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Sharp asymptotics for the generalized Burgers equations

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1 Introduction

This note is concerned with large time behavior of the global solutions to the generalized Burgers equations:

$$u_t + (f(u))_x = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

where $u_0 \in L^1(\mathbb{R})$ and $f(u) = \frac{b}{2}u^2 + \frac{c}{3}u^3$ with $b \neq 0$, $c \in \mathbb{R}$. The subscripts t and x stand for the partial derivatives with respect to t and x, respectively. In Kawashima [7] and Nishida [11], it was shown that the solution of (1.1) and (1.2) tends to a nonlinear diffusion wave defined by

$$\chi(x,t) \equiv \frac{1}{\sqrt{1+t}}\chi_*\left(\frac{x}{\sqrt{1+t}}\right), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.3}$$

where

$$\chi_*(x) \equiv \frac{1}{b}\frac{(e^{b\delta/2}-1)e^{-\frac{x^2}{4}}}{\sqrt{\pi} + (e^{b\delta/2}-1)\int_{x/2}^{\infty} e^{-y^2}dy}, \tag{1.4}$$

$$\delta \equiv \int_{\mathbb{R}} u_0(x)dx. \tag{1.5}$$

By the Hopf -Cole transformation in Hopf [4] and Cole [1], we see that it is a solution of the Burgers equation

$$\chi_t + \left(\frac{b}{2}\chi^2\right)_x = \chi_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.6}$$

satisfying

$$\int_{\mathbb{R}} \chi(x,0)dx = \delta. \tag{1.7}$$

More precisely, if $u_0 \in L^1_\beta(\mathbb{R}) \cap H^1(\mathbb{R})$ for some $\beta \in (0,1/2)$ and $\|u_0\|_{H^1} + \|u_0\|_{L^1}$ is small, then we have

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \le C(1+t)^{-3/4+\alpha}(\|u_0\|_{H^1} + \|u_0\|_{L^1_\beta}), \quad t \ge 0, \tag{1.8}$$

where $\alpha = (1/2-\beta)/2$. Here, $H^1(\mathbb{R})$ denotes the space of functions $u = u(x)$ such that $\partial_x^l u$ are L^2-functions on $\mathbb{R}$ for $l = 0,1$, endowed with the norm $\|\cdot\|_{H^k}$, while $L^1_\beta(\mathbb{R})$ is a subset of $L^1(\mathbb{R})$ whose elements satisfy $\|u\|_{L^1_\beta} \equiv \int_{\mathbb{R}} |u|(1+|x|)^\beta dx < \infty$. They deal with the hyperbolic - parabolic system of conservation laws. If we consider the single equation, then we easily modify the estimate of (1.8). For $\beta \in (0,1)$ and $\|u_0\|_{H^1} + \|u_0\|_{L^1}$ is small, then we have

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1+\alpha}(\|u_0\|_{H^1} + \|u_0\|_{L^1_\beta}), \quad t \ge 0, \tag{1.9}$$

where $\alpha = (1-\beta)/2$. However, the estimate (1.9) leads to a natural question whether it is possible to take $\alpha = 0$ in (1.9) for the extreme case $\beta = 1$ or not. In [9], it was shown that we can't take $\alpha = 0$ in (1.9), unless $\delta = 0$ or $c = 0$. Indeed, the second asymptotic profile of large time behavior of the solutions is given by

$$V(x,t) \equiv -\frac{cd}{6\sqrt{\pi}} V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\log(2+t), \quad t \ge 0, \quad x \in \mathbb{R}, \tag{1.10}$$

where

$$V_*(x) \equiv \partial_x(e^{-\frac{x^2}{4}}\eta_*)(x), \tag{1.11}$$

$$\eta_*(x) \equiv \exp\left(\frac{b}{2}\int_{-\infty}^x \chi_*(y)dy\right), \tag{1.12}$$

$$d \equiv \int_{\mathbb{R}} \eta_*^{-1}(y)\chi_*^3(y)dy. \tag{1.13}$$

The aim of this note is to strengthen the result of [9]in the following two points. One is to show that $V(x,t)$ is the second asymptotic profile not only in the sense of L^∞ but also in the sense of L^p($1 \le p \le \infty$). The other is to show that we can take the initial data from rather wider class $L^1_1(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$. Here we denoted by $\mathcal{B}(\mathbb{R})$ the Banach space of all bounded and uniformly continuous functions on $\mathbb{R}^N$ with the usual supremum norm. And we set $E_\beta \equiv \|u_0\|_{L^\infty} + \|u_0\|_{L^1_\beta}$. Then we have the following result.

Theorem 1.1. *Assume that $u_0 \in L^1(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$ and E_0 is small. Then the initial value problem for* (1.1) *and* (1.2) *has a unique global solution $u(x,t)$ satisfying $u \in$*

$C^0([0,\infty);L^1)\cap C^0([0,\infty);\mathcal{B})$. *Moreover, if* $u_0\in L^1_1(\mathbb{R})\cap\mathcal{B}(\mathbb{R})$ *and* E_1 *is small, then the solution satisfies the estimate*

$$(1.14)\|u(\cdot,t)-\chi(\cdot,t)-V(\cdot,t)\|_{L^p}\le CE_1(1+t)^{-1+1/(2p)},\quad t\ge 0,\quad 1\le p\le\infty.$$

Here $\chi(x,t)$ *is defined by* (1.3), *while* $V(x,t)$ *is defined by* (1.10).

REMARK 1.2. In Liu [8], the initial value problem for the Burgers equations (1.1) and (1.2) is studied, provided $c=0$ implicity at page 42. After the proof of Theorem 2.2.1, it is mentioned, without proof, that if we assume $(1+|x|)^2|u_0(x)|\le\tilde{\delta}$ and $\tilde{\delta}$ is small, then the estimate

$$\|u(\cdot,t)-\chi(\cdot,t)\|_{L^\infty}\le C\tilde{\delta}(1+t)^{-1},\quad t\ge 1$$

holds. However, from our result, the above estimate fails true for the case $c\delta\ne 0$. In Matsumura and Nishihara [10], it was shown that, for some initial data, the estimate

$$\|u(\cdot,t)-\chi(\cdot,t)\|_{L^\infty}\le C(1+t)^{-1}\log(2+t),\quad t\ge 0 \tag{1.15}$$

holds instead of (1.9).

We also remark that the estimate similar to (1.14) was obtained for other types of Burgers equation such as KdV-Burgers in Hayashi and Naumkin [3] and Kaikina and Ruiz-Paredes [5], and Benjamin-Bona-Mahony-Burgers in Hayashi, Kaikina and Naumkin [2].

2 Preliminaries

In order to prove the basic estimates given by Lemma 3.2, Lemma 3.3 and Lemma 3.5, we prepare the following two lemmas. The first one is concerned with the decay estimates for semigroup $e^{t\Delta}$ associated with the heat equation.

Lemma 2.1. *Let* l *be a nonnegative integer and* $1\le q\le p\le\infty$. *Suppose* $q_0\in L^q(\mathbb{R})$. *Then the estimate*

$$\|\partial_x^l e^{t\Delta}q_0\|_{L^p}\le Ct^{-(1/q-1/p+l)/2}\|q_0\|_{L^q},\quad t\ge 0 \tag{2.1}$$

holds.

The second one is related to the diffusion wave $\chi(x,t)$ and the heat kernel $G(x,t)$. The explicit formula of $\chi(x,t)$ and $G(x,t)$ are given by (1.3) and

$$G(x,t)\ =\ \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}},\quad t>0,\quad x\in\mathbb{R}, \tag{2.2}$$

respectively. It is easy to see that

$$|\chi(x,t)| \le C|\delta|(1+t)^{-\frac{1}{2}}e^{-\frac{x^2}{4(1+t)}}, \quad t \ge 0, \quad x \in \mathbb{R}. \tag{2.3}$$

Moreover, we get the following (see e.g. [10]).

Lemma 2.2. *Let α and β be positive integers. Then, for $p \in [1,\infty]$, the estimates*

$$\|\partial_x^\alpha \partial_t^\beta \chi(\cdot,t)\|_{L^p} \le C|\delta|(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{\alpha}{2}-\beta}, \quad t \ge 0, \tag{2.4}$$

$$\|\partial_x^\alpha \partial_t^\beta G(\cdot,t)\|_{L^p} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{\alpha}{2}-\beta}, \quad t > 0 \tag{2.5}$$

hold.

For the latter sake, we introduce η

$$\eta(x,t) \equiv \eta_*\left(\frac{x}{\sqrt{1+t}}\right) = \exp\left(\frac{b}{2}\int_{-\infty}^{x} \chi(y,t)dy\right). \tag{2.6}$$

We easily have

$$\min\{1, e^{\frac{b\delta}{2}}\} \le \eta(x,t) \le \max\{1, e^{\frac{b\delta}{2}}\}. \tag{2.7}$$

Moreover, we can deduce the following. For the proof, see [9].

Corollary 2.3. *Let l be a positive integer and $1 \le p \le \infty$. If $|\delta| \le 1$, then we have*

$$\|\partial_x^l \eta(\cdot,t)\|_{L^p} \le C|\delta|(1+t)^{-(l-1/p)/2}. \tag{2.8}$$

3 Basic estimates

We deal with the following linearized equations which coresponds to (4.4), (4.5) below:

$$z_t = z_{xx} - (b\chi z)_x, \quad t > 0, \quad x \in \mathbb{R}, \tag{3.1}$$

$$z(x,0) = z_0(x). \tag{3.2}$$

The explicit representation formula (3.4) below plays a crucial role in our analysis.

Lemma 3.1. *If we set*

$$U[w](x,t,\tau) = \int_{\mathbb{R}} \partial_x(G(x-y,t-\tau)\eta(x,t))\eta^{-1}(y,\tau)\int_{-\infty}^{y} w(\xi)d\xi dy, \quad 0 \le \tau < t, \quad x \in \mathbb{R}, \tag{3.3}$$

then the solutions for (3.1) and (3.2) is given by

$$z(x,t) = U[z_0](x,t,0), \quad t > 0, \quad x \in \mathbb{R}. \tag{3.4}$$

PROOF. If we put

$$r(x,t) = \int_{-\infty}^{x} z(y,t)dy, \tag{3.5}$$

then we see from (3.1), (3.2) that $r(x,t)$ satisfies

$$r_t = r_{xx} - b\chi r_x, \quad t > 0, \quad x \in \mathbb{R}, \tag{3.6}$$

$$r(x,0) = \int_{-\infty}^{x} z_0(y)dy. \tag{3.7}$$

Then a direct computation yields

$$\left(\frac{r(x,t)}{\eta(x,t)}\right)_t = \left(\frac{r(x,t)}{\eta(x,t)}\right)_{xx}, \tag{3.8}$$

where η is defined by (2.6). Therefore, we have

$$r(x,t) = \eta(x,t) \int_{\mathbb{R}} G(x-y,t)\eta(y,0)^{-1}r(y,0)dy. \tag{3.9}$$

Hence (3.9), (3.5) and (3.7) yield (3.4). □

Next we derive the decay estimates (3.10) and (3.11) below for the homogenous equation (3.1). For the proof of Lemma 3.2, see [9].

Lemma 3.2. *Let $\beta \in [0,1]$ and $1 \le p \le \infty$. Assume that $z_0 \in L^1_\beta(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then, the estimate*

$$\|U[z_0](\cdot,t,0)\|_{L^p} \le Ct^{-(1-\frac{1}{p}+\beta)/2}\|z_0\|_{L^1_\beta}, \quad t > 0 \tag{3.10}$$

holds.

We modify the L^2 estimate of [9] to the following.

Lemma 3.3. *Let $1 \le p \le \infty$. Assume that $z_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then the estimate*

$$\|U[z_0](\cdot,t,0)\|_{L^p} \le C(1+t)^{-(1-1/p)/2}(\|z_0\|_{L^1} + \|z_0\|_{L^p}), \quad t > 0 \tag{3.11}$$

holds.

PROOF. We have from (3.3)

$$\begin{aligned} U[z_0](x,t,0) &= \int_{\mathbb{R}} \partial_x(G(x-y,t)\eta(x,t))\eta^{-1}(y,0) \int_{-\infty}^{y} z_0(\xi)d\xi dy \\ &= \partial_x\eta(x,t) \int_{\mathbb{R}} G(x-y,t)\eta^{-1}(y,0) \int_{-\infty}^{y} z_0(\xi)d\xi dy \\ &\quad + \eta(x,t) \int_{\mathbb{R}} G(x-y,t)\partial_y J(y)dy, \end{aligned} \tag{3.12}$$

where we put

$$J(y) = \eta^{-1}(y,0)\int_{-\infty}^{y} z_0(\xi)d\xi. \tag{3.13}$$

Therefore we have from (2.5) and Corollary 2.3

$$\begin{aligned} \|U[z_0](\cdot,t,0)\|_{L^p} &\le C\|\partial_x\eta(\cdot,t)\|_{L^p}\|G(\cdot,t)\|_{L^1}\|z_0\|_{L1} \\ &\quad +C\|\eta(\cdot,t)\|_{L^\infty}\|e^{t\Delta}[\partial_x J]\|_{L^p} \\ &\le C(1+t)^{-(1-1/p)/2}\|z_0\|_{L1} \\ &\quad +C\|e^{t\Delta}[\partial_x J]\|_{L^p}. \end{aligned} \tag{3.14}$$

From (3.13), (2.4) and Corollary 2.3, we have

$$\begin{aligned} \|\partial_x J\|_{L^1} &\le C\left\|\chi(x,0)\int_{-\infty}^{x} z_0(\xi)d\xi\right\|_{L^1(\mathbb{R}_\infty)} + C\|z_0\|_{L^1} \\ &\le C\|z_0\|_{L^1}, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \|\partial_x J\|_{L^p} &\le C\sum_{n=0}^{1}\left\|\partial_x^{1-n}\eta^{-1}(x,0)\partial_x^n\int_{-\infty}^{x} z_0(\xi)d\xi\right\|_{L^p(\mathbb{R}_\infty)} \\ &\le C\|\partial_x\eta^{-1}(\cdot,0)\|_{L^p}\|z_0\|_{L^1} + C\|\eta^{-1}(\cdot,0)\|_{L^\infty}\|z_0\|_{L^p} \\ &\le C(\|z_0\|_{L^1} + \|z_0\|_{L^p}). \end{aligned} \tag{3.16}$$

Hence, from (3.15), (3.16) and Lemma 2.1, we have

$$\begin{aligned} \|e^{t\Delta}[\partial_x J]\|_{L^p} &\le C(1+t)^{-(1-1/p)/2}(\|\partial_x J\|_{L^1} + \|\partial_x J\|_{L^p}) \\ &\le C(1+t)^{-(1-1/p)/2}(\|z_0\|_{L^1} + \|z_0\|_{L^p}). \end{aligned} \tag{3.17}$$

Therefore by (3.14) and (3.17), we obtain (3.11). This completes the proof. □

From Lemma 3.2 and Lemma 3.3, we get the following uniform estimate.

Corollary 3.4. *Let* $1 \le p \le \infty$. *Assume that* $z_0 \in L^1_1(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$ *and* $\int_{\mathbb{R}} z_0(x)dx = 0$. *Then the estimate*

$$\|U[z_0](\cdot,t,0)\|_{L^p} \le CE_1(1+t)^{-(1-1/p)/2-1/2}, \quad t>0$$

holds.

By using Lemma 2.1 and Corollary 2.3, we derive the decay estimate (3.18) below for the imhomogenous equation (4.4) below. The estimate will be used to get the decay rate of the solution $w(x,t)$ for the problem (4.4) and (4.5).

Lemma 3.5. *Let* $1 \le p \le \infty$. *Suppose* $w, \partial_x w \in C^0(0,\infty;L^1) \cap C^0(0,\infty;\mathcal{B})$. *Then the estimate*

$$\left\| \int_0^t U[\partial_x w(\tau)](\cdot,t,\tau)d\tau \right\|_{L^p}$$

$$\le C\int_0^{t/2} (t-\tau)^{-(1-1/2p)} \|w(\cdot,\tau)\|_{L^1} d\tau$$

$$+C\int_{t/2}^t (t-\tau)^{-1/2} \|w(\cdot,\tau)\|_{L^p} d\tau \tag{3.18}$$

holds.

4 Proof of Theorem 1

In order to prove our result, we introduce the following auxiliary problem:

$$v_t = v_{xx} - (b\chi v)_x - (\frac{c}{3}\chi^3)_x, \quad t>0, \quad x \in \mathbb{R}, \tag{4.1}$$

$$v(x,0) = 0. \tag{4.2}$$

By using Lemma 3.5 and Lemma 2.2, we derive the decay estimate for the solution $v(x,t)$ to the above problem.

Lemma 4.1. *Let* $1 \le p \le \infty$. *Then we have*

$$\|v(\cdot,t)\|_{L^p} \le C|\delta|^3(1+t)^{-1+1/(2p)} \log(2+t), \quad t \ge 0. \tag{4.3}$$

Our first step to prove Theorem 1.1 is the following.

Proposition 4.2. *Let* $1 \le p \le \infty$. *Assume that* $u_0 \in L^1(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$ *and* E_0 *is small. Then the initial value problem for* (1.1) *and* (1.2) *has a unique global solution* $u(x,t)$ *satisfying* $u \in C^0([0,\infty);L^1) \cap C^0([0,\infty);L^p)$. *Moreover, if* $u_0 \in L^1_1(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$ *and* E_1 *is small, then the estimate*

$$\|u(\cdot,t) - \chi(\cdot,t) - v(\cdot,t)\|_{L^p} \le CE_1(1+t)^{-1+1/(2p)}, \quad t \ge 0,$$

holds. Here $\chi(x,t)$ *is defined by* (1.3), *while* $v(x,t)$ *is the solution for the problem* (4.1) *and* (4.2).

PROOF. We shall prove only the decay estimate. We put

$$w(x,t) = u(x,t) - \chi(x,t) - v(x,t).$$

Then $w(x,t)$ satisfies

$$w_t = w_{xx} - (b\chi w)_x + (g(w,\chi,v))_x, \quad t>0, \quad x\in\mathbb{R}, \tag{4.4}$$

$$w(x,0) = w_0(x), \tag{4.5}$$

where we have set $w_0(x) = u_0(x) - \chi(x,0)$ and

$$\begin{aligned} g(w,\chi,v) &= -\frac{b}{2}(w+v)^2 \\ &\quad -\frac{c}{3}[w^3+v^3+3(w+v)(w+\chi)(\chi+v)]. \end{aligned} \tag{4.6}$$

Since $u_0(x)$, $\chi(x,0) \in L^1_1(\mathbb{R})\cap\mathcal{B}(\mathbb{R})$, we have $w_0(x)\in L^1_1\cap\mathcal{B}$. Bisides by (1.5) and (1.7),

$$\int_{\mathbb{R}} w_0(x)dx = 0. \tag{4.7}$$

Now, we define $N(T)$ by

$$N(T) = \sup_{0\le t\le T}\left\{(1+t)^{1/2}\|w(\cdot,t)\|_{L^1} + (1+t)^1\|w(\cdot,t)\|_{L^\infty}\right\}. \tag{4.8}$$

First of all, we shall show that

$$\|g(\cdot,t)\|_{L^p} \le C(1+t)^{-2+1/(2p)}((|\delta|\log(2+t))^2 + N(T)^2). \tag{4.9}$$

Here and below, $|\delta|$ and $N(T)$ is assumed to be small. We put $h_1(x,t) = w(x,t)+v(x,t)$, $h_2(x,t) = w(x,t)+\chi(x,t)$ and $h_3(x,t) = \chi(x,t)+v(x,t)$. Then, we have from (4.8), (4.3) and (2.4)

$$\|(w+v)^2(\cdot,t)\|_{L^p} \le C(1+t)^{-2+1/(2p)}((|\delta|\log(2+t))^2 + N(T)^2), \tag{4.10}$$

$$\|w^3(\cdot,t)\|_{L^p} \le C(1+t)^{-3+1/(2p)}N(T)^3, \tag{4.11}$$

$$\|v^3(\cdot,t)\|_{L^p} \le C(1+t)^{-2+1/(2p)}(|\delta|\log(2+t))^2, \tag{4.12}$$

$$\|(h_1h_2h_3)(\cdot,t)\|_{L^p} \le C(1+t)^{-2+1/(2p)}((|\delta|\log(2+t))^2 + N(T)^2). \tag{4.13}$$

Suming up these estimates, we obtain (4.9) from(4.6).

Applying the Duhamel principle for the problem (4.4) and (4.5), we have

$$w(x,t) = U[w_0](x,t,0) + \int_0^t U[\partial_x g(w,\chi,v)(\tau)](x,t,\tau)d\tau, \quad t>0, \quad x\in\mathbb{R}. \tag{4.14}$$

We have from (4.14), Corollary 3.4 and Lemma 3.5

$$\begin{aligned} \|w(\cdot,t)\|_{L^p} &\le C(1+t)^{-1+1/(2p)}E_1 + C\int_0^{t/2}(t-\tau)^{-1+1/(2p)}\|g(\cdot,\tau)\|_{L^1}d\tau \\ &\quad + C\int_{t/2}^t (t-\tau)^{-1/2}\|g(\cdot,\tau)\|_{L^p}d\tau \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{4.15}$$

First we evaluate I_2. From (4.9), we have

$$\begin{aligned} I_2 &\le C\int_0^{t/2}(t-\tau)^{-1+1/(2p)}(1+\tau)^{-\frac{3}{2}}\left((|\delta|\log(2+\tau))^2+N(T)^2\right)d\tau \\ &\le C(1+t)^{-1+1/(2p)}\left(|\delta|^2+N(T)^2\right). \end{aligned} \tag{4.16}$$

Next we evaluate I_3. From (4.9), we have

$$\begin{aligned} I_3 &\le C\int_{t/2}^{t}(t-\tau)^{-1/2}(1+\tau)^{-2+1/(2p)}\left((|\delta|\log(2+\tau))^2+N(T)^2\right)d\tau \\ &\le C(1+t)^{-(3-1/p)/2}\left((|\delta|\log(2+t))^2+N(T)^2\right). \end{aligned} \tag{4.17}$$

Since $|\delta|\le E_1$, if E_1 is small, then we obtain the inequality

$$(1+t)^{1-1/(2p)}\|\partial_x^l w(\cdot,t)\|_{L^p}\le C(E_1+N(T)^2). \tag{4.18}$$

Therefore, (4.18) gives the desired estimate $N(T)\le CE_1$. This completes the proof. □

To complete the proof of Theorem 1.1, it is sufficent to show Proposition 4.3 below by virtue of Proposition 4.2. Although the similer estimate was shown by Lemma 3 in [5], but we need to modify the proof of it, in order to avoid the logarithmic term in the right-hand side.

Proposition 4.3. *Assume that* $|\delta|\le 1$. *Then the estimate*

$$\|v(\cdot,t)-V(\cdot,t)\|_{L^p}\le C|\delta|^3(1+t)^{-1+1/(2p)},\quad t\ge 1 \tag{4.19}$$

holds. Here, $v(x,t)$ *is the solution for the problem* (4.1) *and* (4.2), *while* $V(x,t)$ *is defined by* (1.10).

PROOF. By the Duhamel principle, we have

$$\begin{aligned} v(x,t) &= -\frac{c}{3}\int_0^t U[\partial_x\chi^3(\tau)](x,t,\tau)d\tau \\ &= -\frac{c}{3}\int_{t/2}^t\int_{\mathbf{R}}\partial_x(G(x-y,t-\tau)\eta_1(x,t))\eta_2(y,\tau)\chi^3(y,\tau)dyd\tau \\ &\quad -\frac{c}{3}\int_0^{t/2}\int_{\mathbf{R}}\partial_x(G(x-y,t-\tau)\eta_1(x,t))\eta_2(y,\tau)\chi^3(y,\tau)dyd\tau \\ &\equiv I_1+I_2. \end{aligned} \tag{4.20}$$

First we evaluate I_1. By the integration by parts with respects to y, we have

$$\begin{aligned} I_1 &= -\frac{c}{3}\eta_1(x,t)\int_{t/2}^t\int_{\mathbf{R}}\left(\partial_x G(x-y,t-\tau)+\frac{b}{2}\chi(x,t)G(x-y,t-\tau)\right) \\ &\quad \times\eta_2(y,\tau)\chi^3(y,\tau)dyd\tau \\ &= -\frac{c}{3}\eta_1(x,t)\int_{t/2}^t\int_{\mathbf{R}}G(x-y,t-\tau)\Big(\partial_y(\eta_2(y,\tau)\chi^3(y,\tau)) \\ &\quad +\frac{b}{2}\eta_2(y,\tau)\chi(x,t)\chi^3(y,\tau)\Big)dyd\tau. \end{aligned}$$

Therefore, we get from Lemma 2.2 and (2.7)

$$\begin{aligned}\|I_1(\cdot,t)\|_{L^p} &\le C\int_{t/2}^{t}\|G(\cdot,t-\tau)\|_{L^1}\Big\{\|\chi^4(\cdot,\tau)\|_{L^p}+\|\chi(\cdot,\tau)\|_{L^\infty}^2\|\partial_x\chi(\cdot,\tau)\|_{L^p}\\ &\quad +\|\chi(\cdot,t)\|_{L^\infty}\|\chi^3(\cdot,\tau)\|_{L^p}\Big\}d\tau\\ &\le C|\delta|^3\int_{t/2}^{t}(1+\tau)^{-2+1/(2p)}d\tau\\ &\le C|\delta|^3(1+t)^{-1+1/(2p)}. \qquad (4.21)\end{aligned}$$

Next we evaluate I_2. If we put

$$\Lambda(x,t,y,\tau)\equiv-\frac{c}{3}\partial_x(G(x-y,t-\tau)\eta_1(x,t)), \qquad (4.22)$$

then we have

$$I_2=\int_0^{t/2}\int_{\mathbf{R}}\Lambda(x,t,y,\tau)\eta_2(y,\tau)\chi^3(y,\tau)dyd\tau. \qquad (4.23)$$

Spliting the y-integral at $y=0$ and making the integration by parts, we have

$$\begin{aligned}I_2 &= \int_0^{t/2}\int_0^{\infty}\partial_y\Lambda(x,t,y,\tau)\int_y^{\infty}\eta_2(\xi,\tau)\chi^3(\xi,\tau)d\xi dyd\tau\\ &\quad -\int_0^{t/2}\int_{-\infty}^{0}\partial_y\Lambda(x,t,y,\tau)\int_{-\infty}^{y}\eta_2(\xi,\tau)\chi^3(\xi,\tau)d\xi dyd\tau\\ &\quad +\int_0^{t/2}\Lambda(x,t,0,\tau)\int_{\mathbf{R}}\eta_2(\xi,\tau)\chi^3(\xi,\tau)d\xi d\tau\\ &\equiv I_3+I_4+I_5. \qquad (4.24)\end{aligned}$$

First we consider I_3. By using the Young inequality, (2.5) and (2.8), from (4.24), we have

$$\|I_3(\cdot,t)\|_{L^p} \le Ct^{-(3-1/p)/2}\int_0^{t/2}\int_0^{\infty}\int_y^{\infty}|\chi(\xi,\tau)|^3d\xi dyd\tau.$$

Then, by the integration by parts with respect to y, it follows from Lemma 2.2 and (2.3) that

$$\begin{aligned}\|I_3(\cdot,t)\|_{L^p} &\le Ct^{-(3-1/p)/2}\int_0^{t/2}\int_0^{\infty}y|\chi(y,\tau)|^3dyd\tau\\ &\le C|\delta|^3t^{-(3-1/p)/2}\int_0^{t/2}(1+\tau)^{-1}\int_{\mathbf{R}}\frac{|y|}{\sqrt{1+\tau}}e^{-\frac{y^2}{4(1+\tau)}}dyd\tau\\ &\le C|\delta|^3(1+t)^{-1+1/(2p)}. \qquad (4.25)\end{aligned}$$

Similarly, we have

$$\|I_4(\cdot,t)\|_{L^p} \le C|\delta|^3(1+t)^{-1+1/(2p)}. \tag{4.26}$$

Next we consider I_5. From (2.6), (1.3) and (1.13), we have

$$\int_{\mathbf{R}} \eta_2(\xi,\tau)\chi^3(\xi,\tau)d\xi = d(1+\tau)^{-1},$$

it follows from (4.22) and (4.24) that

$$\begin{aligned} I_5 &= d\int_0^{t/2} \Lambda(x,t,0,\tau)(1+\tau)^{-1}d\tau \\ &= -\frac{cd}{3}\eta_1(x,t)\int_0^{t/2}(1+\tau)^{-1}\Big((\partial_x G(x,t-\tau)-\partial_x G(x,t)) \\ &\quad +\frac{b}{2}\chi(x,t)(G(x,t-\tau)-G(x,t))\Big)d\tau \\ &\quad -\frac{cd}{3}\eta_1(x,t)\left(\partial_x G(x,t)+\frac{b}{2}\chi(x,t)G(x,t)\right)\log\left(\frac{2+t}{2}\right) \\ &\equiv I_{5,1}+I_{5,2}. \end{aligned} \tag{4.27}$$

In order to evaluate $I_{5,1}$, we shall use

$$\|\partial_x^l G(\cdot,t-\tau)-\partial_x^l G(\cdot,t)\|_{L^p} \le C(t-\tau)^{-(3-1/(2p)+l)/2}\tau \tag{4.28}$$

for $l=0,1$ and $0\le\tau\le t/2$. This estimate can be shown by observing that

$$\partial_x^l G(x,t-\tau)-\partial_x^l G(x,t) = -\tau\int_0^1(\partial_t\partial_x^l G)(x,t-\theta\tau)d\theta$$

and by recalling (2.5). Since $|d|\le C|\delta|^3$ by (1.13), we have from(4.28)

$$\begin{aligned} \|I_{5,1}(\cdot,t)\|_{L^p} &\le C|\delta|^3\int_0^{t/2}(1+\tau)^{-1}\{(t-\tau)^{-2+1/(2p)}\tau+(1+t)^{-\frac{1}{2}}(t-\tau)^{-(3-1/p)/2)}\tau\}d\tau \\ &\le C|\delta|^3\int_0^{t/2}(t-\tau)^{-2+1/(2p)}d\tau \le C|\delta|^3(1+t)^{-1+1/(2p)}. \end{aligned} \tag{4.29}$$

Finally, we evaluate $I_{5,2}$. From (4.27), (2.2), (1.3) and (2.6), it follows that

$$I_{5,2} = -\frac{cd}{12\sqrt{\pi}}\eta_*\left(\frac{x}{\sqrt{1+t}}\right)\left(b\frac{\sqrt{t}}{\sqrt{1+t}}\chi_*\left(\frac{x}{\sqrt{1+t}}\right)-\frac{x}{\sqrt{t}}\right)e^{-\frac{x^2}{4t}}t^{-1}(\log(t+2)-\log 2).$$

Since

$$\left|\frac{\sqrt{t}}{\sqrt{t+1}}-1\right| \le C(1+t)^{-1},\quad \left\|\eta_*\left(\frac{\cdot}{\sqrt{1+t}}\right)-\eta_*\left(\frac{\cdot}{\sqrt{t}}\right)\right\|_{L^\infty} \le C(1+t)^{-1}$$

and

$$\left\|\chi_*\left(\frac{\cdot}{\sqrt{1+t}}\right)-\chi_*\left(\frac{\cdot}{\sqrt{t}}\right)\right\|_{L^\infty}\le C(1+t)^{-1},$$

we have from (1.10), (1.11), (1.4) and (1.12),

$$\begin{aligned}\|I_{5,2}(\cdot,t)-V(\cdot,t)\|_{L^p} &\le C|\delta|^3t^{-1+1/(2p)}+C|\delta|^3\left|\frac{\sqrt{t}}{\sqrt{t+1}}-1\right|t^{-1+1/(2p)}\log(t+2)\\ &\quad +C|\delta|^3\left\|\eta_*\left(\frac{\cdot}{\sqrt{1+t}}\right)-\eta_*\left(\frac{\cdot}{\sqrt{t}}\right)\right\|_{L^\infty}t^{-1+1/(2p)}\log(2+t)\\ &\quad +C|\delta|^3\left\|\chi_*\left(\frac{\cdot}{\sqrt{1+t}}\right)-\chi_*\left(\frac{\cdot}{\sqrt{t}}\right)\right\|_{L^\infty}t^{-1+1/(2p)}\log(2+t)\\ &\le C|\delta|^3(1+t)^{-1+1/(2p)}. \end{aligned}\tag{4.30}$$

Summarizing (4.20), (4.21), (4.25), (4.26), (4.29) and (4.30), we obtain (4.19). This completes the proof. □

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