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Simultaneously lowering operators

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Abstract

The paper considers Terwilliger's problem 11.26 in [Ter06], about simultaneously lowering maps on finite-dimensional polynomial spaces with respect to two specific bases. The problem is related to construction of Leonard pairs and relation between its split bases. We show a family of counterexamples of simultaneously lowering maps that do not correspond to Leonard pairs.

1 Introduction

Our setting is the following. Let d denote a nonnegative integer, and let $\mathbb{K}$ denote e field of characteristic not equal to 2. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote a sequence of mutually distinct scalars in $\mathbb{K}$. Let x denote an indeterminante, and let V denote the linear space over $\mathbb{K}$ consisting of all polynomials in $\mathbb{K}[x]$ that have degree at most d. We consider the following two sequences of polynomials from V. One sequence consists of the polynomials

$$\begin{aligned} \tau_0 &= 1, \\ \tau_k &= (x-\theta_0)(x-\theta_1)\ldots(x-\theta_{k-1}), \qquad \text{for } i = 1,2,\ldots,d. \end{aligned}$$

The other sequence consists of the polynomials

$$\begin{aligned} \rho_0 &= 1, \\ \rho_k &= (x-\theta_d)(x-\theta_{d-1})\ldots(x-\theta_{d-k+1}), \qquad \text{for } i = 1,2,\ldots,d. \end{aligned}$$

The polynomials in each sequence are monic, of different degrees. Each sequence forms a basis for V.

Definition 1.1 By a *simultaneously lowering* map on V we mean a linear transformation $\Psi : V \to V$ that acts on both bases $\{\tau_k\}_{k=0}^d$ and $\{\rho_k\}_{k=0}^d$ as follows:

$$\begin{aligned} &\Psi\tau_0 = 0, \qquad \Psi\tau_k \in \text{span}(\tau_{k-1}) \quad \text{for } k = 1,2,\ldots,d, \\ &\Psi\rho_0 = 0, \qquad \Psi\rho_k \in \text{span}(\rho_{k-1}) \quad \text{for } k = 1,2,\ldots,d. \end{aligned}$$

By a *proper lowering* map we mean a simultaneously lowering map whose kernel is generated by $\tau_0 = \rho_0$ only.

Observe that simultaneously lowering maps form a linear space.

Definition 1.2 By a *weakly lowering* map on V we mean a linear transformation $\Psi : V \to V$ that acts on both bases $\{\tau_k\}_{k=0}^d$ and $\{\rho_k\}_{k=0}^d$ as follows:

$$\begin{aligned} &\Psi\tau_0 \in \operatorname{span}(\tau_0), \qquad \Psi\tau_k \in \operatorname{span}(\tau_k, \tau_{k-1}) \quad \text{for } k = 1, 2, \ldots, d, \\ &\Psi\rho_0 \in \operatorname{span}(\rho_0), \qquad \Psi\rho_k \in \operatorname{span}(\rho_k, \rho_{k-1}) \quad \text{for } k = 1, 2, \ldots, d. \end{aligned}$$

Observe that weakly lowing maps form a linear space. It includes the identity map and the space of simultaneously lowing maps.

Paul Terwilliger suggested the following conjecture [Ter06, Problem 11.26]:

Conjecture 1.3 *There is a nonzero simultaneously lowering map on V if and only if the quotient*

$$\frac{\theta_{k-2} - \theta_{k+1}}{\theta_{k-1} - \theta_k} \tag{1}$$

is independent of k for $2 \le k \le d-1$.

We prove this conjecture in the generic case of proper lowering maps. That is, we prove that if proper lowering maps exist, then the quotient (1) is independent of k. Vice versa, if (1) is independent of k, then a simultaneously lowering map exists, though not necessarily a proper lowering map.

If the kernel of a simultaneously lowering is allowed to contain polynomials of positive degree (that is, some of the polynomials $\tau_1, \ldots, \tau_d$, $\rho_1, \ldots, \rho_d$), the conjecture is false. We present a family of counterexamples.

Under the condition on the quotient (1), the linear space of simultaneously lowering maps has dimension 1. A generating lowering map has the form

$$\Psi\tau_k = \varphi_k\tau_{k-1}, \qquad \Psi\rho_k = \varphi_k\rho_{k-1}, \qquad \text{for } k = 1, 2, \ldots, d, \tag{2}$$

where

$$\varphi_k = \sum_{j=0}^{k-1} \frac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d}. \tag{3}$$

The space of weakly lowering maps has dimension 4; see Section 4.

In our family of counter-examples, we have the integer d odd, and the sum $\theta_{2i} + \theta_{2i+1}$ independent of i for $i = 0, 1, \ldots, \lfloor d/2 \rfloor$. The linear space of simultaneously lowering is generated by the lowering map of the form (2) with

$$\varphi_k = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}. \end{cases} \tag{4}$$

The space of weakly lowering maps has dimension 3 in general; see Section 5. The space of weakly lowering maps can have dimension 2 as well; in particular, if $d = 5$ then the generic dimension for the space of weakly lowering maps is 2, instead of expectable 1.

2 Restrictions on the transition matrix

Suppose that the two bases $\{\tau_k\}_{k=0}^d$ and $\{\rho_k\}_{k=0}^d$ of V are related as follows:

$$\rho_k = \tau_k + a_{k,1}\tau_{k-1} + a_{k,2}\tau_{k-2} + \ldots + a_{k,k}\tau_0, \qquad \text{for } k = 1, 2, \ldots, d. \tag{5}$$

The transformation matrix from the ρ-basis to the τ-basis is therefore

$$T = \begin{pmatrix} 1 & a_{1,1} & a_{2,2} & a_{3,3} & \cdots & a_{d,d} \\ & 1 & a_{2,1} & a_{3,2} & \cdots & a_{d,d-1} \\ & & 1 & a_{3,1} & & \vdots \\ & & & 1 & \ddots & \vdots \\ & & & & \ddots & a_{d,1} \\ & & & & & 1 \end{pmatrix}. \tag{6}$$

For convenience, we define $a_{k,0} = 1$ for $0 \le k \le d$, and $a_{k,j} = 0$ for $j < 0$, $j > k$ or $k > d$.

The transition coefficients $a_{k,j}$ are nicely related by the "multiplication by x" map $X : V \to V$. When polynomials in V are viewed as functions on the finite set $\{\theta_0, \theta_1, \ldots, \theta_d\}$, the map X multiplies the value of $p \in V$ at θ_k (for $k = 0, 1, \ldots, d$) by θ_k, so that $(Xp)(\theta_k) = \theta_k p(\theta_k)$. More algebraically, the map X multiplies the polynomials in V by x modulo the polynomial $(x-\theta_0)(x-\theta_1)\ldots(x-\theta_d)$.

Here is how the map X acts on the bases $\{\tau_k\}_{k=0}^d$ and $\{\rho_k\}_{k=0}^d$:

$$X\tau_k = \tau_{k+1} + \theta_k\tau_k \qquad \text{for } 0 \le k < d; \qquad X\tau_d = \theta_d\tau_d, \tag{7}$$
$$X\rho_k = \rho_{k+1} + \theta_{d-k}\rho_k \qquad \text{for } 0 \le k < d; \qquad X\rho_d = \theta_0\rho_d. \tag{8}$$

By letting X act on both sides of (5), and after expressing both sides of the equalities in the τ-basis, we get the equations

$$a_{k+1,j+1} - a_{k,j+1} = (\theta_{k-j} - \theta_{d-k})a_{k,j}. \tag{9}$$

In particular, for $j = 0$, $j = k$ and $k = d$ we have, respectively, $a_{k+1,1} - a_{k,1} = \theta_k - \theta_{d-k}$, $a_{k+1,k+1} = (\theta_0 - \theta_{d-k})a_{k,k}$ and $a_{d,k+1} = (\theta_0 - \theta_{d-k})a_{d,k}$, so that

$$a_{k,1} = (\theta_0 + \theta_1 + \ldots + \theta_{k-1}) - (\theta_{d-k-1} + \ldots + \theta_{d-1} + \theta_d), \tag{10}$$
$$a_{k,k} = a_{d,k} = (\theta_0 - \theta_d)(\theta_0 - \theta_{d-1})\cdots(\theta_0 - \theta_{d-k+1}). \tag{11}$$

Since we have mutually distinct θ_k's, all entries in the first row and the last column of the transformation matrix T are non-zero.

More generally, we may observe that $a_{k,j} = a_{d+j-k,j}$ by the symmetry of recursion relations (9). In other words, the transformation matrix T is symmetric with respect to the diagonal going from the upper right corner to the lower left corner.

Now suppose that there exists a weakly lowering map $\Psi : V \to V$ satisfying

$$\Psi\tau_0 = \theta_0^*\tau_0, \qquad \Psi\tau_k = \theta_k^*\tau_k + \varphi_k\tau_{k-1} \quad \text{for } k = 1, 2, \ldots, d, \tag{12}$$

$$\Psi\rho_0 = \theta_0^*\rho_0, \qquad \Psi\rho_k = \theta_k^*\rho_k + \phi_k\rho_{k-1} \quad \text{for } k = 1, 2, \ldots, d. \tag{13}$$

for some scalars

$$\theta_0^*, \theta_1^*, \ldots, \theta_d^*;\ \varphi_1, \ldots, \varphi_d;\ \phi_1, \ldots, \phi_d. \tag{14}$$

By letting Ψ act on both sides of (5), and after expressing both sides of the equalities in the τ-basis, we get the equations

$$a_{k,1}\left(\theta_k^* - \theta_{k-1}^*\right) = \varphi_k - \phi_k, \tag{15}$$

$$a_{k,j+1}\left(\theta_k^* - \theta_{k-j-1}^*\right) = a_{k,j}\varphi_{k-j} - a_{k-1,j}\phi_k. \tag{16}$$

If we set $j = k-1$ and use(11), we arrive at the equation

$$\phi_k = \frac{a_{k,k-1}}{a_{k-1,k-1}}\varphi_1 + (\theta_k^* - \theta_0^*)(\theta_{d-k+1} - \theta_0). \tag{17}$$

3 The generic case

Here we look for proper lowering maps. We show that when such lowering maps exist, the statement of Conjecture 1.3 is true.

For simultaneously lowering maps, we have to take all θ_k^*'s to be equal to zero in equations (15)–(16), so we have

$$\varphi_k = \phi_k, \qquad a_{k,j}\varphi_{k-j} = a_{k-1,j}\phi_k. \tag{18}$$

For proper lowering maps, none of the φ_k's is equal to zero. Since the $a_{k,k}$'s are non-zero, existence of a proper lowering maps implies all $a_{i,j}$'s are non-zero.

Theorem 3.1 *Suppose that a proper lowering map exists, in the setting of Section* 1. *Then the quotient in* (1) *is independent of k for* $2 \le k \le d-1$, *and the linear space of simultaneously lowering maps is spanned by the map* (2)-(3).

Proof. Let Ψ denote a proper lowering map. In both bases $\{\tau_k\}_{k=0}^d$ and $\{\rho_k\}_{k=0}^d$, it has the form

$$\Psi = \begin{pmatrix} 0 & \varphi_1 & & & \\ & 0 & \varphi_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_d \\ & & & & 0 \end{pmatrix}.$$

Equations in (18) mean that

$$\frac{a_{k,j}}{a_{k-1,j}} = \frac{\varphi_k}{\varphi_{k-j}}. \tag{19}$$

In particular, these equations with $j = 1$ mean that the vectors

$$(a_{1,1}, a_{2,1}, \ldots, a_{d,1}), \qquad (\varphi_1, \varphi_2, \ldots, \varphi_d) \tag{20}$$

are proportional. The form (2)–(3) of a possible lowering map follows from (10).

Equations (19) with $j = 2$ mean that the vector $(a_{2,2}, a_{3,2}, \ldots, a_{d,2})$ is proportional to the vector $(\varphi_1\varphi_2, \varphi_2\varphi_3, \ldots, \varphi_{d-1}\varphi_d)$, etc. All together, the equations in (19) mean that the transformation matrix T is a polynomial in Ψ:

$$T = I + c_1\Psi + c_2\Psi^2 + \ldots + c_d\Psi^d, \tag{21}$$

where $c_1, c_2, \ldots, c_d$ are non-zero scalars. The entries $a_{k,j}$ of T can be expressed as

$$a_{k,j} = c_j\, \varphi_{k-j+1} \cdots \varphi_{k-1}\varphi_k. \tag{22}$$

After substituting (22) into (9) and dividing out by $\varphi_{k-j+1} \cdots \varphi_{k-1}\varphi_k$, we get

$$c_{j+1}\,(\varphi_{k+1} - \varphi_{k-j}) = c_j\,(\theta_{k-j} - \theta_{d-k}). \tag{23}$$

Using $\varphi_i = a_{i,1}/c_1$ and (15) we conclude that the quotient

$$\frac{(\theta_{k-j} + \ldots + \theta_k) - (\theta_{d-k} + \ldots + \theta_{d-k+j})}{\theta_{k-j} - \theta_{d-k}} = \frac{c_j c_1}{c_{j+1}} \tag{24}$$

is independent of k for any fixed j. (The undeterminance $0/0$ for $k = (d+j)/2$ is not important, since we use this fact in the form of linear equations between the θ_k's.)

Here are a few equations equivalent to (24) with $j = 1$, $j = 2$:

$$\begin{aligned} u_1\theta_{k-1} + \theta_k &= u_1\theta_{d-k} + \theta_{d-k+1}, \\ u_1\theta_k + \theta_{k+1} &= u_1\theta_{d-k-1} + \theta_{d-k}, \\ u_2\theta_{k-1} + \theta_k + \theta_{k+1} &= u_2\theta_{d-k-1} + \theta_{d-k} + \theta_{d-k+1}, \end{aligned} \tag{25}$$

where $u_1 = 1 - c_1^2/c_2$ and $u_2 = 1 - c_1c_2/c_3$. Solving for $\theta_{d-k-1}, \theta_{d-k}, \theta_{d-k+1}$ gives, in particular,

$$\theta_{d-k-1} = \frac{(u_2 - u_1)\theta_{k-1} + u_1(u_1 - 1)\theta_k + u_1\theta_{k+1}}{u_1^2 - u_1 + u_2}, \tag{26}$$

$$\theta_{d-k} = \frac{u_1(u_1 - u_2)\theta_{k-1} + u_1u_2\theta_k + (u_2 - u_1)\theta_{k+1}}{u_1^2 - u_1 + u_2}. \tag{27}$$

After substitution of k by $k-1$ in the former formula we have two expressions for θ_{d-k}. Elimintation of θ_{d-k} gives the recurrence relation

$$(u_1 - u_2)(\theta_{k+1} - \theta_{k-2}) = u_1(u_2 - 1)(\theta_k - \theta_{k-1}). \tag{28}$$

Hence the quotient in (1) is constant, unless $u_1 = u_2$ or the denominator in (26) or (27) is zero. In the former case, we can't have $\theta_k = \theta_{k-1}$ nor $u_2 = 1$ (since $c_1c_2 \neq 0$) nor $u_1 = 0$ (see the system (25)). In the latter case, the numerator in (26) gives a three-term recurrence relation $\theta_{k+1} = (1 - u_1)\theta_k + u_1\theta_{k-1}$, and the quotient in (1) turns out to be a constant anyway. □

Example 3.2 Suppose that d is an odd integer. Let

$$\theta_k = a + b\,(-1)^k + \left(\tfrac{d}{2} - k\right)(-1)^k, \tag{29}$$

for $k = 0, 1, 2, \ldots, d$. We assume $b \neq 0$ so that $\theta_0 \neq \theta_d$. We have $\theta_{2i} + \theta_{2i+1} = 2a + 1$ for $i = 0, 1, \ldots \lfloor d/2 \rfloor$. With these θ_k's, all even-indexed polynomials τ_{2i} and ρ_{2i} are symmetric with respect to the transformation $x \mapsto 2a + 1 - x$, and form a linear space of dimension $\lceil d/2 \rceil$. The even-indexed ρ_{2i}'s are linear combinations of even indexed τ_{2i}'s, hence

$$a_{k,j} = 0 \qquad \text{whenever } k \text{ is even and } j \text{ is odd.} \tag{30}$$

Particularly,

$$a_{k,1} = \begin{cases} 2b, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even,} \end{cases} \tag{31}$$

since $\theta_k - \theta_{d-k} = 2b(-1)^k$. Equations (18) with $j = 1$ imply that $\varphi_k = 0$ if k is even; equations (18) with other odd j concur. The odd-indexed φ_k's are related by equations (18) with even j and odd k. For $i = 1, 2, \ldots, \ldots \lfloor d/2 \rfloor$, we have

$$a_{2i,2} = a_{2i+1,2} = 2bi(d + 1 - 2i). \tag{32}$$

Equations (18) with $j = 2$ imply that all odd-indexed φ_k's must be equal. For other equations (18) with even j and odd k we have $a_{k,j} = a_{k-1,j}$ by (9) and (37). It follows that the space of simultaneously lowering maps is spanned by the map Ψ given as follows:

$$\varphi_k = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even.} \end{cases} \tag{33}$$

This is not a proper lowering map. Incidentally, the quotient in (1) is independent of k (and equal to -1), and the vectors in (20) are proportional. But the transition matrix T is not a polynomial in Ψ. In fact, $\Psi^2 = 0$.

Now we compute the space of weakly lowering maps for sequence (36). Equation (15) with even $k = 2i$ implies $\varphi_{2i} = \phi_{2i}$. Consequently, equation (16) with $j = 1$ and $k = 2i$ or $k = 2i + 1$ gives two expressions for φ_{2i}:

$$\varphi_{2i} = i(2i - d - 1)(\theta^*_{2i} - \theta^*_{2i-2}), \qquad \varphi_{2i} = i(d + 1 - 2i)(\theta^*_{2i+1} - \theta^*_{2i-1}).$$

It follows that the sum $\theta^*_{2i} + \theta^*_{2i+1}$ is independent of i. Additionally, equation (17) with $k = 2i$ gives $\varphi_{2i} = (2i - d - 1)(\theta^*_{2i} - \theta^*_0)$. It follows that the even-indexed θ^*_{2i}'s form an arithmetic progression. Equations (15) and (17) with odd $k = 2i + 1$ give, respectively,

$$\phi_{2i+1} = \varphi_{2i+1} - 2b(\theta^*_{2i+1} - \theta^*_{2i}), \qquad \phi_{2i+1} = \varphi_1 - 2(b + i)(\theta^*_{2i+1} - \theta^*_0),$$

because $a_{2i+1,2i} = a_{2i,2i}$. This determines all odd-indexed ϕ_k's and φ_k's once φ_1 is fixed. Without assuming any additional relation between $\theta^*_0, \theta^*_1, \theta^*_2$ and φ_1, one can check that relations (9) and (16) with $j \geq 2$ for $a_{k,j}$'s are compatible. It follows that the space of weakly lowering maps has dimension 4.

4 The expected picture

The motivating perspective of Conjecture 1.3 was a possible new characterization of Leonard pairs. Let us recall a few definitions.

Definition 4.1 Let V be a linear space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair (A, B), where $A : V \to V$ and $B : V \to V$ are linear transformations which satisfy the following two conditions:

(i) There exists a basis for V with respect to which the matrix representing A is diagonal, and the matrix representing B is irreducible tridiagonal (that is, all entries on the first subdiagonal and the first superdiagonal are nonzero).

(ii) There exists a basis for V with respect to which the matrix representing B is diagonal, and the matrix representing A is irreducible tridiagonal.

Leonard pairs are specified by parameter arrays.

Definition 4.2 [Ter06, Definition 5.4] By a *parameter array* over $\mathbb{K}$, of diameter d, we mean a sequence

$$(\theta_0, \theta_1, \ldots, \theta_d;\ \theta_0^*, \theta_1^*, \ldots, \theta_d^*;\ \varphi_1, \ldots, \varphi_d;\ \phi_1, \ldots, \phi_d) \tag{34}$$

of scalars taken from $\mathbb{K}$, that satisfy the following conditions:

PA1. $\theta_k \neq \theta_j$ and $\theta_k^* \neq \theta_j^*$ if $k \neq j$, for $0 \leq k, j \leq d$.

PA2. $\varphi_k \neq 0$ and $\phi_k \neq 0$, for $1 \leq k \leq d$.

PA3. $\varphi_k = \phi_1 \sum_{j=0}^{k-1} \dfrac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d} + (\theta_k^* - \theta_0^*)(\theta_{k-1} - \theta_d)$, for $1 \leq k \leq d$.

PA4. $\phi_k = \varphi_1 \sum_{j=0}^{k-1} \dfrac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d} + (\theta_k^* - \theta_0^*)(\theta_{d-k+1} - \theta_0)$, for $1 \leq i \leq d$.

PA5. The expressions

$$\frac{\theta_{k-2} - \theta_{k+1}}{\theta_{k-1} - \theta_k}, \qquad \frac{\theta_{k-2}^* - \theta_{k+1}^*}{\theta_{k-1}^* - \theta_k^*}$$

are equal and independent of k, for $2 \leq k \leq d-1$.

Particularly [Ter06, Section 5.1], if sequence (41) is a parameter array, then the following two matrices form a Leonard pair:

$$\begin{pmatrix} \theta_0 & & & & \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \theta_d \end{pmatrix}, \qquad \begin{pmatrix} \theta_0^* & \varphi_1 & & & \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ & & & & \theta_d^* \end{pmatrix}. \tag{35}$$

Theorem 4.3 *In the setting of Section* 1*, suppose that the quotient in* (1) *is independent of k. Then a simultaneously lowering map exists, the linear space of weakly lowering maps has dimension* 4*, and a sequence of scalars in* (14) *defines a weakly lowering map by* (12)-(13) *if and only if conditions* PA3–PA5 *of Definition* 4.2 *are satisfied.*

Proof. This largely matches computations in the proof of [Ter06, Theorem 10.1]. Let q denote a scalar such that $1 + q + q^{-1}$ is equal to the quotient in (1). If $q \neq \pm 1$, then the θ_k's have the form

$$\theta_k = u + vq^k + wq^{-k}, \qquad \text{for some scalars } u, v, w. \tag{36}$$

If $q = 1$ or $q = -1$, then for some scalars u, v, w we have, respectively,

$$\theta_k = u + vk + wk^2 \qquad \text{or} \qquad \theta_k = u + v(-1)^k + wk(-1)^k. \tag{37}$$

In each of these three cases, the values in (3) satisfy equations in (18) and define a simultaneously lowering map. Equations (15)–(16) for weakly lowering maps are linear in the scalars in (14). Particular equation (17) coincides with condition PA4. Condition PA3 follows from the symmetry of the τ- and ρ-bases. This eliminates all φ_k's and ϕ_k's except one, say φ_1. In particular,

$$\varphi_k - \psi_k = (\theta_0 + \theta_{k-1} - \theta_{d-k+1} - \theta_d)(\theta^*_k - \theta^*_0) - a_{k,1}(\theta^*_1 - \theta^*_0). \tag{38}$$

Three equations (15) with consecutive k allows us to eliminate θ^*_0, θ^*_1 linearly and get a recurrence relation for θ^*_k's for whatever sequence of θ_k's of the forms in (43) or (44), except in the case of $q = -1$ and odd d. (Only in the exceptional case some $a_{k,1}$'s are zero.) The recurrence relation is the restriction on the quotient of θ^*_k's in condition PA5; by the recurrence relation and elimination expressions for φ_k's and ϕ_k's we can choose $\theta^*_0, \theta^*_1, \theta^*_2, \varphi_1$ freely, while relations (9) and (16) with $j \geq 2$ for $a_{k,j}$'s work out to be compatible. The conclusions now follow except for the case of $q = -1$ and odd d. The exceptional case is considered in Example 3.3, with $u = a$, $v = b + \frac{d}{2}$ and (inconsequentially) $w = -1$. The same restriction on θ^*_k's holds, and the dimension is 4 anyway. □

Corollary 4.4 *In the setting of Section* 1*, suppose that a proper lowering map exists. Then the linear space of weakly lowering maps has dimension* 4*, and a sequence of scalars in* (14) *defines a weakly lowering map by* (12)-(13) *if and only if conditions* PA3–PA5 *of Definition* 4.2 *are satisfied.*

Conclusions of this corollary were expected to be true whenever a non-zero simultaneously lowering map exists. However, in the next section we present a family of counterexamples to this expectation. We find (non-proper) lowering maps that are not related to Leonard pairs; the dimension of weakly lowering maps is generally 3.

5 The counterexamples

In the setting of Section 1, let us assume that d is odd, $d = 2n + 1$. We define the following maps on the linear space of polynomials of degree at most d:

$$Lx^{2i} = 0, \qquad Lx^{2i+1} = x^{2i}, \qquad \text{for } i = 0, 1, \ldots, n, \tag{39}$$

$$Px^{2i} = x^{2i}, \qquad Px^{2i+1} = 0, \qquad \text{for } i = 0, 1, \ldots, n. \tag{40}$$

If we view polynomials as functions on the real line, the map L annihilates even polynomial functions, and divides odd functions by x. The map P fixes even functions, and annihilates odd polynomial functions.

Let $\mu_0, \mu_1, \ldots, \mu_n$ be a sequence of distinct scalars. We set $d = 2n + 1$ and

$$\theta_{2i} = \mu_i, \qquad \theta_{2i+1} = -\mu_i, \qquad \text{for } i = 0, 1, \ldots, n. \tag{41}$$

We define the polynomials $\tau_0, \tau_1, \ldots, \tau_d$ and $\rho_0, \rho_1, \ldots, \rho_d$ as in Section 1 from this data. Note that the even-indexed polynomials $\tau_0, \tau_2, \ldots \tau_{2n}$ and $\rho_0, \rho_2, \ldots, \rho_{2n}$ are even polynomial functions.

The action of L on the τ and ρ bases is the following:

$$L\tau_{2i} = 0, \qquad L\tau_{2i+1} = \tau_{2i}, \qquad \text{for } i = 0, 1, \ldots, n, \tag{42}$$

$$L\rho_{2i} = 0, \qquad L\rho_{2i+1} = \rho_{2i}, \qquad \text{for } i = 0, 1, \ldots, n. \tag{43}$$

We see that L is a lowering map with, but it is not a proper lowering map. The map is given by (4). We have no other restriction on the θ_k's except $\theta_{2i} + \theta_{2i+1} = 0$. The quotient in (1) is equal to -1 for even k, and is variable for odd k.

The action of P on the τ and ρ bases is the following:

$$P\tau_{2i} = \tau_{2i}, \qquad P\tau_{2i+1} = -\mu_i\tau_{2i}, \qquad \text{for } i = 0, 1, \ldots, n, \tag{44}$$

$$P\rho_{2i} = \rho_{2i}, \qquad P\rho_{2i+1} = \mu_{n-i}\rho_{2i}, \qquad \text{for } i = 0, 1, \ldots, n. \tag{45}$$

We see that P is a weakly lowering map. The space of weakly lowering maps contains L, P and the identity, hence its dimension is at least 3. For $d = 5, 7$ this appears to be the general dimension.

This example can be generalized by adding a fixed scalar m to each member of the sequence of θ_k's; the 'evenness' symmetry is then the transformation $x \mapsto m - x$. The general relation on the θ_k's is the condition that the sum $\theta_{2i} + \theta_{2i+1}$ must be independent of i. Example 3.3 is a special case of this setting; it arises when the sequence of μ_i's is an arithmetic progression.

References

[Ter06] P. Terwilliger. An algebraic approach to the Askey scheme of orthogonal polynomials. In F. Marcellan and W. Van Assche, editors, *Orthogonal Polynomials and Special Functions: Computation and Applications*, volume 1883 of *Lecture Notes in Mathematics*, pages 225–330. Springer, 2006.