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Weighted Estimates for Exterior Nonstationary Navier-Stokes Flows

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1. Introduction

In an exterior domain $\Omega \subset \mathbb{R}^n$ $(n=3)$ with smooth boundary $\partial\Omega$, we study the space-time decay properties of solutions to the Navier–Stokes initial value problem

$$\begin{aligned} \partial_t u + u\cdot\nabla u = \Delta u - \nabla p \qquad & (x\in\Omega,\ t>0),\\ \nabla\cdot u = 0 \qquad & (x\in\Omega,\ t\geq 0),\\ u|_{\partial\Omega}=0, \quad u\to 0\ \ (|x|\to\infty) & \\ u|_{t=0}=a, & \end{aligned} \tag{1.1}$$

for unknown velocity $u = u(x,t) = (u_1,\dots,u_n)$, unknown pressure $p = p(x,t)$ and a prescribed initial velocity $a = a(x)$. The kinematic viscosity is normalized to be one.

There is an extensive literature dealing with decay properties of weak and strong solutions to (1.1) (see, e.g., [6], [7], [8], [17], [24], [25], [28], [29], [31], [32], [33], [37]). For weak solutions, L^2 decay properties have been studied and the algebraic decay rates, similar to those for solutions of the heat equation, are obtained. The results show that for each $a \in L^2_\sigma(\Omega)$, the space of the L^2 solenoidal vector fields, there is a weak solution u defined for all $t \geq 0$ such that $\|u(t)\|_2 \to 0$ as $t\to\infty$. Hereafter, $\|\cdot\|_r$ denotes the norm of $L^r(\Omega)$. If $a \in L^2_\sigma(\Omega)\cap L^r(\Omega)$ for some $1\leq r<2$, then the weak solution satisfies

$$\|u(t)\|_2 \leq c(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}. \tag{1.2}$$

See [6], [7] and [11]. For strong solutions, L^q-theory was developed by Iwashita [25] and Chen [11] for $n\geq 3$, and by Dan and Shibata [12] for

$n = 2$ (see also [1] and [16]). They proved the estimates

$$\|u_0(t)\|_q \le ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_p \qquad (1 < p \le q < \infty,\ 1 \le p < q \le \infty), \qquad (1.3)$$

$$\|\nabla u_0(t)\|_q \le ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_p \qquad (1 < p \le q \le n,\ 1 \le p < q \le n), \quad (1.4)$$

on solutions u_0 of the Stokes problem, i.e., the linearized version of (1.1). These estimates were applied by [8], [11] and [25] to extend results of Kato [26] for the Cauchy problem to the case of (1.1), and we know that if $n \ge 3$, if a is in the space $L^n_\sigma(\Omega)$ of L^n solenoidal vector fields and if $\|a\|_n$ is sufficiently small, then (1.1) admits a unique strong solution u defined for all $t \ge 0$. Moreover, if $a \in L^r(\Omega) \cap L^n_\sigma(\Omega)$ for some $1 < r \le n$, then

$$t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}u \in BC([0,\infty); L^q(\Omega)) \qquad (r \le q \le \infty), \qquad (1.5)$$

$$t^{\frac{1}{2}+\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\nabla u \in BC([0,\infty); L^q(\Omega)) \quad (r \le q \le n). \qquad (1.6)$$

In [18] we extended (1.5) and (1.6) to the case where $r = 1 < q$.

In this paper we first discuss estimates for the L^2-moments of weak solutions of the form:

$$\int_\Omega |x|^{2\alpha}|u(x,t)|^2dx + \int_0^t\int_\Omega |x|^{2\alpha}|\nabla u(x,\tau)|^2dxd\tau \le c. \qquad \text{(M)}$$

For the Cauchy problem, E. M. Schonbek and T. Schonbek [38] proved (M) with $\alpha = 3/2$ for smooth solutions on $\mathbb{R}^3$ (see also [15]). He and Xin [23] proved (M) for weak solutions, with $\alpha = 3/2$, assuming $a \in L^1(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$ and $|x|^{3/2}a \in L^2(\mathbb{R}^3)$. Bae and Jin [3] proved (M) for weak solutions, with $1 < \alpha < 5/2$, assuming $a \in L^2_\sigma(\mathbb{R}^3)$, $(1+|x|)a \in L^1(\mathbb{R}^3)$ and $|x|^\alpha a \in L^2(\mathbb{R}^3)$. Brandolese [9] found a local smooth solution $u \in C([0,T); \mathbb{Z}_\alpha)$, with some $T > 0$, assuming $a \in \mathbb{Z}_\alpha$ for $3/2 < \alpha < 9/2$ ($\alpha \ne 5/3, 7/2$). Here, $f \in \mathbb{Z}_\alpha$ means that

$$(1+|x|^2)^{\alpha-2}f \in L^2, \quad (1+|x|^2)^{\alpha-1}\nabla f \in L^2, \quad (1+|x|^2)^\alpha \Delta f \in L^2.$$

For problem (1.1), the corresponding results are still incomplete. Farwig and Sohr [14] found a class of weak solutions u with associated pressures p such that

$$|x|^\alpha \partial_t u, \quad |x|^\alpha \partial_x^2 u, \quad |x|^\alpha \nabla p \in L^s(0,\infty; L^q(\Omega)) \quad (n = 3),$$

for $1 < q < 3/2$ and $1 < s < 2$ with $3/q + 2/s - 4 \le \alpha < \min\{1/2, 3 - 3/q\}$. Farwig [13] then gave another class of weak solutions such that

$$\||x|^{\frac{\alpha}{2}}u(t)\|_2^2 + c_\alpha\int_s^t \||x|^{\frac{\alpha}{2}}\nabla u\|_2^2 d\tau \le \||x|^{\frac{\alpha}{2}}u(s)\|_2^2 \qquad (1 < \alpha < 1);$$

$$\||x|^{\frac{1}{2}}u(t)\|_2^2 + 2\int_s^t \||x|^{\frac{1}{2}}\nabla u\|_2^2 d\tau \le \||x|^{\frac{1}{2}}u(s)\|_2^2 + c_{a,\delta}|t-s|^\delta,$$

for $s = 0$, a.e. $s > 0$ and all $t \geq s$, where $\delta > 0$ is arbitrary. Recently, Bae and Jin have studied decay rates of L^2-moments. When $n = 2$, they prove in [4] that there is a weak solution u satisfying

$$\| |x|^\alpha u(t) \|_p = O(t^{-\frac{1}{2}+\frac{1}{p}+\frac{\alpha}{2}+\delta}) \quad \text{for large } t$$

for all $\delta > 0$ and $0 < \alpha \leq 1$, if $a \in L^r(\Omega) \cap L^2(\Omega)$ and $|x|a \in L^{\frac{2r}{2-r}}(\Omega)$ with $1 < r \leq 2p/(p+2) < 2 \leq p < \infty$. Moreover, in case $n = 3$ they prove in [5] that there is a weak solution such that

$$\| |x| u(t) \|_2 \leq c_\delta (1+t)^{\frac{5}{4}-\frac{3}{2r}+\delta}$$

for all $\delta > 0$, if $a \in L^r(\Omega) \cap L^2(\Omega)$ for some $1 < r < 6/5$, $|x|a \in L^{6/5}(\Omega)$ and $|x|^2 a \in L^2(\Omega)$.

This paper improves the above results on L^2-moments and gives weak solutions satisfying

$$\| |x|^\alpha u(t) \|_2^2 + \int_0^t \| |x|^\alpha \nabla u(\tau) \|_2^2 d\tau \leq c \qquad (1 < \alpha < n/2),$$

$$\| |x|^\beta u(t) \|_2 \leq c(1+t)^{-\frac{n(\alpha-\beta)}{4\alpha}} \qquad (0 \leq \beta \leq \alpha < n/2),$$

for all $t \geq 0$. The restriction $\alpha < n/2$ comes from our estimates on pressures. But, this condition on α is optimal in the following sense: in Theorem 2.5 (see section 2), we will show that strong solutions behave in general as $|u(x,t)| \approx |x|^{-n}$ for large $|x|$. So $|x|^\alpha u$ is in $L^2(\Omega)$ only when $\alpha < n/2$. In a special case, however, this restriction on α is relaxed. Indeed, we show that one can take $\alpha < 1 + n/2$ if the associated pressure p satisfies

$$G(t) = \int_{\partial\Omega} (y \partial_\nu p - p\nu)(y,t) dS_y = 0, \qquad t \in (0,\infty) \tag{1.7}$$

where ν is the unit outward normal to $\partial\Omega$.

We next disuss the behavior of weighted L^q-norms of strong solutions. For the Cauchy problem, the estimates $t^{\frac{\beta}{2}} \| |x|^\alpha u \|_q + t^{\frac{1+\beta}{2}} \| |x|^\alpha \nabla u \|_q \leq c$ are known to be valid if $\alpha \geq 0$, $\beta \geq 0$ and

$$\alpha + 2\beta = n - n/q \quad \text{or} \quad \alpha + 2\beta = n + 1 - n/q; \quad n < q \leq \infty. \tag{1.8}$$

See [2], [3] [15], [23], [35] and [36] for the details. See also [19] for solutions with some symmetries. The balance relation (1.8) agrees with that for solutions of the linear heat equation on $\mathbb{R}^n$.

On the other hand, for (1.1) with $n = 3$, He and Xin [22] gave strong solutions such that $\| |x|^\alpha u(t) \|_q \leq c$ for $\alpha = 3/7 - 3/q$, $7 < q \leq \infty$. Recently, Bae and Jin have adapted the ideas of [22] and proved

$$\| |x|^2 u(t) \|_p \leq c_\delta t^{1-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})+\delta} \quad \text{for large } t > 0,$$

with an arbitrary $\delta > 0$, assuming that $a \in L^r(\Omega) \cap L^3(\Omega)$ for some $1 < r < 6/5$, and

$$|x|a, \ |x|^2 a \in L^r(\Omega), \qquad |x|a \in L^{6/5}(\Omega), \qquad |x|^2 a \in L^2(\Omega).$$

However, these results are not optimal. In this paper we deduce the optimal decay rates in space and time and establish a balance relation between these two kinds of decays which is similar to that of solutions to the Cauchy problem.

It should be noticed that for (1.1), the spatial decay property of a solution is closely connected with the vanishing of the total net force exerted by the fluid to the body $\mathbb{R}^n \setminus \Omega$. Indeed, it is shown in [18] that the following three statements are equivalent:

(a) The total net force vanishes, i.e., we have

$$\mathcal{F}(t) = \int_{\partial\Omega} (T[u,p]\cdot\nu)(y,t)dS_y = 0, \tag{1.9}$$

where $T[u,p] = (T_{jk}[u,p])_{j,k=1}^n = (\partial_j u_k + \partial_k u_j - \delta_{jk} p)$ is the stress tensor.

(b) The solution u is in $C([0,T); L^1(\Omega))$.

(c) Assertion (1.7) holds, i.e., $G(t) = 0$.

In this paper we further show that if $|x|^{n(1-\frac{1}{r})} a \in L^r(\Omega)$ for some $1 \le r < \infty$, then in general we have $t^{\frac{n}{2}(1-\frac{1}{r})}|x|^{n(1-\frac{1}{r})}u \in L^\infty_{\rm loc}(0,\infty; L^r_w(\Omega))$, where L^r_w is the weak L^r-space; and that

$$\text{(1.9) holds if and only if } t^{\frac{n}{2}(1-\frac{1}{r})}|x|^{n(1-\frac{1}{r})}u \in L^\infty_{\rm loc}(0,\infty; L^r(\Omega)).$$

See Theorem 2.5 in section 2.

Finally, we give a class of initial data a such that the corresponding strong solutions satisfy $\mathcal{F} \neq 0$ (or, equivalently, $G \neq 0$). For such data, our moment estimates (Theorem 2.2) and the time-decay rates (Theorem 2.4) are optimal. But, we do not know if our class is vacuous or not.

Throughout the paper we assume $n = 3$; but we use the notation n to denote the space dimension. Indeed, our results on strong solutions are valid for all dimensions $n \ge 3$ and, moreover, our notation (of using n) would be convenient for the reader to understand the nature of assumptions in our main results (Theorems 2.1–2.6 below).

2. Notation and Main Results

We always assume that $n = 3$ and that the origin of $\mathbb{R}^n$ is in $\mathbb{R}^n \setminus \bar{\Omega}$. $L^q(\Omega)$, $1 \le q \le \infty$, denotes the Lebesgue space of real-valued functions as well as that of vector functions, with norm $\|\cdot\|_q$, and $C_{0,\sigma}^\infty(\Omega)$ the set of smooth solenoidal vector fields with compact support in Ω. $L_\sigma^q(\Omega)$, $1 < q < \infty$, is the closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm $\|\cdot\|_q$. Given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X)$, $1 \le p \le \infty$, the set of strongly measurable functions $f : (0,T) \to X$ such that $\int_0^T \|f(t)\|_X^p dt < \infty$ (obvious modification when $p = \infty$). $P : L^q(\Omega) \to L_\sigma^q(\Omega)$ is the bounded projection as defined in [33], and the Stokes operator $A = -P\Delta$ is the closed linear operator in in $L_\sigma^q(\Omega)$, with (dense) domain $D(A) = D(A_q) = H^{2,q}(\Omega) \cap H_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$. We know that $-A_q$ generates in $L_\sigma^q(\Omega)$ a bounded analytic semigroup $\{e^{-tA}\}_{t\ge 0}$. Using this we define $v \in D_q^{1-1/s,s}$ if and only if

$$\|v\|_{D_q^{1-1/s,s}} = \|v\|_q + \Big(\int_0^\infty \|t^{\frac{1}{s}} A e^{-tA} v\|_q^s dt/t\Big)^{\frac{1}{s}} < +\infty,$$

where $1 < s < \infty$. We need these spaces for specifying our initial data.

Definition 2.1. Let $a \in L_\sigma^2(\Omega)$. A vector function u on $\Omega \times [0,\infty)$ is called a *weak solution* to problem (1.1) if

1) $u \in L^\infty(0,T; L_\sigma^2(\Omega) \cap L^2(0,T; H_0^{1,2}(\Omega))$ for all $T > 0$.

2) For every $\phi \in C_0([0,\infty); H_0^{1,2}(\Omega)) \cap C_0^1([0,\infty); L_\sigma^2(\Omega))$, we have

$$\int_0^\infty \int_\Omega \big(- u \cdot \partial_\tau \phi + \nabla u \cdot \nabla \phi + (u \cdot \nabla) u \cdot \phi \big) dx d\tau = \int_\Omega \phi(x,0) \cdot a(x) dx.$$

3) u satisfies $\nabla \cdot u = 0$ in Ω in the sense of distributions.

Definition 2.2. Let $a \in L_\sigma^n(\Omega)$. A vector function u is called a *strong solution* to problem (1.1) if $u \in BC([0,\infty); L_\sigma^n(\Omega))$ and if 2) and 3) in Definition 2.1 hold for u.

Our main results are as follows. The first result deals with the existence and estimates of weak solutions in weighted L^2-spaces.

Theorem 2.1. *For each $a \in L_\sigma^2(\Omega)$, there exists a weak solution u such that*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u\|_2^2 d\tau \le \|u(s)\|_2^2 \tag{2.1}$$

for $s = 0$, a.e. $s > 0$, and all $t \ge s$. Moreover, if $a \in L^1(\Omega) \cap L_\sigma^2(\Omega) \cap D_p^{1-1/s,s}$, $n + 1 = 2/s + n/p$, and $6/5 \le p < n/(n-1)$ and if $|x|^\alpha a \in L^2(\Omega)$ for some

$1 < \alpha < n/2$, *the weak solution given above satisfies*

$$\| |x|^\alpha u(t)\|_2^2 + \int_0^t \| |x|^\alpha \nabla u\|_2^2 d\tau \le c, \qquad \| |x|^\beta u(t)\|_2 \le c(1+t)^{-\frac{n(\alpha-\beta)}{4\alpha}} \tag{2.2}$$

for $0 \le \beta \le \alpha$ *and* $t \ge 0$, *with* c *depending only on* α, $\|a\|_1$, $\|a\|_{D_q^{1-1/s,s}}$ *and* $\| |x|^\alpha a\|_2$.

As will be seen from the proof, the restriction $\alpha < n/2$ comes from our estimates on the pressures. But, condition $\alpha < n/2$ is optimal, as mentioned in Introduction, since our weak solutions behave like $|x|^{-n}$ as $|x| \to \infty$. On the other hand, if p satisfies $G = 0$, where G is the function defined in (1.7), then u will behave like $|x|^{-n-1}$. We now discuss the validity of this conjecture. However, it is now known that condition $G = 0$ is closely connected with some symmetry conditions on $\{u, p\}$; so we state our result in the following form.

Theorem 2.2. *Suppose* Ω *is invariant under the reflection* $x \mapsto -x$. *Let* $a \in L^1(\Omega) \cap L^2_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $n+1 = 2/s + n/p$, *and* $6/5 \le p < n/(n-1)$. *If* $a(-x) = -a(x)$ *and* $|x|^\alpha a \in L^2(\Omega)$ *for some* $1 < \alpha < 1 + n/2$, *then a weak solution* u *exists, satisfying* $G = 0$ *and*

$$\| |x|^\alpha u(t)\|_2^2 + \int_0^t \| |x|^\alpha \nabla u\|_2^2 d\tau \le c, \qquad \| |x|^\beta u(t)\|_2 \le c(1+t)^{-\frac{n(\alpha-\beta)}{4\alpha}} \tag{2.3}$$

for $0 \le \beta \le \alpha$.

As shown in [18], $G = 0$ is equivalent to (1.9). The result above is the same as those given in [3] and [9] for solutions to the Cauchy problem.

We next deal with strong solutions and prove the existence of those solutions which decay more rapidly than those treated, e.g., in [7], [8], [11] and [25].

Theorem 2.3. *Let* $a \in L^1(\Omega) \cap L^{\frac{n}{2}}_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $2/s + n/p = n+1$, *and* $6/5 \le p < n/(n-1)$. *There is a constant* $\lambda > 0$ *so that* $\|a\|_n \le \lambda$ *implies the existence of a strong solution* u *defined for all* $t \ge 0$ *such that*

$$\begin{aligned} &\|u\|_r \le ct^{-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad (1 \le \ell \le \min\{n, r\},\ 1 < r \le \infty) \\ &\|\nabla u\|_r \le ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \quad (1 \le \ell \le r \le n) \\ &\|\partial^2 u\|_r + \|\partial_t u\|_r + \|\nabla p\|_r \le ct^{-1-\frac{n}{2}(\frac{1}{\ell}-\frac{1}{r})} \\ &\qquad (1 \le \ell \le r \le n/2,\ r > 1) \end{aligned} \tag{2.4}$$

and

$$\|\nabla u(t)\|_{L^r(\Omega_\delta)} \le ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} + c_\delta t^{-\frac{n}{2}} \quad (n < r < \infty) \tag{2.5}$$

for all $t > 0$, where $\Omega_\delta = \{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) > \delta\}$, $\delta > 0$.

(2.4) is given in [18] (see Theorem 1), and (2.5) will be proved in section 5. The last term in (2.5) comes from a boundary integral in the representation formula of u, which does not appear in the case of the Cauchy problem.

The result below deals with the time-decay of weighted norms of strong solutions.

Theorem 2.4. *Let $a \in L^1(\Omega) \cap L^n_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $2/s + n/p = n+1$, and $6/5 \le p < n/(n-1)$. Suppose $|x|^\alpha a \in L^r(\Omega)$ with $\alpha = n(1-1/r)$ for some $1 \le r < \infty$. Then, there is a number $\lambda_1 > 0$ so that $\|a\|_n \le \lambda_1$ ensures the existence of a strong solution u satisfying*

$$\||x|^\alpha u(t)\|_q \le ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \quad (\max\{r, n/(n-1)\} < q \le \infty) \tag{2.6}$$

for all $t > 0$.

For the Cauchy problem, there are strong solutions u satisfying $t^\beta \||x|^\alpha u\|_q \le c$, $n < q \le \infty$, with $\alpha = n(1-1/r)$, $\beta = (n/2)(1/r - 1/q)$ and $1 < r \le q \le \infty$. See [23]. Our result above is similar to that of [23] and improves that of [5]. The relation between the space and time decays given above agrees with that of the Cauchy problem.

We finally discuss the relation between the decay properties of solutions u and the vanishing of the associated total net force, i.e., the validity of (1.9). Define $V(x,t) = (V_{jk}(x,t))$ by

$$V_{jk}(x,t) = E_t(x)\delta_{jk} + \partial_j\partial_k(\mathcal{N} * E_t)(x), \tag{2.7}$$

with $\mathcal{N} = c_n|x|^{2-n}$ is the Newtonian potential and $E_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Moreover, recall the function $\mathcal{F}(t) = (\mathcal{F}_j(t))_{j=1}^n$ defined in (1.9). We shall prove

Theorem 2.5. *Let $a \in L^1(\Omega) \cap L^n_\sigma(\Omega) \cap D_p^{1-1/s,s}$, $2/s + n/p = n+1$, and $6/5 \le p < n/(n-1)$. Suppose $|x|^\alpha a \in L^r(\Omega)$ with $\alpha = n(1-1/r)$ for some $1 \le r < \infty$. If $\|a\|_n \le \lambda_1$, the strong solution obtained in Theorem 2.4 satisfies*

$$\Big\||x|^\alpha\Big(u(t) - V(\cdot,t)\cdot\int_0^t \mathcal{F}\,d\tau\Big)\Big\|_r \le c(1 + t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \textit{for all } t > 0. \tag{2.8}$$

This implies that $t^{\frac{\alpha}{2}}|x|^\alpha u \in L^\infty_{\mathrm{loc}}(0,\infty; L^r_w(\Omega))$ and

$$t^{\frac{\alpha}{2}}|x|^\alpha u \in L^\infty_{\mathrm{loc}}(0,\infty; L^r(\Omega)) \quad \textit{if and only if } (1.9) \textit{ holds.}$$

To see that (2.8) is in general optimal, we need to construct a velocity field a for which the corresponding solution does not satisfy (1.9). To this end, the following result would be useful. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be the unit outward normal to $\partial\Omega$ and consider the functions h_k, $k = 1, 2, 3$, satisfying

$$\Delta h_k = 0, \qquad \partial h_k/\partial\nu|_{\partial\Omega} = -\nu_k, \qquad |h_k(x)| = O(|x|^{-1}) \quad (|x| \to \infty).$$

Now, we know ([30]) that if $a \in L^1(\Omega) \cap D(A_2)$, a (unique) strong solution u exists at least locally in time, satisfying $\|u(t) - a\|_{H^{2,2}(\Omega)} \to 0$ as $t \to 0$. In this situation we prove

Theorem 2.6. *A strong solution u and the associated pressure p satisfy* (1.9) *if and only if*

$$\int_{\partial\Omega} (\partial_\nu u_k + \partial_\nu u_i \cdot \partial_i h_k) dS_x + \int_\Omega (u_i u_j) \partial_{ij}^2 h_k dx = 0 \quad \textit{for all } k \in \{1, 2, 3\}.$$

Therefore, if $a \in L^1(\Omega) \cap D(A_2)$ satisfies

$$\int_{\partial\Omega} (\partial_\nu a_k + \partial_\nu a_i \cdot \partial_i h_k) dS_x + \int_\Omega (a_i a_j) \partial_{ij}^2 h_k dx \neq 0 \quad \textit{for some } k \in \{1, 2, 3\},$$

then the corresponding $\{u, p\}$ does not satisfy (1.9). *In particular, if $a \in C_{0,\sigma}^\infty(\Omega)$ and*

$$\int_\Omega (a_i a_j) \partial_{ij}^2 h_k dx \neq 0 \quad \textit{for some } k \in \{1, 2, 3\}, \tag{2.9}$$

then $\{u, p\}$ does not satisfy (1.9).

Let Ω be the exterior to the unit ball, and so $h = -c\nabla|x|^{-1}$. If

$$a(-x) = -a(x),$$

the corresponding $\{u, p\}$ satisfies

$$u(-x, t) = -u(x, t), \quad p(-x, t) = p(x, t).$$

Direct calculation then gives

$$\int_{\partial\Omega} (\partial_\nu u_k + \partial_\nu u_i \cdot \partial_i h_k) dS_x + \int_\Omega (u_i u_j) \partial_{ij}^2 h_k dx = 0 \quad \text{for all } k \in \{1, 2, 3\}.$$

Hence, (1.9) holds by Theorem 2.6. However, by now we have no examples of a satisfying (2.9).

Theorems 2.1–2.2 are proved by establishing necessary estimates for approximate solutions which are uniform in approximation parameter and then invoking the fact that our weak solutions become strong after a finite time. Theorems 2.3–2.6 are obtained by directly estimating the strong solutions whose existence is now well known. In dealing with strong solutions, we freely make use of the results obtained in our previous paper [18]. Detailed proofs are given in [21].

References

[1] Abels, H.: Bounded imaginary powers and H_∞-calculus of the Stokes operator in two dimensional exterior domains. Math. Z. **251**, 589–605 (2005).

[2] Amrouche, C., Girault, V., Schonbek, M. E., Schonbek, T. P.: Pointwise decay of solutions and of higher derivatives to Navier-Stokes equations. SIAM J. Math. Anal. **31**, 740–753 (2000).

[3] Bae, H. O., Jin, B. J.: Temporal and spatial decays for the Navier-Stokes equations. Proc. Royal Soc. Edinburgh: Sect. A, **135**, 461–477 (2005).

[4] Bae, H. O., Jin, B. J.: Asymptotic behavior for the Navier-Stokes equations in 2D exterior domains. J. Funct. Anal. **240**, 508–529 (2006).

[5] Bae, H. O., Jin, B. J.: Temporal and spatial decay rates of the Navier-Stokes solutions in exterior domains. Bull. Korean Math. Soc. **44**, 547–567 (2007).

[6] Borchers, W., Miyakawa, T.: Algebraic L^2 decay for Navier-Stokes flows in exterior domains. Acta Math. **165**, 189–227 (1990).

[7] Borchers, W., Miyakawa, T.: Algebraic L^2 decay for Navier-Stokes flows in exterior domains, II. Hiroshima Math. J. **21**, 621–640 (1991).

[8] Borchers, W., Miyakwa, T.: On stability of exterior stationary Navier-Stokes flows. Acta Math. **174**, 311–382 (1995).

[9] Brandolese, L.: Application of the realization of homogeneous Sobolev spaces to Navier-Stokes. SIAM J. Math. Anal. **37**, 673–683 (2005).

[10] Caffarelli, L., Kohn, R., Nirenberg, L.: First order interpolation inequalities with weights. Compositio Math. **53**, 259–275 (1984).

[11] Chen, Z-M.: Solutions of the stationary and nonstationary Navier-Stokes equations in exterior domains. Pacific J. Math. **159**, 227–240 (1993).

[12] Dan, W., Shibata, Y.: On the L_q-L_r estimates of the Stokes semigroup in a two-dimensional exterior domain. J. Math. Soc. Japan **51**, 181–207 (1999).

[13] Farwig, R.: Partial regularity and weighted energy estimates of global weak solutions of the Navier-Stokes equations. π, *Pitman Research Notes in Math. Series 345*, M. Chipot and I. Shafrir Eds, Longman, 1996.

[14] Farwig, R., Sohr, H.: Global estimates in weighted spaces of weak solutions of the Navier-Stokes equations in exterior domains. Arch. Math. **67**, 319–330 (1996).

[15] Fujigaki, Y., Miyakawa, T.: Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in the whole space. SIAM J. Math. Anal. **33**, 523–544 (2001).

[16] Giga, Y., Sohr, H.: Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. J. Funct. Anal. **102**, 72–94 (1991).

[17] Galdi, G. P., Maremonti, P.: Monotonic decreasing and asymptotic behavior of the kinetic energy for weak solutions of the Navier- Stokes equations in exterior domains. Arch. Rational Mech. Anal. **94**, 253–266 (1986).

[18] He, C., Miyakawa, T.: On L^1-summability and asymptotic profiles for smooth solutions to Navier-Stokes equations in a 3D exterior domain. Math. Z. **245**, 387–417 (2003).

[19] He, C., Miyakawa, T.: On two-dimensional Navier-Stokes flows with rotational symmetries. Funkcial. Ekvac. **49**, 163–192 (2006).

[20] He, C., Miyakawa, T.: Nonstationary Navier-Stokes flows in a two-dimensional exterior domain with rotational symmetries. Indiana Univ. Math. J. **55**, 1483–1556 (2006).

[21] He, C., Miyakawa, T.: On weighted-norm estimates for nonstationary Navier-Stokes flows in a 3D exterior domain. Preprint, Kanazawa University, 2008.

[22] He, C., Xin, Z.: Weighted estimates for nonstationary Navier-Stokes equations in exterior domains. Methods and Appl. Anal. **7**, 443–458 (2000).

[23] He, C., Xin, Z.: On the decay properties for solutions to nonstationary Navier-Stokes equations in $\mathbb{R}^3$. Proc. Royal Soc. Edinburgh: Sect. A, **131**, 597–619 (2001).

[24] Heywood, J. G.: The Navier-Stokes equations: On the existence, regularity and decay of solutions. Indiana Univ. Math. J. **29**, 639–681 (1980).

[25] Iwashita, H.: L^p-L^q estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problem in L^q spaces. Math. Ann. **285**, 265–288 (1989).

[26] Kato, T.: Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with application to weak solutions. Math. Z. **187**, 471–480 (1984).

[27] Kellogg, O. D.: *Foundations of Potential Theory.* Springer-Verlag, Berlin, 1929.

[28] Kozono, H., Ogawa, T.: Some L^p estimate for the exterior Stokes flow and an application to the non-stationary Navier-Stokes equations. Indiana Univ. Math. J. **41**, 789–808 (1992).

[29] Kozono, H., Ogawa, T., Sohr, H.: Asymptotic behavior in L^r for weak solutions of the Navier-Stokes equations in exterior domains. Manuscripta Math. **74**, 253–275 (1992).

[30] Ladyzhenskaya, O. A.: *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York, 1969.

[31] Maremonti, P.: On the asymptotic behaviour of the L^2-norm of suitable weak solutions to the Navier-Stokes equations in three-dimensional exterior domains. Commun. Math. Phys. **118**, 385–400 (1988).

[32] Masuda, K.: L^2-decay of solutions of the Navier-Stokes equations in the exterior domains. *Proc. Symp. in Pure Math.* **45**, Part 2, pp. 179–182., Amer. Math. Soc., Providence, 1986.

[33] Miyakawa, T.: On nonstationary solutions of the Navier-Stokes equations in an exterior domains. Hiroshima Math. J. **12**, 115–140 (1982).

[34] Miyakawa, T.: Hardy spaces of solenoidal vector fields, with applications to the Navier-Stokes equations. Kyushu J. Math. **50**, 1–64 (1996).

[35] Miyakawa, T.: On space-time decay properties of nonstationary incompressible Navier-Stokes flows in $\mathbb{R}^n$. Funkcial. Ekvac. **43**, 541–557 (2000).

[36] Miyakawa, T.: Notes on space-time decay properties of nonstationary incompressible Navier-Stokes flows in $\mathbb{R}^n$. Funkcial. Ekvac. **45**, 271–289 (2002).

[37] Miyakawa, T., Sohr, H.: On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains. Math. Z. **199**, 455–478 (1988).

[38] Schonbek, M. E., Schonbek, T. P.: On the boundedness and decay of moments of solutions of the Navier-Stokes equations. Advances in Diff. Eq. **5**, 861–898 (2000).

[39] Stein, E. M.: Note on singular integrals. Proc. Amer. Math. Soc. **8**, 250–254 (1957).

[40] Stein, E. M.: *Harmonic Analysis*, Princeton University Press, Princeton, 1993.

[41] Stein, E. M., Weiss, G.: Fractional integrals on n-dimensional Eulicdean space. J. Math. Mech. **17**, 503–514 (1958).