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Vanishing of Vacuum States and Blow-up Phenomena of the Compressible Navier-Stokes Equations

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1 Introduction

In this short paper, we report our recent results, adapted from [6], on the vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations.

A one-dimensional compressible flow model, called the viscous Saint-Venant system for laminar shallow water, derived rigorously from incompressible Navier-Stokes system with a moving free surface by Gerbeau-Perthame recently in [4], has the form:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x - a(\rho u_x)_x + (\rho^2)_x = 0. \end{cases} \tag{1.1}$$

Such models appear naturally and often in geophysical flows [1,2].

The compressible isentropic Navier-Stokes equations, which are the basic models describing the evolution of a viscous compressible fluid, read as follows

$$\begin{cases} \rho_t + div(\rho u) = 0, \\ (\rho u)_t + div(\rho u \otimes u) - 2div(\mu D(u)) - \nabla(\xi divu) + \nabla p(\rho) = 0, \end{cases} \tag{1.2}$$

where $x \in \Omega \subset R^N, t \in (0,T)$, $D(u) = (\nabla u + (\nabla u)^{\mathrm{T}})/2$, and $p(\rho) = a\rho^\gamma, a > 0, \gamma \geq 1$, the viscosity coefficients μ, ξ are assumed to satisfy $\mu \geq 0$ and $\xi + 2\mu/N \geq 0$.

If μ and ξ are both constants, there is huge literature on the studies of the global existence and behavior of solutions to (1.2). For instance, the one-dimensional (1D) problems were addressed by Kazhikhov et al [9] for sufficiently smooth data, where the data are uniformly away from the vacuum; the multidimensional problems (1.2) were investigated by Matsumura et al [13, 14], who proved global existence of smooth solutions for data close to a non-vacuum equilibrium. And for the existence of solutions for arbitrary data(which may include vacuum states), Lions [12] (see also Feireisl et al [3]) obtained global existence of weak solutions - defined as solutions with finite energy - when the exponent γ is suitably large, where the only restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish.

Despite the important progress, the regularity, uniqueness and behavior of these weak solutions remain largely open. As emphasized in many papers related to compressible fluid dynamics, the possible appearance of vacuum is one of the major difficulties when trying to prove global existence and strong regularity results. Hoff and Smoller [5] proved that weak solutions of the compressible Navier-Stokes equations (1.2) in one space dimension do not exhibit vacuum states in a finite time provided that no vacuum is present initially under fairly general conditions on the data. On the other hand, the results of Xin [17] showed that there is no global smooth solution to Cauchy problem for (1.2) with a nontrivial compactly supported initial density, which gives results for finite time blow-up in the presence of vacuum.

For $\mu = \mu(\rho), \xi = \xi(\rho)$ and the multidimensional case, Vaigant et al [16] first proved that for the 2D case and for the case μ is a constant and $\xi(\rho) = a\rho^{\beta}$, with $a > 0, \beta > 3$, (1.2) with periodic boundary condition has a unique strong and classical solution with density away from vacuum. More recently, Bresch and Desjardins [1, 2] (see also [15]) have made important progress. Under the condition that $\xi(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$, they establish a new Bresch-Desjardins (BD) entropy inequality which can not only be applied to the vacuum case but also be used to get the compactness results for (1.2) which extended the compactness results due to Lions [12] to the case $\gamma \geq 1$.

We study mainly the initial-boundary-value problem (IBVP) for (1.2), where $\mu = \rho^{\alpha}$ with $\alpha > 1/2$, on spatial one-dimensional bounded spatial domains or periodic domains. This contains the physical important model for shallow water equations (1.1). We first establish the global existence of entropy weak solutions for the compressible Navier-Stokes equations (1.2), with pressure $p = \rho^{\gamma}$ and $\gamma \geq \alpha/2$, for general initial data with finite entropy and vacuum. The key in our analysis is the construction non-vacuum approximate solutions so that we can make use of the stability analysis in [15], where the Bresch-Desjardins (BD) entropy inequality was used to obtain the compactness results. In general, it seems rather difficult to investigate the dynamics of vacuum states due to the degeneracy of nonlinear diffusion and the density function connecting to vacuum continuously. Therefore, we further consider the cases of more regular initial data containing point vacuum or continuous vacuum of one piece, and we show that there is a global entropy weak solution which is unique and regular with well-defined velocity field at least for short time, and the vacuum states remain for the short time. Then, we use some ideas due to [7, 10, 11] to prove that any possible vacuum state in such global weak solutions which satisfy the BD entropy must vanish within finite time. This shows that such short time structure and vacuum states of weak solutions can not be maintained all the time. And as the vacuum states vanish, the spatial derivative of velocity (if it exists) has to blow up even if the velocity is regular enough and well-defined before. After the vanishing of vacuum states, we can redefine the velocity field and recover the nonlinear diffusion term in terms of density and velocity. In addition, the global entropy weak solution is shown to become a strong solution and tends to the non-vacuum equilibrium state exponentially in time. This phenomena, applied to the compressible shallow water equations (1.1), seems to be never observed for the compressible Navier-Stokes equations before.

2 Main results

We consider the initial-boundary-value-problem (IBVP) for the 1D compressible Navier-Stokes equations with density-dependent viscosity

$$\rho_t + (\rho u)_x = 0, \tag{2.1}$$
$$(\rho u)_t + (\rho u^2 + p(\rho))_x - (\mu(\rho) u_x)_x = 0, \tag{2.2}$$

with $\rho \geq 0$ the density, ρu the momentum. The pressure and viscosity are assumed to have the form:

$$p(\rho) = a_1 \rho^\gamma, \ \mu(\rho) = a_2 \rho^\alpha \tag{2.3}$$

where $\gamma \geq 1,\ a_1 > 0,\ a_2 > 0$, and $\alpha > 1/2$ are constants.

For simplicity we only state our results on the IBVP of shallow water equations (1.1) with periodic boundary conditions,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x - (\rho u_x)_x + (\rho^2)_x = 0, \\ \rho,\ u \text{ are periodic in } x \text{ of period } 1, \\ \rho(x,0) = \rho_0(x) \geq 0, \quad \rho u(x,0) = m_0(x). \end{cases} \tag{2.4}$$

In fact, our results still hold for (2.1)-(2.3) and the following Dirichlet type boundary conditions

$$\rho u(0,t) = \rho u(1,t) = 0, \quad t \geq 0.$$

Throughout the present paper the initial data is assumed to satisfy

$$\begin{cases} \rho_0 \geq 0 \text{ a.e. in } \Omega, \quad \rho_0 \in L^1(\Omega), \quad (\sqrt{\rho_0})_x \in L^2(\Omega), \\ m_0 = 0, \quad \text{a.e. on } \{x \in \Omega \,|\, \rho_0(x) = 0\}, \ \frac{|m_0|^2}{\rho_0} \in L^1(\Omega). \end{cases} \tag{2.5}$$

Remark 2.1 *It should be clear that a large class of initial data satisfy the conditions in* (2.5). *In particular, the assumptions* (2.5) *are satisfied for following initial data*

$$\rho_0(x) = (x - x_0)^2, \quad m_0(x) = 0, \quad x \in \Omega.$$

We define the set of test functions Φ as follows,

$$\Phi = \{\varphi \in C^\infty(\mathbb{R} \times [0,T)) | \ \varphi \text{ is periodic in } x \text{ of period } 1\}.$$

Definition 2.2 (global weak solutions) *For any $T > 0$, (ρ, u) is said to be a weak solution to (2.4) if*

$$\begin{cases} 0 \le \rho \in L^\infty(0,T; L^1(\Omega) \cap L^\gamma(\Omega)), & (\sqrt{\rho})_x \in L^\infty(0,T; L^2(\Omega)), \\ \sqrt{\rho}u \in L^\infty(0,T; L^2(\Omega)), & \rho u_x \in L^2(0,T; W^{-1,1}_{\text{loc}}(\Omega)), \end{cases} \tag{2.6}$$

and (ρ, u) satisfies

$$\int_\Omega \rho_0 \varphi(x,0)dx + \int_0^T \int_\Omega \rho \varphi_t dxdt + \int_0^T \int_\Omega \sqrt{\rho}\sqrt{\rho}u\varphi_x dxdt = 0, \tag{2.7}$$

and

$$\begin{aligned} &\int_\Omega m_0 \varphi(x,0)dx + \int_0^T \int_\Omega \sqrt{\rho}(\sqrt{\rho}u)\varphi_t dxdt \\ &+ \int_0^T \int_\Omega \left((\sqrt{\rho}u)^2 + \rho^2\right) \varphi_x dxdt - \langle \rho u_x, \varphi_x \rangle = 0 \end{aligned}$$

for all $\varphi \in \Phi$. The nonlinear diffusion term ρu_x is defined as

$$\begin{aligned} \langle \rho u_x, \varphi \rangle &= -\int_0^T \int \sqrt{\rho}\sqrt{\rho}u\varphi_x dxdt \\ &\quad -2\int_0^T \int (\sqrt{\rho})_x \sqrt{\rho}u\varphi dxdt \end{aligned} \tag{2.8}$$

where $\rho \in L^\infty(\Omega \times (0,T))$ due to (2.6). Moreover, $(\rho, \sqrt{\rho}u)$ is periodic.

Definition 2.3 (global entropy weak solutions) *Let (ρ, u) be a global weak solution (in the sense of Definition 2.2) to (2.4). Then, (ρ, u) is said to be a global entropy weak solution if there exists some function $\Lambda \in L^2(\Omega \times (0,T))$ satisfying (2.8), i.e.,*

$$\int_0^T \int \Lambda \varphi dxdt = -\int_0^T \int \sqrt{\rho}\sqrt{\rho}u\varphi_x dxdt - 2\int_0^T \int (\sqrt{\rho})_x \sqrt{\rho}u\varphi dxdt$$

for any $\varphi \in \Phi$, and the following uniform entropy inequality holds

$$\begin{aligned} &\sup_{0\le t\le T} \int_\Omega \left(|\sqrt{\rho}u|^2 + |(\sqrt{\rho})_x|^2 + \rho^2\right)(x,t)\,dx + \int_0^T \int_\Omega (\rho_x^2 + \Lambda^2)(x,t)\,dxdt \\ &\le C_0 \int_\Omega (\tfrac{|m_0|^2}{\rho_0} + |(\sqrt{\rho_0})_x|^2 + \rho_0^2(x)\,dx \end{aligned} \tag{2.9}$$

with $C_0 > 0$ independent of T.

We have the following result on the existence of global entropy weak solutions.

Theorem 2.1 (**Global existence**) *Assume that the initial data* (ρ_0, m_0) *satisfies* (2.5) *and* $\frac{|m_0|^{2+\nu}}{\rho_0^{1+\nu}} \in L^1(\Omega)$ *for some positive constant* ν*. Then for any* $T > 0$, *there exists a global entropy weak solution* (ρ, u) *to the IBVP (2.4) in the sense of Definition 2.3.*

Next, we show that there is a global entropy weak solution (ρ, u) in the sense of Definition 2.3 for which the vacuum states and the structure of interface, if existing initially, can be preserved for a short time, so long as the initial data has additional regularity besides (2.5) and the fluids and the vacuum states in initial data are connected "smoothly". In addition, the weak solution (ρ, u) is actually a unique regular solution for the short time. For simplicity, we consider the case of one point vacuum state contained at $x = x_0 \in (0,1)$ in the initial data $(\rho_0, m_0) = (\rho_0, \rho_0 u_0)$ with additional regularity

$$A_0|x - x_0|^\sigma \le \rho_0(x) \le A_1|x - x_0|^\sigma, \text{ for any } x \in \Omega, \tag{2.10}$$

$$u_0 \in C^1(\bar{\Omega}), \quad (\rho_0^{\gamma-1+1/2j})_x \in L^{2j}(\Omega), \ \rho_0^{-1+1/2j}(\rho_0^\alpha u_{0x})_x \in L^{2j}(\Omega), \ j = 1, n, \tag{2.11}$$

with $n \ge 2$ an integer; and in the case of continuous vacuum state of one piece initially on $\Omega^0 = [x_0, x_1] \subset (0,1)$ in the initial data, we require

$$\begin{cases} A_0(x_0 - x)^\sigma \le \rho_0(x) \le A_1(x_0 - x)^\sigma, & x \in [0, x_0), \\ \rho_0(x) = 0, m_0(x) = \rho_0 u_0(x) = 0, & x \in [x_0, x_1], \\ B_0(x - x_1)^\sigma \le \rho_0(x) \le B_1(x - x_1)^\sigma, & x \in (x_1, 1] \end{cases} \tag{2.12}$$

and

$$\begin{cases} (\rho_0^{\gamma-1+1/2j})_x \in L^{2j}(\Omega), \ j = 1, n, \quad u_0 \in C^1(\bar{\Omega} \setminus \Omega^0), \\ \rho_0^{-1+1/2j}(\rho_0^\alpha u_{0x})_x \in L^{2j}(\Omega \setminus \Omega^0), \ j = 1, n, \end{cases} \tag{2.13}$$

with $n \ge 2$ an integer. Here, σ, A_0, A_1, and B_0, B_1 are positive constants, and the power $\sigma \in (\sigma_-, \sigma_+)$ with positive constants $\sigma_\pm$ given in (2.14) later.

We have the following results on short time structure of global entropy weak solutions.

Theorem 2.2 (**Short time structure of vacuum states**) *In addition to the assumptions of Theorem 2.1, assume further that there is either one point vacuum state in initial data* (ρ_0, u_0) *with* (2.10)–(2.11) *satisfied or a piece of continuous vacuum states in initial data* (ρ_0, u_0) *with* (2.12)–(2.13) *satisfied. Then, there exists a global entropy weak solution* (ρ, u) *to the IBVP (2.4) in the sense of Definition 2.3.*

Moreover, there is a short time $T_* > 0$, *so that the global entropy weak solution* (ρ, u) *is unique*[1] *and regular on the domain* $\Omega \times [0, T_*]$, *and the initial structure of vacuum states is maintained for* $t \in [0, T_*]$ *in the following sense:*

[1]Here the uniqueness is specified for density ρ and momentum $\rho u = \sqrt{\rho}\sqrt{\rho} u$ for continuous vacuum states of one piece.

For the case of one point vacuum state initially, (2.10), *the solution* (ρ,u) *is regular and unique on the domain* $\bar{\Omega}\times[0,T_*]$,

$$(\rho,u)\in C^0(\bar{\Omega}\times[0,T_*]),\quad u_x\in L^\infty(0,T_*;C^0(\bar{\Omega})),$$
$$\|u\|_{L^\infty(\bar{\Omega}\times[0,T_*])}+\|u_x\|_{L^\infty([0,T_*];C^0(\bar{\Omega}))}\le C(T_*).$$

The one point vacuum state propagates along particle path, namely, there is one particle path $x=X_0(t):[0,T_*]\to\overline{\Omega}$ *with* $X_0(t)\in C([0,T_*])$ *defined by*

$$\dot{X}_0(t)=u(X_0(t),t),\quad X_0(0)=x_0\in(0,1),$$

so that

$$a_-|x-X_0(t)|^\sigma\le\rho(x,t)\le a_+|x-X_0(t)|^\sigma$$

for $(x,t)\in\Omega\times[0,T_*]$, *where the two positive constants* $a_\pm$ *are independent of time* T_*.

In the case of a piece of continuous vacuum states initially, (2.12), *there are two particle pathes* $x=X_i(t):[0,T_*]\to\overline{\Omega}$ *with* $X_i(t)\in C([0,T_*]), i=0,1$ *defined by*

$$\dot{X}_i(t)=u(X_i(t),t),\quad X_i(0)=x_i\in(0,1), i=0,1,$$

so that it holds for some positive constants $a_\pm$, $b_\pm$ *independent of the time* T_*,

$$a_-(X_0(t)-x)^\sigma\le\rho(x,t)\le a_+(X_0(t)-x)^\sigma,$$

for $(x,t)\in[0,X_0(t))\times[0,T_*]$, *and*

$$b_-(x-X_1(t))^\sigma\le\rho(x,t)\le b_+(x-X_1(t))^\sigma$$

for $(x,t)\in(X_1(t),1]\times[0,T_*]$ *respectively, and the interfaces separating the fluid and vacuum coincide with the particle pathes*

$$\rho(x,t)=0,\quad \rho u(x,t)=0,\quad (x,t)\in[X_0(t),X_1(t)]\times[0,T_*].$$

The solution (ρ,u) *is regular and unique up to the vacuum boundary*

$$\rho\in C^0(\bar{\Omega}\times[0,T_*]),\quad u\in C^0(\bar{\Omega}\times[0,T_*]\setminus\Omega^0_{T_*}),$$
$$\|u\|_{L^\infty(\bar{\Omega}\times[0,T_*]\setminus\Omega^0_{T_*})}+\|u_x\|_{L^\infty(\bar{\Omega}\times[0,T_*]\setminus\Omega^0_{T_*})}\le C(T_*).$$

where $\Omega^0_{T_*}=(X_0(t),X_1(t))\times[0,T_*]$.

Remark 2.4 (1). *The constant exponents* $\sigma_\pm$ *are defined as* $\sigma_\pm=\beta_\pm/(1-\beta_\pm)>0$ *with* $\beta_\pm$ *determined by*

$$\beta_-=\max\{\tfrac{1}{2\alpha},\ \tfrac{1}{\gamma}(1-\tfrac{1}{2n})\},\quad \beta_+=\min\{1,\ \tfrac{1}{\alpha}(1-\tfrac{1}{2n}),\ \tfrac{1}{1+3\alpha}(4-\tfrac{1}{n})\},$$

while the positive constants $a_\pm$ *are independent of the time* T_*.

(2). *The regularity assumptions* (2.10)–(2.11) *are satisfied for the following initial data*

$$\rho_0(x) = \tfrac{1}{2}(A_0 + A_1)(|x - x_0|^2)^{(\sigma_- + \sigma_+)/4}, \quad u_0(x) = 0, \quad x \in \Omega,$$

and the regularity assumptions (2.12)–(2.13) *are satisfied for the initial data*

$$\rho_0(x) = \begin{cases} \frac{1}{2}(A_0 + A_1)(x_0 - x)^{(\sigma_- + \sigma_+)/2}, & x \in [0, x_0), \\ 0, & x \in (x_0, x_1), \\ \frac{1}{2}(B_0 + B_1)(x - x_1)^{(\sigma_- + \sigma_+)/2}, & x \in (x_1, 1], \end{cases} \quad u_0(x) = 0, \quad x \in \Omega.$$

Next, we prove that for any global entropy weak solution (ρ, u) to the IBVP (2.4) in the sense of Definition 2.3, even though in some cases that the vacuum states may exist for some finite time, for instance, in the cases as shown by Theorem 2.2, any possible vacuum state has to vanish within finite time after which the density is always away from vacuum. Simultaneously, not only can the velocity field be defined in terms of the density and momentum, and the nonlinear diffusion is represented in terms of the density and velocity, but also the global entropy weak solution (ρ, u) is shown to be a unique and strong solution after the vanishing of vacuum states. We have the following result.

Theorem 2.3 (Vanishing of vacuum states) *Let (ρ, u) be any global entropy weak solution to the IBVP (2.4) in the sense of Definition 2.3. Then, there exist some time $T_0 > 0$ (depending on initial data) and a constant ρ_- so that*

$$\inf_{x \in \bar{\Omega}} \rho(x, t) \geq \rho_- > 0, \quad t \geq T_0, \tag{2.14}$$

and the global entropy weak solution (ρ, u) becomes a unique strong solution (ρ, u) for $t \geq T_0$ and satisfies

$$\begin{cases} \rho \in L^\infty(T_0, t; H^1(\Omega)), \quad \rho_t \in L^\infty(T_0, t; L^2(\Omega)), \\ u \in H^1(T_0, t; L^2(\Omega)) \cap L^2(T_0, t; H^2(\Omega)), \end{cases} \tag{2.15}$$

with velocity u and nonlinear diffusion term given by

$$u \triangleq \frac{\sqrt{\rho} u}{\sqrt{\rho}}, \quad (\rho^\alpha u_x)_x = \Lambda_x,$$

respectively. In addition, for

$$u_s \triangleq \frac{1}{\bar{\rho}_0} \int_\Omega m_0 dx$$

there exist two positive constants μ_0, c_0 both depending on initial data (ρ_0, m_0) and ρ_-, such that

$$\|(\rho - \bar{\rho}_0, u - u_s)(\cdot, t)\|_{L^2(\Omega)} \leq c_0 e^{-\mu_0 (t - T_0)}, \quad t > T_0,$$

where and what follows $\bar{f}$ denotes the average of f over the bounded domain Ω, i.e.,

$$\bar{f} = \frac{1}{|\Omega|} \int_\Omega f(x) dx = \int_\Omega f(x) dx.$$

Remark 2.5 *Theorem 2.3 shows that any possible vacuum states must vanish in finite time.*

Finally, for any global entropy weak solution (ρ, u) to the IBVP (2.4) in the sense of Definition 2.3, the density is continuous, i.e., $\rho \in C(\overline{\Omega} \times [0,T])$ for any $T > 0$, due to (2.6) and (2.7). Thus, the continuity of ρ and Theorem 2.3 imply that if the density contains vacuum states at least at one point, then there exists some critical time $T_1 \in [0, T_0)$ with $T_0 > 0$ given by (2.14) and a nonempty subset $\Omega^0 \subset \bar{\Omega}$ such that

$$\begin{cases} \rho(x,T_1) = 0, & \forall\, x \in \Omega^0 \\ \rho(x,T_1) > 0, & \forall\, x \in \bar{\Omega} \setminus \Omega^0, \\ \rho(x,t) > 0, & \forall\, (x,t) \in \bar{\Omega} \times (T_1, T_0]. \end{cases} \tag{2.16}$$

It follows from (2.15) easily that for any $\delta > 0$, it holds

$$\int_{T_1+\delta}^{T_0} \|u_x\|_{L^\infty} ds < \infty.$$

Under the condition that vacuum states appear, we shall prove that the spatial derivative of velocity (if regular enough and definable) blows up in finite time as the vacuum states vanish, even if the solution is regular enough for short time so that the velocity field and its derivatives are bounded as shown by Theorem 2.2.

Theorem 2.4 **(Finite time blow-up)** *Let (ρ, u) be any global entropy weak solution, which contains vacuum states at least at one point for some finite time, to the IBVP* (2.4) *in the sense of Definition 2.3. Let $T_0 > 0$ and $T_1 \in [0, T_0)$ be the time such that* (2.14) *and* (2.16) *holds respectively.*

Then, the solution (ρ, u) blows up as vacuum states vanish. Namely, for T_1 satisfying (2.16) *and for given any fixed $\eta > 0$, it holds*

$$\lim_{t \to T_1^+} \int_t^{T_1+\eta} \|u_x\|_{L^\infty} ds = \infty. \tag{2.17}$$

On the other hand, if there exists some $T_2 \in (0, T_0)$ such that the weak solution (ρ, u) satisfies

$$\|u\|_{L^1(0,T_2;W^{1,\infty}(\Omega))} < \infty,$$

then, there is a time $T_3 \in [T_2, T_0)$ so that the blowup phenomena happens for (ρ, u), i.e.,

$$\lim_{t \to T_3^-} \int_0^t \|u_x\|_{L^\infty} ds = \infty.$$

Remark 2.6 *Theorem 2.4 implies that for any global entropy weak solution (ρ, u) to the IBVP* (2.4) *in the sense of Definition 2.3, which contains vacuum states at least at one point initially, the finite time blowup phenomena* (2.17) *happens for such solution (ρ, u).*

Remark 2.7 *Theorems 2.1–2.4 provide a complete dynamical description on the vanishing of vacuum states and blow-up phenomena for the global entropy weak solutions to the compressible Navier-Stokes equations with density-dependent viscosity. That is, a global entropy weak solution exists for general large initial data with finite entropy. For short time, such weak solution is unique and regular with well-defined velocity field subject to additional initial regularity, and any existing vacuum state is maintained with the same interface structure as initial. Then, within finite time the vacuum states vanish definitely and the velocity blows up (even if it is regular enough and definable along the interfaces). After the vanishing of vacuum states, the global entropy weak solution becomes a strong one and tends to the non-vacuum equilibrium state exponentially in time. This dynamical phenomena is quite similar to those well-known for the 3-D incompressible Navier-Stokes equations. However, before the time of vacuum-vanishing, the uniqueness of the global entropy weak solution to the compressible Navier-Stokes equations with density-dependent viscosity subject to the initial data is not known yet.*

3 Outline of proof of Theorem 2.3

Here, we only prove (2.14).

We will employ an idea due to [7,10,11]) to obtain (2.14). Let $T \in (0,\infty)$ be fixed and C denote some generic positive constant independent of T. First, it is noted that the total mass is conserved, i.e., for any $t \in (0,T]$

$$\int_\Omega \rho(x,t)dx = \int_\Omega \rho_0(x)dx.$$

The entropy inequality (2.9) leads to

$$\sup_{0\le t\le T}\left(\|\rho\|_{L^\infty} + \|\rho_x\|_{L^2}\right) + \int_0^T \|\rho_x\|_{L^2}^2\, dt \le C. \tag{3.1}$$

It will be shown below that

$$g(t) \triangleq \|(\rho-\overline{\rho_0})(\cdot,t)\|_{L^4(\Omega)}^4 \to 0 \text{ as } t\to\infty. \tag{3.2}$$

Now, we assume that (3.2) holds, and continue the proof of (2.14). In fact, the inequality (3.1) and the Poincáre-Sobolev inequality imply that

$$\begin{aligned}&\|(\rho-\overline{\rho_0})(\cdot,t)\|_{C(\overline{\Omega})}\\ &\quad\le C\,\|(\rho-\overline{\rho_0})(\cdot,t)\|_{L^4(\Omega)}^{2/3}\,\|\rho_x(\cdot,t)\|_{L^2}^{1/3} \to 0, \text{ as } t\to\infty.\end{aligned}$$

This finishes the proof of (2.14). It remains to prove (3.2). First, it follows directly from (3.1) and the Poincáre-Sobolev inequality that

$$\begin{aligned}\int_0^T g(t)dt &\le C\sup_{0\le t\le T}\|\rho-\overline{\rho_0}\|_{L^\infty}^2\int_0^T \|\rho_x\|_{L^2}^2 dt\\ &\le C.\end{aligned}$$

Note that (2.7) implies that

$$\begin{aligned}
\int_0^T |g'(t)|dt &= 4\int_0^T \left|\left\langle (\rho-\overline{\rho_0})^3, \rho_t\right\rangle_{H^1\times H^{-1}}\right| dt \\
&= 12\int_0^T \left|\int_\Omega (\rho-\overline{\rho_0})^2 \rho_x\sqrt{\rho}\sqrt{\rho}u dx\right| dt \\
&\leq C\int_0^T \|\rho_x\|_{L^2}^2 dt \\
&\leq C.
\end{aligned}$$

The above two estimates easily yields (3.2). □

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