operator, $\mathcal{A}^* = -\mathcal{A}$ (stemming from the anti-symmetry of the Lie bracket), and $\mathcal{H}$ is a symmetric operator, $\mathcal{H}^* = \mathcal{H}$. Since this paper will focus on the linear theory, we will omit the subscript e in what follows and u will always refer to the equilibrium state satisfying (4). For the MHD case, these operators are computed as shown in Appendix A.

We consider the kenamatically accessible perturbations $\tilde{u} \in \{\mathcal{A}\zeta; \zeta \in \mathfrak{g}\}$ to the equilibrium state, which span the range of $\mathcal{A}$. It is of interest to note that these perturbations constitute an invariant subspace for the evolution of $\tilde{u}(t)$. The following assumption and the subsequent theorem are fundamental to the later sections.

Assumption 1. *The initial data $\tilde{u}_0 \in \mathfrak{g}^*$ of the linearized equation* (E) *is kenamatically accessible, i.e.,*

$$\exists \zeta_0 \in \mathfrak{g} \quad \textit{such that} \quad \tilde{u}_0 = \mathcal{A}\zeta_0. \tag{A}$$

Theorem 2. *Under the assumption* (A), *the solution $\tilde{u}(t)$ of* (E) *remains kenamatically accessible; $\tilde{u}(t) = \mathcal{A}\zeta(t)$ for all $t > 0$ where $\zeta(t)$ is a solution of the adjoint problem*

$$\partial_t \zeta = (\mathcal{H}\mathcal{A} - \mathcal{B}^*)\zeta, \quad \zeta(0) = \zeta_0. \tag{E*}$$

In addition, a symmetric quadratic form $\delta^2 H : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *defined by*

$$\delta^2 H := -\langle(\mathcal{A}\mathcal{H} + \mathcal{B})\mathcal{A}\zeta, \zeta\rangle = \langle\tilde{u}, \partial_t\zeta\rangle \tag{6}$$

is a constant of motion.

This theorem can be proved by noting the symmetry $(\mathcal{B}\mathcal{A})^* = \mathcal{B}\mathcal{A}$ stemming from the Jacobi identity of the Lie bracket. The proof of $(\mathcal{B}\mathcal{A})^* = \mathcal{B}\mathcal{A}$ and the subsequent claim, $\delta^2 H =$ const., has already given in Ref. 11 with some different notations. It must be emphasized here that there is an explicit *duality* between the variables $\tilde{u}$ and ζ. The solution of the linearized Lie-Poisson equation (E) is closely related to that of the adjoint problem (E*) via the mapping $\mathcal{A}$. This fact and the underlying assumption (A) are required for the second (kenamatically accessible) variation $\delta^2 H$ of H to be invariant. Therefore, we refer to this $\delta^2 H = \langle\tilde{u}, \partial_t\zeta\rangle$ as the *energy of perturbation.*

III. WAVE ENERGY AND WAVE ACTION

In the linear theory, it is useful to regard the perturbations $\tilde{u} \in \mathfrak{g}^*$ and $\zeta \in \mathfrak{g}$ as complex variables and invoke the Fourier-Laplace transform. In what follows, we will naturally identify $\mathfrak{g}^*$ as $\mathfrak{g}$ and extend them into a complex Hilbert space L^2. The inner product is then given by $\langle\overline{\tilde{u}}, \zeta\rangle$ for any $\tilde{u}, \zeta \in L^2$, where the bar $(\bar{\ })$ denotes complex conjugate. By just multiplying the linearized systems by the imaginary unit i, we get Schrödinger-like equations,

$$i\partial_t\tilde{u} = \mathcal{L}\tilde{u}, \quad \tilde{u}(0) = \tilde{u}_0, \tag{E$'$}$$

$$i\partial_t\zeta = \mathcal{L}^*\zeta, \quad \zeta(0) = \zeta_0. \tag{E$'^*$}$$

where we defined two pure-imaginary operators, $\mathcal{L} := i(\mathcal{A}\mathcal{H} + \mathcal{B})$ and $\mathcal{L}^* := i(\mathcal{H}\mathcal{A} - \mathcal{B}^*)$.

The linear waves and their frequencies will respectively correspond to the eigenfunctions and the eigenvalues of the linear operator $\mathcal{L}$. Since fluids and plasmas have infinite degree-of-freedom, the spectrum of $\mathcal{L}$ generally includes the continuous spectrum as well as the point (or discrete) spectrum. We will define wave energy (= energy of eigenmode) for both kinds of spectra in this section.

Let $\mathsf{U}(\Omega)$ and $\mathsf{Z}(\Omega)$ $(\Omega \in \mathbb{C})$ be the Laplace transform of $\tilde{u}(t)$ and $\zeta(t)$, which are the solutions of

$$(\Omega - \mathcal{L})\mathsf{U}(\Omega) = i\tilde{u}_0, \tag{LE}$$

$$(\Omega - \mathcal{L}^*)\mathsf{Z}(\Omega) = i\zeta_0. \tag{LE*}$$

The spectrum $\mathrm{Sp}(\mathcal{L}) \subset \mathbb{C}$ of $\mathcal{L}$ is then characterized by the singularities of $\mathsf{U}(\Omega)$ on the complex Ω-plane;

$$\mathrm{Sp}(\mathcal{L}) = \{\omega \in \mathbb{C} : \mathsf{U}(\Omega) = i(\Omega - \mathcal{L})^{-1}\tilde{u}_0 \text{ is not regular at } \Omega = \omega\}. \tag{7}$$

Similarly, the singularities of $\mathsf{Z}(\Omega)$ correspond to the spectrum $\mathrm{Sp}(\mathcal{L}^*)$, which is generally known to be the complex conjugate of $\mathrm{Sp}(\mathcal{L})$; $\mathrm{Sp}(\mathcal{L}) = \overline{\mathrm{Sp}(\mathcal{L}^*)}$.

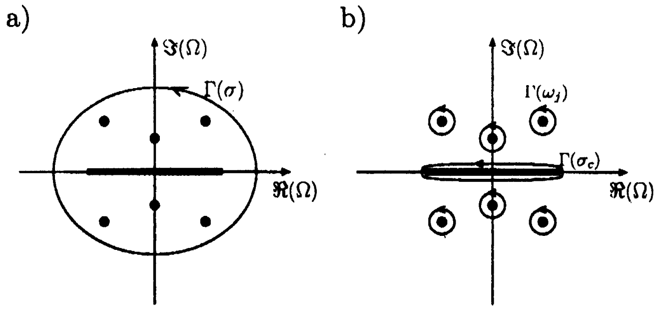

FIG. 1: Path of the Dunford integration

When the initial data is restricted by the assumption (A), the relation $\mathsf{U}(\Omega) = \mathcal{A}\mathsf{Z}(\Omega)$ holds and the resultant solution may not include all eigenmodes. Hence, its spectrum is possibly a subset of $\mathrm{Sp}(\mathcal{L})$, which will be denoted by $\sigma \subset \mathrm{Sp}(\mathcal{L})$;

$$\sigma = \{\omega \in \mathbb{C} : \mathcal{A}\mathsf{Z}(\Omega) = i(\Omega - \mathcal{L})^{-1}\mathcal{A}\zeta_0 \text{ is not regular at } \Omega = \omega\}. \tag{8}$$

The solution $\tilde{u}(t)$ is expressed by the inverse Laplace transform (or the Dunford integral),

$$\tilde{u}(t) = -\frac{1}{2\pi}\oint_{\Gamma(\sigma)} \mathsf{U}(\Omega)e^{-i\Omega t}d\Omega, \tag{9}$$

where the path of integration $\Gamma(\sigma)$ encircles the all spectrum σ in the counterclockwise direction as illustrated in Figure 1a). This $\Gamma(\sigma)$ can be analytically deformed into the neighborhood of each point and continuous spectra as in Figure 1b).

For simplicity, suppose that the spectrum $\sigma = \sigma_p \cup \sigma_c$ is composed of *semi-simple* point spectra $\omega_j \in \sigma_p$, $j = 1, 2, \ldots$, and *real* continuous spectrum $\sigma_c \subset \mathbb{R}$, as is common with ideal fluids and plasmas. For $\omega_j \in \sigma_p$, the residue theorem leads to the corresponding eigenfunction, denoted by

$$\hat{\tilde{u}}(\omega_j) := -\frac{1}{2\pi}\oint_{\Gamma(\omega_j)} \mathsf{U}(\Omega)d\Omega. \tag{10}$$

As for the continuous spectrum $\sigma_c \subset \mathbb{R}$, the path of integration is deformed into the two paths that run parallel to σ_c at the slightly upper and lower sides. Hence, it is reasonable to define the generalized eigenfunction for $\omega \in \sigma_c$ by

$$\hat{\tilde{u}}(\omega) := \frac{1}{2\pi}\left[\mathsf{U}(\omega + i0) - \mathsf{U}(\omega - i0)\right]. \tag{11}$$

This definition of $\hat{\tilde{u}}(\omega)$ is consistent with the Fourier transform of $\tilde{u}(t)$ according to the Sato's hyperfunction theory [7] (see also the Appendix of Ref. [12]). The eigenfunction $\hat{\tilde{u}}(\omega)$ for the continuous spectrum $\omega \in \sigma_c$ is therefore a generalized (or singular) function. This fact has been pointed out in many literatures; for example, see Case [13, 14], Sedláček [15] and Tataronis [16].

The solution is formally represented by

$$\tilde{u}(t) = \sum_{\omega_j \in \sigma_p} \hat{\tilde{u}}(\omega_j)e^{-i\omega_j t} + \int_{\sigma_c} \hat{\tilde{u}}(\omega)e^{-i\omega t}d\omega. \tag{12}$$

In the same way, the eigenfunctions, $\hat{\zeta}(\omega)$ for $\omega \in \sigma$, are generated by $\mathsf{Z}(\Omega)$. Since we have $\mathsf{U}(\Omega) = \mathcal{A}\mathsf{Z}(\Omega)$ under the assumption (A), the following relation is satisfied;

$$\hat{\tilde{u}}(\omega) = \mathcal{A}\hat{\zeta}(\omega) \quad \text{for } \omega \in \sigma. \tag{13}$$

This spectral decomposition can be applied to the theorem 2, which enables us to define energy of each eigenmode (=*wave energy*) as follows.

Theorem 3. *Let $\tilde{u}(t)$ be a solution of* (E) *under the assumption* (A). *If the spectrum σ is composed of semi-simple point spectra $\sigma_p = \{\omega_j \in \mathbb{C} : j = 1, 2, \dots\}$ and real continuous spectrum $\sigma_c \subset \mathbb{R}$, the energy of perturbation $\delta^2 H = \langle \tilde{u}(t), \partial_t \zeta(t) \rangle$ =const. is decomposed into the energy of eigenmodes,*

$$\delta^2 H = \sum_{\omega_j \in \sigma_p} \Re[\omega_j \mu_p(\omega_j)] + \int_{\sigma_c} \omega \mu_c(\omega) d\omega, \tag{14}$$

where

$$\mu_p(\omega_j) = \left\langle \overline{\zeta_0}, i\mathcal{A}\hat{\zeta}(\omega_j) \right\rangle \quad \text{for } \omega_j \in \sigma_p, \tag{15}$$

$$\mu_c(\omega) = \left\langle \overline{\zeta_0}, i\mathcal{A}\hat{\zeta}(\omega) \right\rangle \quad \text{for } \omega \in \sigma_c, \tag{16}$$

and the real part ($\Re$) needs to be taken when ω_j is complex.

This theorem is proved by inserting the solution (12) into $\delta^2 H$ and by using the following two lemmas.

Lemma 4. *The spectrum $\sigma \subset \mathrm{Sp}(\mathcal{L})$ satisfies $\sigma = \overline{\sigma} = -\overline{\sigma} = -\sigma$.*

Lemma 5. *The generalized eigenfunctions $\hat{\tilde{u}}(\omega)$ and $\hat{\zeta}(\omega)$, defined in* (11), *satisfy*

$$\left\langle \overline{\hat{\tilde{u}}(\omega)}, \hat{\zeta}(\omega') \right\rangle = \delta(\omega - \omega') \left\langle \overline{\tilde{u}_0}, \hat{\zeta}(\omega) \right\rangle = \delta(\omega - \omega') \left\langle \overline{\hat{\tilde{u}}(\omega)}, \zeta_0 \right\rangle, \tag{17}$$

for $\omega, \omega' \in \sigma_c$.

Recall that, for point spectra $\sigma_p = \{\omega_j\}$, the eigenfunctions $\{\hat{\tilde{u}}(\omega_j)\}$ of $\mathcal{L}$ are non-orthogonal to each other, but the dual basis is provided by the eigenfunctions $\{\hat{\zeta}(\omega_j)\}$ of $\mathcal{L}^*$. The above lemma 5 implies that such "orthogonality" holds also for continuous spectrum. The proofs of these lemmas will be given in the upcoming paper by us.

We call the quantities $\mu_p(\omega_j)$, $j = 1, 2, \dots$, wave actions since they will prove to be the action variables of the eigenmodes. The magnitude of $\mu_p(\omega_j)$ depends on the square of the corresponding modal amplitude and measures the activity of the wave. The sign of $\mu_p(\omega_j)$ can be either positive or negative (if the symmetric operator, $i\mathcal{A}$, is indefinite), which is of particular interest in the bifurcation theory of Hamiltonian systems [5].

Before ending this section, we introduce another expression for the wave action which is related to the dispersion relation. Let us define a linear operator $\mathcal{E}(\Omega) : L^2 \to L^2$ by

$$\mathcal{E}(\Omega) = i(\Omega - \mathcal{L})\mathcal{A}, \tag{18}$$

with $\Omega \in \mathbb{C}$ being a complex parameter. A symmetric property $\mathcal{E}^*(\Omega) = \mathcal{E}(\overline{\Omega})$ follows from $\mathcal{A} = -\mathcal{A}^*$ and $\mathcal{L}\mathcal{A} = \mathcal{A}\mathcal{L}^*$. In this work, we will refer to the equation

$$\mathcal{E}(\Omega)\mathsf{Z}(\Omega) = -\tilde{u}_0, \tag{19}$$

as *symmetric response equation*, for the solution $\mathsf{Z}(\Omega)$ represents the frequency response to some (kenamatically accessible) initial data $\tilde{u}_0$. If our problem was a system of finite degree of freedom, the space L^2 would be a finite-dimensional vector space and $\mathcal{E}(\Omega)$ be a matrix of *numbers*. Then, the dispersion relation would be given by the determinant, $D(\Omega) = \det|\mathcal{E}(\Omega)| = 0$. However, since we are considering the system of infinite degree of freedom, $\mathcal{E}(\Omega)$ is generally a differential operator and the continuous spectrum shows up. It is no longer possible to obtain the dispersion relation algebraically. We, therefore, introduce *generalized dispersion relation* as follows.

Proposition 6. *Define the generalized dispersion relation* $D : \mathbb{C} \times L^2 \times L^2 \to \mathbb{C}$ *for* (19) *as*

$$D(\Omega, \zeta_1, \zeta_2) := \left\langle \overline{\zeta_1}, \mathcal{E}(\Omega)\zeta_2 \right\rangle . \tag{20}$$

Then, the wave actions for a semi-simple point spectrum $\omega_j \in \sigma_p$ *and real continuous spectrum* $\omega \in \sigma_c$ *are, respectively, given by*

$$\mu_p(\omega_j) = \frac{1}{2\pi i} \oint_{\Gamma(\omega_j)} D(\Omega, \mathsf{Z}(\overline{\Omega}), \mathsf{Z}(\Omega)) d\Omega, \tag{21}$$

$$\mu_c(\omega) = \frac{i}{2\pi} \left[\lim_{\Omega \to \omega + i0} D(\Omega, \mathsf{Z}(\overline{\Omega}), \mathsf{Z}(\Omega)) - \lim_{\Omega \to \omega - i0} D(\Omega, \mathsf{Z}(\overline{\Omega}), \mathsf{Z}(\Omega)) \right] . \tag{22}$$

While this proposition is straightforward from the definitions of μ_p and μ_c, the above expressions will play an important role in the subsequent sections.

IV. MAGNETOHYDRODYNAMIC WAVES

We apply the general theorems discussed so far to the MHD equation. Under the assumption (A), the linearized equation (E) for the MHD case can be reduced to the equation derived by Frieman & Rotenberg [17], which is easier to solve especially when the basic flow is absent. Since the theorem 2 assures $\tilde{u}(t) = \mathcal{A}\zeta(t)$ for all $t > 0$, the corresponding relations holds between $\tilde{u} = (\tilde{\boldsymbol{M}}, \tilde{B}, \tilde{\rho}, \tilde{s})^T$ and $\zeta = (\boldsymbol{\xi}, \boldsymbol{\eta}, \alpha, \beta)^T$ as follows,

$$\begin{aligned} \tilde{\boldsymbol{v}} &= \boldsymbol{\xi} \times (\nabla \times \boldsymbol{v}) - \rho^{-1}\boldsymbol{B} \times (\nabla \times \boldsymbol{\eta}) - \nabla(\boldsymbol{\xi} \cdot \boldsymbol{v} + \alpha) + \beta\rho^{-1}\nabla s, && (23) \\ \tilde{\boldsymbol{B}} &= \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}), && (24) \\ \tilde{\rho} &= -\nabla \cdot (\rho\boldsymbol{\xi}), && (25) \\ \tilde{s} &= -\boldsymbol{\xi} \cdot \nabla s. && (26) \end{aligned}$$

By exploiting these relations, we can eliminate $\boldsymbol{\eta}$, α and β from the evolution equation (E) and reproduce the Frieman-Rotenberg equation for $\boldsymbol{\xi} \in \mathfrak{X}$ with *restricted* initial data;

$$\left\{\begin{aligned} &\rho\partial_t^2\boldsymbol{\xi} + 2\rho(\boldsymbol{v}\cdot\nabla)\partial_t\boldsymbol{\xi} = \mathcal{F}\boldsymbol{\xi}, \\ &(\partial_t\boldsymbol{\xi})(0) = -2(\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi}_0 - \boldsymbol{v}\times(\nabla\times\boldsymbol{\xi}_0) \\ &\qquad\qquad -\rho^{-1}\boldsymbol{B}\times(\nabla\times\boldsymbol{\eta}_0) - \nabla\alpha_0 + \rho^{-1}\beta_0\nabla s, \\ &\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \end{aligned}\right. \tag{27}$$

where $\mathcal{F}:\mathfrak{X}\to\mathfrak{X}$ is a selfadjoint operator called the *Force operator* [17]. The variable $\boldsymbol{\xi}(t)\in\mathfrak{X}$ automatically agrees with the conventional definition of the *Lagrangian displacement field*, which is the displacement vector field of the fluid particle orbits [17]. Therefore, the relations (24)-(26) implies that magnetic field, mass and entropy are *frozen in* the fluid particle motion.

The energy of kenamatically accessible perturbation $\delta^2 H$, defined by the theorem 2, also agrees with the expression of energy derived in Ref. 17 (see Hameiri [18] for detailed comparison). The positive definiteness of the potential energy, $-\int\boldsymbol{\xi}\cdot\mathcal{F}\boldsymbol{\xi}d^3x \geq 0$, gives a *sufficient* stability condition. This criterion is known to works very well for static equilibria ($\boldsymbol{v}=0$), for which (27) is analogous to the Newton's second law. The energy principle [19] claims that a static equilibrium is stable *if and only if* the potential energy is positive definite. Linear instabilities always emerge from the zero eigenvalue of the selfadjoint operator $\mathcal{F}$, whose spectral decomposition can be discussed by the well-established methods (like the Von Neumann theorem) in the quantum mechanics.

However, the potential energy (and also the total energy $\delta^2 H$) of perturbation often turns out to be indefinite in the presence of the basic flow $\boldsymbol{v}$ [20]. The basic flow, moreover, plays the role of the gyroscopic term $2\rho(\boldsymbol{v}\cdot\nabla)\partial_t\boldsymbol{\xi}$ in (27), which allows the existence of neutrally stable modes with negative energy. These facts imply that the energy criterion is difficult to be satisfied for flowing plasmas. It, then, seems to be important to evaluate the wave energy (or action) for the purpose of predicting and understanding various instabilities.

Now, let us proceeds to the derivation of wave energy for the MHD case. The reduction to the Frieman-Rotenberg equation (27) is similarly applicable to the symmetric response equation (19). By denoting the Laplace transform of $\boldsymbol{\xi}(t)$ by $\Xi(\Omega)$, the Laplace-transformed Frieman-Rotenberg equation is written as

$$\mathcal{E}_{\mathrm{FR}}(\Omega)\Xi(\Omega) = -\mathbf{m}_0(\Omega), \tag{28}$$

where the operator $\mathcal{E}_{\mathrm{FR}}(\Omega)$ is defined by

$$\mathcal{E}_{\mathrm{FR}}(\Omega)\Xi(\Omega) := \Omega^2\rho\Xi(\Omega) + 2i\Omega\rho(\boldsymbol{v}\cdot\nabla)\Xi(\Omega) + \mathcal{F}\Xi(\Omega). \tag{29}$$

The right hand side of (28) has a nontrivial expression,

$$\mathbf{m}_0(\Omega) := -i\Omega\rho\boldsymbol{\xi}_0 - \rho\boldsymbol{v}\times(\nabla\times\boldsymbol{\xi}_0) - \boldsymbol{B}\times(\nabla\times\boldsymbol{\eta}_0) - \rho\nabla\alpha_0 + \beta_0\nabla s, \tag{30}$$

reflecting that the initial data was restricted by the assumption (A). Since the operator $\mathcal{E}_{\mathrm{FR}}(\Omega)$ again satisfies $\mathcal{E}^*_{\mathrm{FR}}(\Omega) = \mathcal{E}_{\mathrm{FR}}(\overline{\Omega})$ in terms of the inner bracket $\int\overline{\circ}\cdot\circ d^3x$ of the reduced functional space, one can regard (28) as a new symmetric response equation and define

$$D_{\mathrm{FR}}(\Omega,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \int\overline{\boldsymbol{\xi}}_1\cdot\mathcal{E}_{\mathrm{FR}}(\Omega)\boldsymbol{\xi}_2 d^3x. \tag{31}$$

Theorem 7. *In the MHD case, the wave action is represented by*

$$\begin{aligned}\mu_p(\omega_j) =& i\int \overline{\hat{\boldsymbol{\xi}}(\overline{\omega_j})}\cdot \mathbf{m}_0(\omega_j)d^3x, &(32)\\ =&\frac{\partial D_{\rm FR}}{\partial\Omega}\left(\omega_j,\hat{\boldsymbol{\xi}}(\overline{\omega_j}),\hat{\boldsymbol{\xi}}(\omega_j)\right), &(33)\\ =&\int \overline{\hat{\boldsymbol{\xi}}(\overline{\omega_j})}\cdot 2\rho[\omega_j\hat{\boldsymbol{\xi}}(\omega_j)+i(\boldsymbol{v}\cdot\nabla)\hat{\boldsymbol{\xi}}(\omega_j)]d^3x.\end{aligned}$$

for a semi-simple point spectrum $\omega_j\in\sigma_p$, and

$$\mu_c(\omega) = i\int \overline{\hat{\boldsymbol{\xi}}(\omega)}\cdot \mathbf{m}_0(\omega)d^3x, \tag{34}$$

for real continuous spectrum $\omega\in\sigma_c\subset\mathbb{R}$, where $\hat{\boldsymbol{\xi}}(\omega_j)$ and $\hat{\boldsymbol{\xi}}(\omega)$ are the corresponding eigenfunctions.

Proof. It is essential to notice that the relation,

$$D_{\rm FR}(\Omega,\Xi(\overline{\Omega}),\Xi(\Omega)) = D(\Omega,\mathrm{Z}(\overline{\Omega}),\mathrm{Z}(\Omega)) + \int \rho|\boldsymbol{\xi}_0|^2d^3x, \tag{35}$$

holds. One may, therefore, replace $D(\Omega,\mathrm{Z}(\overline{\Omega}),\mathrm{Z}(\Omega))$ by $D_{\rm FR}(\Omega,\Xi(\overline{\Omega}),\Xi(\Omega))$ in the formulae (21) and (22), because the term $\int\rho|\boldsymbol{\xi}_0|^2d^3x$ independent of Ω does not affect the results. Since $\mathbf{m}_0(\Omega)$ depends on Ω analytically, these formulae yield (32) and (34) respectively.

To derive another expression (33), we invoke the fact that $\Xi(\Omega)$ has semi-simple poles at $\Omega=\omega_j$ and $\Omega=\overline{\omega_j}$ (from Lemma 4). The Laurent expansion at these points must be

$$\Xi(\Omega) = \frac{i\hat{\boldsymbol{\xi}}(\omega_j)}{\Omega-\omega_j}+\dots \quad\text{and}\quad \Xi(\Omega) = \frac{i\hat{\boldsymbol{\xi}}(\overline{\omega_j})}{\Omega-\overline{\omega_j}}+\dots, \tag{36}$$

where the dots $(\dots)$ represent the analytic parts of expansions. By expanding $\mathcal{E}_{\rm FR}(\Omega)$ also at $\Omega=\omega_j$, we get

$$\begin{aligned}D_{\rm FR}(\Omega,\Xi(\overline{\Omega}),\Xi(\Omega)) &= \left\langle\overline{\frac{\hat{\boldsymbol{\xi}}(\overline{\omega_j})}{\overline{\Omega}-\overline{\omega_j}}}, \left[\mathcal{E}_{\rm FR}(\omega_j)+(\Omega-\omega_j)\frac{\partial\mathcal{E}_{\rm FR}}{\partial\Omega}(\omega_j)\right]\frac{\hat{\boldsymbol{\xi}}(\omega_j)}{\Omega-\omega_j}\right\rangle+\dots,\\ &= \left\langle\overline{\hat{\boldsymbol{\xi}}(\overline{\omega_j})}, \frac{\partial\mathcal{E}_{\rm FR}}{\partial\Omega}(\omega_j)\hat{\boldsymbol{\xi}}(\omega_j)\right\rangle\frac{1}{\Omega-\omega_j}+\dots,\end{aligned}$$

where we used the fact that $\hat{\boldsymbol{\xi}}(\omega_j)$ is an eigenfunction, $\mathcal{E}_{\rm FR}(\omega_j)\hat{\boldsymbol{\xi}}(\omega_j)=0$. By putting this form into (21), the residue theorem leads to the required result (33). □

Since both $\mathcal{E}_{\rm FR}(\Omega)$ and $\mathbf{m}_0(\Omega)$ are regular with respect to Ω, the equation (28) and the related dispersion relation $D_{\rm FR}$ can be used anytime in place of (19). The wave energy for a point spectrum $\omega_j\in\sigma_p$ is now written by

$$\omega_j\frac{\partial D_{\rm FR}}{\partial\Omega}\left(\omega_j,\hat{\boldsymbol{\xi}}(\overline{\omega_j}),\hat{\boldsymbol{\xi}}(\omega_j)\right). \tag{37}$$

The formula (33) is more useful than (32) because it is symmetric and independent of initial data. For the continuous spectrum, however, the same formula does not seem to be applicable. The singularity of $\Xi(\Omega)$ at $\Omega = \omega \in \sigma_c$ is far from the simple pole and varies depending on the profiles of mean fields. We will seek the counterpart of the formula (33) for the continuous spectrum by assuming a simple geometry in the next section.

It is obvious from (33) that, in the absence of basic flow $\boldsymbol{v} = 0$, the wave action of any neutrally stable wave $\omega_j \in \mathbb{R}$ is simply $\mu_p(\omega_j) = 2\omega_j \int \rho |\hat{\boldsymbol{\xi}}(\omega_j)|^2 d^3x$. Since the sign of the wave action corresponds to that of ω_j, any linear instability occurs through the zero eigenvalue $\omega_j = 0$ (static bifurcation). In other words, the Hopf bifurcation in fluids and plasmas is necessarily attributed to the presence of basic flow.

V. SLAB EQUILIBRIA

Further reduction of variables can be performed if the equilibrium state has a specific symmetry. As the simplest (but fundamental) example, we restrict our analysis to the slab geometry, i.e. the equilibrium is inhomogeneous only in the x direction as follows,

$$\boldsymbol{v} = (0, v_y(x), v_z(x)), \quad \boldsymbol{B} = (0, B_y(x), B_z(x)), \quad \rho(x) \quad \text{and} \quad s(x) \tag{38}$$

on a bounded domain $x \in [x_1, x_2]$, where the boundary walls are located at $x = x_1, x_2$ and both flow and magnetic field are always tangential to them. This is indeed an equilibrium state if the external force is absent and the total pressure $p_{\text{total}} := p(\rho, s) + B^2/2$ satisfies

$$p'_{\text{total}}(x) = 0 \tag{39}$$

where the prime ($'$) denotes the x-derivative of the equilibrium fields.

For fully three-dimensional perturbations, it is useful to adopt the spatial Fourier transform in the y and z directions;

$$\boldsymbol{\xi}(x, y, z, t) = \frac{1}{2\pi} \int\int \check{\boldsymbol{\xi}}(x, k_y, k_z, t) e^{i(k_y y + k_z)z} dk_y dk_z. \tag{40}$$

Our task is to find the (x, t)-dependences of $\check{\boldsymbol{\xi}}(x, k_y, k_z, t)$ for fixed wavenumbers k_y and k_z. To simplify the notations, we will denote this $\check{\boldsymbol{\xi}}(x, k_y, k_z, t)$ by $\boldsymbol{\xi}(x, t)$, omitting the check ($\check{}$) and the (k_y, k_z)-dependences, which does not cause confusion in many cases. With respect to the fixed wavenumber vector $\boldsymbol{k} := (0, k_y, k_z)$, we introduce the parallel and perpendicular components of $\boldsymbol{\xi}$ as follows.

$$\xi_\parallel := \frac{\boldsymbol{k} \cdot \boldsymbol{\xi}}{k}, \quad \xi_\perp := \frac{(\boldsymbol{e}_x \times \boldsymbol{k}) \cdot \boldsymbol{\xi}}{k}, \tag{41}$$

where $k = |\boldsymbol{k}|$ and $\boldsymbol{e}_x = (1, 0, 0)$. In this manner, we shall use $(x, \parallel, \perp)$ components rather than (x, y, z).

Let $\Xi(x, \Omega)$ be the Laplace transform of $\boldsymbol{\xi}(x, t)$ again. We can algebraically eliminate $\Xi_\parallel$ and $\Xi_\perp$ from (28) after the spatial Fourier transformation. Such the elimination of variables results in a new symmetric response equation;

$$\mathcal{E}_{\text{I}}(\Omega) \Xi_x(\Omega) = -\mathsf{m}_{0\text{I}}(\Omega), \tag{42}$$

with the property $\mathcal{E}_{\rm I}^*(\Omega) = \mathcal{E}_{\rm I}(\overline{\Omega})$ in terms of the one-dimensional inner product $\int_{x_1}^{x_2} \overline{\circ} \circ dx$. The left hand side of this equation corresponds to the well-known eigenvalue problem of the Sturm-Liouville type

$$\mathcal{E}_{\rm I}\Xi_x := \rho\Pi_A\Xi_x + \partial_x \left[\rho \frac{\Pi_S\Pi_A}{\Pi_{sf}} \partial_x \Xi_x\right], \tag{43}$$

which was derived by Hain and Lüst [21], and Goedbloed [22] for the case of static equilibrium (but the generalization to the steady equilibrium (38) is straightforward). The right hand side of (42) represents the initial data which takes the form of

$$\mathsf{m}_{0\rm I} := \mathsf{m}_{0x} - \left(\frac{\mathsf{M}_0}{\Pi_{sf}}\right)', \tag{44}$$

where $\mathsf{M}_0 = i\Pi_S k \mathsf{m}_{0\parallel} - i\Pi_v k b_\parallel \boldsymbol{b} \cdot \mathbf{m}_0$. In the above expressions, we defined some functions of x and Ω as follows.

$$\Pi_v(x,\Omega) = [\Omega - kv_\parallel(x)]^2, \tag{45}$$
$$\Pi_A(x,\Omega) = \Pi_v(x,\Omega) - \omega_A^2(x), \tag{46}$$
$$\Pi_S(x,\Omega) = \left[b^2(x) + c_s^2(x)\right]\left[\Pi_v(x,\Omega) - \omega_S^2(x)\right], \tag{47}$$
$$\begin{aligned}\Pi_{sf}(x,\Omega) &= \Pi_v^2(x,\Omega) - k^2\Pi_S(x,\Omega)\\ &= [\Pi_v(x,\Omega) - \omega_s^2(x)][\Pi_v(x,\Omega) - \omega_f^2(x)],\end{aligned} \tag{48}$$

where $\boldsymbol{b} = \boldsymbol{B}/\sqrt{\rho}$ denotes the Alfvén speed and $c_s = \sqrt{\partial p/\partial \rho}$ the sound speed. The following characteristic frequencies are conventional,

$$\omega_A^2 = k^2 b_\parallel^2 : \text{Alfvén frequency},$$

$$\omega_S^2 = k^2 \frac{b_\parallel^2 c_s^2}{b^2 + c_s^2} : \text{slow magneto-sonic frequency},$$

$$\omega_{s,f}^2 = \frac{k^2}{2}\left[(b^2 + c_s^2) \pm \sqrt{(b^2 + c_s^2)^2 - 4b_\parallel^2 c_s^2}\right]$$

: slow $(-)$ and fast $(+)$ turning point frequencies.

The functions Π_A, Π_S and Π_{sf} may vanish when Π_v is equal to the square of these frequencies. The set of $\omega \in \mathbb{R}$ for which there exists $x_0 \in [x_1, x_2]$ satisfying either $\Pi_A(x_0, \omega) = 0$ or $\Pi_S(x_0, \omega) = 0$ corresponds to the continuous spectrum because such a point x_0 is a regular singular point of the ordinary differential equation (42). On the other hand, the singular point x_0 satisfying $\Pi_{sf}(x_0, \omega) = 0$ is known to be apparent [23], that is, the solution $\Xi_x(x, \Omega)$ remains regular at such (x_0, ω). The continuous spectrum σ_c consists of

$$\text{Alfvén continuous spectrum} \quad \sigma_A = \{kv_\parallel(x) \pm \omega_A(x) \in \mathbb{R} : x \in [x_1, x_2]\}, \tag{49}$$
$$\text{Slow continuous spectrum} \quad \sigma_S = \{kv_\parallel(x) \pm \omega_S(x) \in \mathbb{R} : x \in [x_1, x_2]\}, \tag{50}$$

which may overlap each other and may fold by itself depending on the profiles of the mean fields and the wave number $\boldsymbol{k}$. Moreover, if either $b_\parallel(x)$ or $b_\perp(x)$ vanishes somewhere in $[x_1, x_2]$, some frequencies out of $\omega_A(x)$, $\omega_S(x)$, $\omega_s(x)$ and $\omega_f(x)$ would degenerate and yield a different type of singularity in the equation (even in the incompressible case, these continuous spectra exhibit a nontrivial singularity when $b_{\|}(x)=0$ [12]). In this paper, we focus on the typical case, $b_{\|}(x)\neq 0$ and $b_{\perp}(x)\neq 0$ for all x, where the above four frequencies are separated.

Again, let us define the generalized dispersion relation for (42) by

$$D_{\mathrm{I}}(\Omega,\xi_{x1},\xi_{x2})=\int_{x_1}^{x_2}\overline{\xi_{x1}}\mathcal{E}_{\mathrm{I}}(\Omega)\xi_{x2}dx \quad \text{for } \forall\xi_{x1},\xi_{x2}. \tag{51}$$

The following theorem holds in the same manner as Theorem 7.

Theorem 8. *For the slab MHD equilibria, the wave action for a semi-simple point spectrum $\omega_j\in\sigma_p$ is given by*

$$\begin{aligned}\mu_p(\omega_j)&=i\int_{x_1}^{x_2}\overline{\hat{\xi}_x(\overline{\omega_j})}\mathsf{m}_{0\mathrm{I}}(\omega_j)dx, &(52)\\ &=\frac{\partial D_{\mathrm{I}}}{\partial\Omega}\left(\omega_j,\hat{\xi}_x(\overline{\omega_j}),\hat{\xi}_x(\omega_j)\right), &(53)\\ &=\int_{x_1}^{x_2}2\rho(\omega_j-kv_{\|})\left[\overline{\hat{\xi}_x(\overline{\omega_j})}\hat{\xi}_x(\omega_j)+k^2N(x,\omega_j)\overline{\partial_x\hat{\xi}_x(\overline{\omega_j})}\partial_x\hat{\xi}_x(\omega_j)\right]dx, &(54)\end{aligned}$$

where

$$N(x,\omega_j)=\left.\frac{b_{\|}^2b_{\perp}^2\Pi_v^2+\left(\Pi_S-\Pi_v b_{\|}^2\right)^2}{\Pi_{sf}^2}\right|_{\Omega=\omega_j} \tag{55}$$

If $b_{\|}(x)\neq 0$ and $b_{\perp}(x)\neq 0$ for all x, the wave action for continuous spectrum $\omega\in\sigma_c$ is given by

$$\mu_c(\omega)=i\int_{x_1}^{x_2}\overline{\hat{\xi}_x(\omega)}\mathsf{m}_{0\mathrm{I}}(\omega)dx. \tag{56}$$

Proof. Since (42) was derived from (28) by the elimination of variables $(\Xi_{\|},\Xi_{\perp})$, we find that the new dispersion relation D_{I} is related to D_{FR} as follows.

$$\begin{aligned}&D_{\mathrm{FR}}(\Omega,\Xi(\overline{\Omega}),\Xi(\Omega))\\ &=D_{\mathrm{I}}(\Omega,\Xi_x(\overline{\Omega}),\Xi_x(\Omega))\\ &\quad+\int_{x_1}^{x_2}\overline{\begin{pmatrix}\mathsf{m}_{0\|} & \mathsf{m}_{0\perp}\end{pmatrix}}\cdot\frac{1}{\rho\Pi_{sf}}\begin{pmatrix}\Pi_A & -k^2b_{\|}b_{\perp}\\ -k^2b_{\|}b_{\perp} & \Pi_v-k^2(b_{\perp}^2+c_s^2)\end{pmatrix}\begin{pmatrix}\mathsf{m}_{0\|}\\ \mathsf{m}_{0\perp}\end{pmatrix}dx. \quad (57)\end{aligned}$$

The last expression is regular in terms of Ω except for the apparent singularities stemming from $\Pi_{sf}=0$. Since the apparent singularities are isolated from the genuine singularities, $\Pi_A=0$ and $\Pi_S=0$, due to the assumption $b_{\|},b_{\perp}\neq 0$, we can use D_{I} instead of D_{FR} (and also D) when calculating the wave actions by (21) and (22). The remaining part of the proof is the same as Theorem 7. The computation of the Ω-derivative in (53) results in (54). □

For a *real* point spectrum $\omega_j\in\mathbb{R}$, the integrand of (54) is positive except for $\omega_j-kv_{\|}(x)$. It follows that the wave action is positive (respectively, negative) if the phase velocity ω_j/k of the wave is faster (respectively, slower) than $v_\parallel(x)$, i.e. the basic flow along the $\boldsymbol{k}$ direction, everywhere on $[x_1, x_2]$. When the Hopf bifurcation is concerned, linear instability occurs due to the Krein collision (namely, coalescing on the real axis and splitting toward the upper and lower half planes) between a pair of eigenvalues with positive and negative actions. Such the collision is possible only in the region $\{kv_\parallel(x) \in \mathbb{R} : x \in [x_1, x_2]\}$, from which any unstable eigenvalue $\Im(\omega_j) > 0$ must emerge.

There exists the continuous spectrum on the real axis, too. It seems that, besides the conventional Krein collision between point spectra (eigenvalues), the collision between point and continuum or the one between two continua may cause linear instability as was pointed out by Balmforth & Morrison [24]. To derive the sign of $\mu_c(\omega)$ requires some careful treatment of the singularity. We demonstrate this technique below. Our derivation is straightforward by means of the hyperfunction theory. The similar technique probably applies to various continuous spectra, i.e., for the cases of $b_\parallel = 0$ or $b_\perp = 0$, the hydrodynamic limit $\boldsymbol{B} = 0$ and so on, which will be reported elsewhere.

As was observed by Appert *el al.* [23], we note that the equation (42) is equivalent to the following set of first order differential equations;

$$\begin{cases} \mathsf{P} + \frac{\rho\Pi_S\Pi_A}{\Pi_{sf}}\partial_x\Xi_x = \frac{\mathsf{M}_0}{\Pi_{sf}}, \\ -\rho\Pi_A\Xi_x + \partial_x\mathsf{P} = \mathsf{m}_{0x}, \end{cases} \tag{58}$$

in which a new variable $\mathsf{P}(\Omega)$ is physically interpreted as the Laplace transform of the total pressure perturbation $(\tilde{p} + \boldsymbol{B}\cdot\tilde{\boldsymbol{B}})(t)$. Before applying the formula (22), we represent the dispersion relation in terms of $\mathsf{P}(\Omega)$ as follows.

$$D_\mathrm{I}(\Omega, \Xi_x(\overline{\Omega}), \Xi_x(\Omega)) = \left\langle \overline{\Xi_x(\overline{\Omega})}, \rho\Pi_A\Xi_x(\Omega) + \partial_x\left[\frac{\rho\Pi_S\Pi_A}{\Pi_{sf}}\partial_x\Xi_x(\Omega)\right]\right\rangle_x, \tag{59}$$

$$\begin{aligned} &= -\left\langle \overline{\partial_x\mathsf{P}(\overline{\Omega})}, \frac{1}{\rho\Pi_A}\partial_x\mathsf{P}(\Omega)\right\rangle_x + \left\langle \overline{\mathsf{P}(\overline{\Omega})}, \frac{\Pi_{sf}}{\rho\Pi_S\Pi_A}\mathsf{P}(\Omega)\right\rangle_x \\ &\quad + \left\langle \overline{\mathsf{m}_{0x}}, \frac{1}{\rho\Pi_A}\mathsf{m}_{0x}\right\rangle_x - \left\langle \frac{\overline{\mathsf{M}_0}}{\Pi_{sf}}, \frac{\Pi_{sf}}{\rho\Pi_S\Pi_A}\frac{\mathsf{M}_0}{\Pi_{sf}}\right\rangle_x, \end{aligned} \tag{60}$$

where $\langle\bar{\circ}, \circ\rangle_x = \int_{x_1}^{x_2} \bar{\circ}\ \circ\, dx$ and the several uses of integration by parts has been made. Moreover, by exploiting the equations (58), one can rewrite this in the form of

$$\begin{aligned} D_\mathrm{I}(\Omega, \Xi_x(\overline{\Omega}), \Xi_x(\Omega)) &= \left\langle \overline{\partial_x F(\Omega)}, \frac{1}{\rho\Pi_A}\partial_x F(\Omega)\right\rangle_x - \left\langle \overline{F(\Omega)}, \frac{\Pi_{sf}}{\rho\Pi_S\Pi_A}F(\Omega)\right\rangle_x \\ &\quad + \left\langle \overline{\partial_x G(\Omega) - \mathsf{m}_{0x}}, \frac{1}{\rho\Pi_A}[\partial_x G(\Omega) - \mathsf{m}_{0x}]\right\rangle_x \\ &\quad - \left\langle \overline{G(\Omega) - \frac{\mathsf{M}_0}{\Pi_{sf}}}, \frac{\Pi_{sf}}{\rho\Pi_S\Pi_A}\left[G(\Omega) - \frac{\mathsf{M}_0}{\Pi_{sf}}\right]\right\rangle_x, \end{aligned} \tag{61}$$

where

$$F(\Omega) := \frac{1}{2}[\mathsf{P}(\Omega) - \mathsf{P}(\overline{\Omega})], \tag{62}$$

$$G(\Omega) := \frac{1}{2}[\mathsf{P}(\Omega) + \mathsf{P}(\overline{\Omega})]. \tag{63}$$

The limit of (22) is now estimated as follows. First, let us define some generalized functions as follows,

$$\pi\hat{p}(\omega) = F(\omega + i0), \tag{64}$$

$$\pi\hat{p}^{\dagger}(\omega) = G(\omega + i0) - \frac{\mathsf{M}_0(\omega)}{\Pi_{sf}(x,\omega)}, \tag{65}$$

$$\pi\hat{\psi}(\omega) = \partial_x F(\omega + i0), \tag{66}$$

$$\pi\hat{\psi}^{\dagger}(\omega) = \partial_x G(\omega + i0) - \mathsf{m}_{0x}(\omega). \tag{67}$$

For the Alfvén continuous spectrum $\omega \in \sigma_A$, the well-known Plemlj formula yields the delta functions,

$$\frac{i}{2\pi}\left[\frac{1}{\Pi_A(x,\omega+i0)} - \frac{1}{\Pi_A(x,\omega-i0)}\right] = \frac{1}{2\omega_A(x)}\left[\delta(\omega - kv_{\parallel}(x) - \omega_A(x)) - \delta(\omega - kv_{\parallel}(x) + \omega_A(x))\right], \tag{68}$$

and we have

$$\frac{\Pi_{sf}(x, kv_{\parallel}(x) \pm \omega_A(x))}{\Pi_S(x, kv_{\parallel}(x) \pm \omega_A(x))} = -\frac{k^2 b_{\perp}^2(x)}{b^2(x)}. \tag{69}$$

Substituting these expressions, the wave action for $\omega \in \sigma_A$ is represented by

$$\mu_c(\omega) = \int_{x_1}^{x_2} \frac{\pi^2}{2\rho\omega_A}\left[\delta(\omega - kv_{\parallel} - \omega_A) - \delta(\omega - kv_{\parallel} + \omega_A)\right] \times \left[|\hat{\psi}(\omega)|^2 + |\hat{\psi}^{\dagger}(\omega)|^2 + \frac{k^2 b_{\perp}^2}{b^2}\left(|\hat{p}(\omega)|^2 + |\hat{p}^{\dagger}(\omega)|^2\right)\right] dx. \tag{70}$$

In order for this expression to make sense, the functions $\hat{p}$, $\hat{p}^{\dagger}$, $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ must be continuous at the positions where $\omega - kv_{\parallel}(x) \pm \omega_A(x)$ vanishes.

Let x_0 be, again, the singular point that satisfies either $\omega = kv_{\parallel}(x_0) \pm \omega_A(x_0)$ or $\omega = kv_{\parallel}(x_0) \pm \omega_S(x_0)$. In either cases, the solution $\Xi_x(x,\Omega)$ of (42) is represented by the Frobenius series expansion in the neighborhood of $x = x_0$ [22], say,

$$\Xi_x(x,\Omega) = c_1 f(s) + c_2[g(s) + f(s)\ln s] \tag{71}$$

where $s \in \Omega$ is the complex continuation of the real variable $x - x_0$, and $f(s) = f_0 + f_1 s + \ldots$ and $g(s) = g_0 + g_1 s + \ldots$ are some analytic functions. Thus, the singularity of $\Xi_x(x,\Omega)$ is at worst logarithmic $\sim \ln s$. It easily follows from $\Pi_S\Pi_A \propto s$ that

$$\Pi_S\Pi_A\partial_x\Xi_x = c_1 s f'(s) + c_2\left[s g'(s) + f(s) + s f'(s)\ln s\right]. \tag{72}$$

Therefore, the singularity of $\mathsf{P}(x,\Omega)$ is at worst $s\ln s$, which implies that $\mathsf{P}(x,\omega \pm i0)$ must be continuous function including the point $s = 0$, namely, $x = x_0$. This argument guarantees that the continuity of $\hat{p}(\omega)$ and $\hat{p}^{\dagger}(\omega)$ at $x = x_0$.

Furthermore, since $\Xi_x \sim \ln s$ and $\Pi_A \propto s$ for Alfvén singularity, we note from (58) that $\partial_x \mathsf{P} - \mathsf{m}_{0x} = \rho\Pi_A\Xi_x$ must be not only continuous but also zero at $s = 0$.

This implies that $\hat{\psi}(\omega) = 0$ and $\hat{\psi}^\dagger(\omega) = 0$ at $x = x_0$, and these functions actually does not appear in (70).

Similarly, as regards the slow continuous spectrum $\omega \in \sigma_S$, we obtain

$$\frac{i}{2\pi}\left[\frac{1}{\Pi_S(\omega + i0, x)} - \frac{1}{\Pi_S(\omega - i0, x)}\right]$$
$$= \frac{1}{2c_s^2(x)\omega_A(x)}\left[\delta(\omega - kv_\parallel(x) - \omega_S(x)) - \delta(\omega - kv_\parallel(x) + \omega_S(x))\right], \tag{73}$$

and

$$\frac{\Pi_{sf}(kv_\parallel(x) \pm \omega_S(x), x)}{\Pi_A(kv_\parallel(x) \pm \omega_S(x), x)} = -\frac{k^2 b_\parallel^2(x) c_s^4(x)}{b^2(x)[b^2(x) + c_s^2(x)]}. \tag{74}$$

Our computations are then summarized as follows.

Theorem 9. *If $b_\parallel(x) \neq 0$, $b_\perp(x) \neq 0$ for all x, the wave action for the Alfvén continuous spectrum $\omega \in \sigma_A = \{kv_\parallel(x) \pm \omega_A(x)\}$ is given by*

$$\mu_c(\omega) = \int_{x_1}^{x_2} \frac{\pi^2}{2\rho\omega_A}\left[\delta(\omega - kv_\parallel - \omega_A) - \delta(\omega - kv_\parallel + \omega_A)\right]$$
$$\times \frac{k^2 b_\perp^2}{b^2}\left[|\hat{p}(\omega)|^2 + |\hat{p}^\dagger(\omega)|^2\right] dx, \tag{75}$$

and that for the slow continuous spectrum $\omega \in \sigma_S = \{kv_\parallel(x) \pm \omega_A(x)\}$ is given by

$$\mu_c(\omega) = \int_{x_1}^{x_2} \frac{\pi^2}{2\rho\omega_A}\left[\delta(\omega - kv_\parallel - \omega_S) - \delta(\omega - kv_\parallel + \omega_S)\right]$$
$$\times \frac{k^2 b_\parallel^2 c_s^2}{b^2(b^2 + c_s^2)}\left[|\hat{p}(\omega)|^2 + |\hat{p}^\dagger(\omega)|^2\right] dx. \tag{76}$$

Since both integrals include the delta function, only the value of each integrand at the singularity $x = x_0$ is of interest to the calculation of $\mu_c(\omega)$. The sign of the wave action is now clear-cut without solving $\hat{p}(\omega)$ or $\hat{p}^\dagger(\omega)$. If a singular mode localized at $x = x_0$ propagates in the positive (or negative) direction of the mean magnetic field, $\boldsymbol{k} \cdot \boldsymbol{B}(x_0) > 0$ [or $\boldsymbol{k} \cdot \boldsymbol{B}(x_0) < 0$], relative to the mean flow, $v_\parallel(x_0)$, then the corresponding wave action is positive (or negative). Note that, without mean shear flow $\boldsymbol{v} \equiv 0$, the spectra of the singular modes with positive and negative actions never collide each other on the real frequency axis (this collision occurs if $b_\parallel = 0$ somewhere which is indeed well-known as the most dangerous place for static MHD equilibria). While the mean shear flow causes only the Doppler shift of each singular mode by $\boldsymbol{k}\cdot\boldsymbol{v}(x)$, it enables the coalescence of continuous spectra with opposite signs of wave action. This finding may explain mechanisms of various instabilities, such as the joint instability [25] and the magnetorotational instability [26], in which the energy exchange between flow and magnetic field seems to be essential. The value of the wave action will be crucial also in the study of the wave-mean field interaction, because it plays the role of an adiabatic invariant when the mean fields undergo slow modification.

APPENDIX A: LINEAR OPERATORS IN THE LINEARIZED MHD EQUATION

The linearized MHD equation for perturbations $\tilde{u} = (\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{B}}, \tilde{\rho}, \tilde{s})^T$ can be represented by the form of (E), where the linear operators $\mathcal{A}$, $\mathcal{H}$ and $\mathcal{B}$ are explicitly written in 4×4 matrix forms as follows.

$$\mathcal{A} = \begin{pmatrix} -\boldsymbol{M}\nabla \cdot \circ - \nabla(\boldsymbol{M} \cdot \circ) - (\nabla \times \boldsymbol{M}) \times \circ & -\boldsymbol{B} \times (\nabla \times \circ) & -\rho\nabla\circ & \circ\nabla s \\ \nabla \times (\circ \times \boldsymbol{B}) & 0 & 0 & 0 \\ -\nabla \cdot (\rho\circ) & 0 & 0 & 0 \\ -\circ \cdot \nabla s & 0 & 0 & 0 \end{pmatrix}, \quad \text{(A1)}$$

$$\mathcal{H} = \begin{pmatrix} \frac{1}{\rho} & 0 & -\frac{\boldsymbol{M}}{\rho^2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\boldsymbol{M}}{\rho^2} & 0 & \frac{|\boldsymbol{M}|^2}{\rho^3} + \frac{\partial^2(\rho e)}{\partial \rho^2} & \frac{\partial^2(\rho e)}{\partial \rho \partial s} \\ 0 & 0 & \frac{\partial^2(\rho e)}{\partial \rho \partial s} & \frac{\partial^2(\rho e)}{\partial s^2} \end{pmatrix}, \quad \text{(A2)}$$

$$\mathcal{B} = \begin{pmatrix} -\nabla \cdot \boldsymbol{v} - \nabla(\circ \cdot \boldsymbol{v}) - (\nabla \times \circ) \times \boldsymbol{v} & (\nabla \times \boldsymbol{B}) \times \circ & \nabla[\frac{|\boldsymbol{v}|^2}{2} - h] & \rho T \nabla\circ \\ 0 & \nabla \times (\boldsymbol{v} \times \circ) & 0 & 0 \\ 0 & 0 & -\nabla \cdot (\circ\boldsymbol{v}) & 0 \\ 0 & 0 & 0 & -\boldsymbol{v} \cdot \nabla\circ \end{pmatrix}, \quad \text{(A3)}$$

where $e(\rho, s)$ is a given function of ρ and s representing the internal energy per unit mass, and $h(\rho, s) = \partial[\rho e(\rho, s)]/\partial\rho$ denotes the enthalpy per unit mass and $T(\rho, s) = \partial e(\rho, s)/\partial s$ the temperature.

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