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A NEW SATURATED FILTER

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ABSTRACT. We construct a new model of ZFC in which ω_1 carries a saturated filter.

1. INTRODUCTION

In the groundbreaking work [8] Kunen established

Theorem 1. *If there is a huge cardinal, there is a forcing extension in which ω_1 carries a saturated filter.*

See [5, 6] for detailed expositions of Kunen's proof.

In these notes we present a model as in Theorem 1 that can be defined simply. This would make it easier to analyze the model in detail. Moreover the method of the proof is expected to work for other problems.

2. PRELIMINARIES

We refer the reader to [7] for background material. Throughout the paper κ denotes a regular cardinal. By a filter on κ we mean a normal one. We say that a filter on κ is saturated if it is κ^+-saturated.

Suppose that P and Q are posets. We say that a map $\pi : P \to Q$ is a projection if

- π is order-preserving, i.e. $p' \leq p$ implies $\pi(p') \leq \pi(p)$, and
- if $q \leq \pi(p)$, then there is $p^* \leq p$ with $\pi(p^*) \leq q$.

Suppose that $\pi : P \to Q$ is a projection. Then it is easy to see that if D is dense open in Q, $\pi^{-1}(D)$ is dense in P. So if $G \subset P$ is generic, π"G generates a generic filter over Q. We say that a projection $\pi : P \to Q$ is total if $\operatorname{ran} \pi$ is dense (or equivalently predense) in Q. Note that a projection $\pi : P \to Q$ is total if $\pi(1_P) = 1_Q$.

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Lemma 1. *If there is a total projection from $\pi : P \to Q$, then Q can be completely embedded into $B(P)$, the completion of P.*

Proof. Since π is total, we can define $e : Q \to B(P) - \{0\}$ by

$$e(q) = \sum\{p : \pi(p) \leq q\}.$$

It is easy to check that e is a complete embedding. □

Suppose that μ is a cardinal and that $\{S_i : i \in I\}$ is a nonempty set of posets. We write $\prod^{\mu}\{S_i : i \in I\}$ for the μ-product of $\{S_i : i \in I\}$, i.e.

$$\prod^{\mu}\{S_i : i \in I\} = \bigcup\left\{\prod\nolimits_{i\in d} S_i : d \in [I]^{<\mu}\right\}.$$

$\prod^{\mu}\{S_i : i \in I\}$ is ordered by: $s' \leq s$ iff $\operatorname{dom} s' \supset \operatorname{dom} s$ and $s'(i) \leq s(i)$ in S_i for every $i \in I$.

3. Modifying the Silver collapse

In [9] Silver defined a variation of the Levy collapse, now called the Silver collapse. This section introduces a modification of the Silver collapse and establishes its basic properties.

Suppose that λ is an inaccessible cardinal $> \kappa$. $S(\kappa, \lambda)$ denotes the set of all functions of the form $s : \delta \times d \to \lambda$, where

- $\delta < \kappa$,
- d is a set of κ-closed cardinals $< \lambda$ of size $\leq \kappa$, and
- $s(\eta, \nu) < \nu$ for every $(\eta, \nu) \in \delta \times d$.

Here a cardinal ν is κ-closed if $\nu^{<\kappa} = \nu > \kappa$. $S(\kappa, \lambda)$ is ordered by reverse inclusion: $s' \leq s$ iff $s' \supset s$. Standard arguments show that

Lemma 2. *$S(\kappa, \lambda)$ is κ-closed, has λ-cc and forces $\lambda = \kappa^+$.*

Also note that if P has κ-cc and size κ, forcing with P does not change the class of κ-closed cardinals.

Here is the main result of this section:

Lemma 3. *Suppose that P has κ-cc and size κ. Then there is a total projection from $P \times S(\kappa, \lambda)$ to $P * \dot{S}(\kappa, \lambda)$ that is the identity on the first coordinate.*

Proof. Since P has κ-cc and size κ, if ν is a cardinal, there exist at most $\nu^{<\kappa}$ representatives from the P-names $\dot{\tau}$ such that $\Vdash_P \dot{\tau} < \nu$. So if ν is a κ-closed cardinal, there exist exactly ν representatives from the P-names $\dot{\tau}$ such that $\Vdash_P \dot{\tau} < \nu$. Note that a κ-closed cardinal has cofinality $\geq \kappa$. Hence if ν is κ-closed and $\Vdash_P \dot{\tau} < \nu$, then $\Vdash_P \dot{\tau} < \gamma$ for some $\gamma < \nu$. Thus we can list as $\{\dot\tau_\xi : \xi < \lambda\}$ a set of P-names so that for every κ-closed cardinal $\nu \le \lambda$

- if $\xi < \nu$, then $\Vdash_P \dot\tau_\xi < \nu$ and
- if $\Vdash_P \dot\tau < \nu$, then $\Vdash_P \dot\tau = \dot\tau_\xi$ for some $\xi < \nu$.

Define

$$\pi : P \times S(\kappa, \lambda) \to P * \dot S(\kappa, \lambda)$$

by

$$\pi(p, s) = (p, \dot s),$$

where $\dot s$ is a P-name such that $\Vdash_P \dot s \in \dot S(\kappa, \lambda)$ as follows: Since $s \in S(\kappa, \lambda)$, there are δ and d such that

- $\operatorname{dom} s = \delta \times d$,
- $\delta < \kappa$ and
- d is a set of κ-closed cardinals $< \lambda$ of size $\le \kappa$.

Define a P-name $\dot s$ so that P forces

- $\operatorname{dom} \dot s = \delta \times d$ and
- $\dot s(\eta, \nu) = \dot\tau_{s(\eta,\nu)}$ for every $(\eta, \nu) \in \delta \times d$.

Note that $\Vdash_P \dot s(\eta, \nu) < \nu$ for every $(\eta, \nu) \in \delta \times d$ by $s(\eta, \nu) < \nu$ and the choice of $\{\dot\tau_\xi : \xi < \lambda\}$. Also d remains a set of κ-closed cardinals after forcing with P. Thus P forces $\dot s \in \dot S(\kappa, \lambda)$.

Claim. *π is a total projection.*

Proof. Since $\pi(1_P, \emptyset) = (1_P, \emptyset)$, it remains to prove that π is a projection. It is easy to see that π is order-preserving.

Suppose that $(p, s) \in P \times S(\kappa, \lambda)$ and $(q, \dot t) \le \pi(p, s)$ in $P * \dot S(\kappa, \lambda)$. We need to find $(p^*, s^*) \in P \times S(\kappa, \lambda)$ such that $(p^*, s^*) \le (p, s)$ and $\pi(p^*, s^*) \le (q, \dot t)$. Let $p^* = q$. It remains to give $s^* \in S(\kappa, \lambda)$ such that $s^* \le s$ and $\pi(p^*, s^*) \le (p^*, \dot t)$.

Since P forces $\dot t \in \dot S(\kappa, \lambda)^P$, there are P-names $\dot\delta$ and $\dot d$ such that P forces

- $\operatorname{dom} \dot t = \dot\delta \times \dot d$,
- $\dot\delta < \kappa$ and
- $\dot d$ is a set of κ-closed cardinals $< \lambda$ of size $\le \kappa$.

Since P has κ-cc, there is $\delta^* < \kappa$ such that $\Vdash_P \dot\delta < \delta^*$. Since P does not change the class of κ-closed cardinals, there is a set d^* of κ-closed cardinals $< \lambda$ of size $\le \kappa$ such that $\Vdash_P \dot d \subset d^*$. Moreover since

$$\Vdash_P \dot t \in \dot S(\kappa, \lambda) \text{ and } \operatorname{dom} \dot t = \dot\delta \times \dot d \subset \delta^* \times d^*,$$

there is a P-name $\dot t^*$ such that

$$\Vdash_P \dot t^* : \delta^* \times d^* \to \lambda \text{ is in } \dot S(\kappa, \lambda) \text{ and } \dot t^* \le \dot t.$$

Since
$$(p^*, \dot t) \le (q, \dot t) \le \pi(p, s) \text{ in } P * \dot S(\kappa, \lambda),$$
we have $p^* \Vdash_P \operatorname{dom} s \subset \operatorname{dom} \dot t$. Hence by $\Vdash_P \operatorname{dom} \dot t \subset \delta^* \times d^*$, we have $\operatorname{dom} s \subset \delta^* \times d^*$. Define $s^* : \delta^* \times d^* \to \lambda$ so that

- $s^* | \operatorname{dom} s = s$ and
- if $(\eta, \nu) \notin \operatorname{dom} s$, then $s^*(\eta, \nu)$ is the minimal ξ such that P forces $\dot\tau_\xi = \dot t^*(\eta, \nu)$.

We claim that $s^* \in S(\kappa, \lambda)$. Note that this implies $s^* \le s$ by $s^* | \operatorname{dom} s = s$. First recall that $\delta^* < \kappa$ and d^* is a set of κ-closed cardinals $< \lambda$ of size $\le \kappa$. It remains to prove that $s^*(\eta, \nu) < \nu$ for every $(\eta, \nu) \in \delta^* \times d^*$. If $(\eta, \nu) \in \operatorname{dom} s$, then $s^*(\eta, \nu) = s(\eta, \nu) < \nu$ by $s \in S(\kappa, \lambda)$. If $(\eta, \nu) \notin \operatorname{dom} s$, the conclusion follows from $\Vdash_P \dot t^*(\eta, \nu) < \nu$ and the choice of $\{\dot\tau_\xi : \xi < \lambda\}$.

Finally we prove that $\pi(p^*, s^*) \le (p^*, \dot t)$ in $P * \dot S(\kappa, \lambda)$. Let $\pi(p^*, s^*) = (p^*, \dot s^*)$. It suffices to show that $p^* \Vdash_P \dot s^* \le \dot t$. First recall that P forces $\operatorname{dom} \dot t \subset \delta^* \times d^* = \operatorname{dom} s^* = \operatorname{dom} \dot s^*$. It remains to prove that $p^* \Vdash_P \dot s^* | \operatorname{dom} \dot t = \dot t$. First note that for every $(\eta, \nu) \in \operatorname{dom} s$
$$p^* \Vdash_P \dot s^*(\eta, \nu) = \dot\tau_{s^*(\eta,\nu)} = \dot\tau_{s(\eta,\nu)} = \dot t(\eta, \nu).$$
The second equality follows from $s^* | \operatorname{dom} s = s$, and the third from $(p^*, \dot t) \le \pi(p, s)$. Next P forces that for every $(\eta, \nu) \in \operatorname{dom} \dot t - \operatorname{dom} s$
$$\dot s^*(\eta, \nu) = \dot\tau_{s^*(\eta,\nu)} = \dot t^*(\eta, \nu) = \dot t(\eta, \nu).$$
To see the second equality, recall that P forces $\operatorname{dom} \dot t \subset \delta^* \times d^*$ and $\dot\tau_{s^*(\eta,\nu)} = \dot t^*(\eta, \nu)$ for every $(\eta, \nu) \in \delta^* \times d^* - \operatorname{dom} s$. The third equality follows from $\Vdash_P \dot t^* \le \dot t$. □

This completes the proof. □

Remark 1. For a P-name $\dot S$ for a poset let $T(\dot S)$ be the term space, i.e. the set of canonical representatives from $\{\dot s : \Vdash_P \dot s \in \dot S\}$ ordered by: $\dot s' \le \dot s$ iff $\Vdash_P \dot s' \le \dot s$ in $\dot S$. It is known (and easy to see) that $\mathrm{id} : P \times T(\dot S) \to P * \dot S$ is a total projection. See [2] for details. The method of the proof of Lemma 3 shows that if P has κ-cc and size κ, $S(\kappa, \lambda)$ is isomorphic to
$$\{\dot s \in T(\dot S(\kappa, \lambda)) : \exists \delta < \kappa \exists d \subset \lambda \Vdash_P \operatorname{dom} \dot s = \delta \times d\},$$
which is dense in $T(\dot S(\kappa, \lambda))$.

Remark 2. The results in this section should be valid with the modified Silver collapse replaced by a suitable modification of the Levy collapse.

4. Main Theorem

This section is devoted to a proof of

Theorem 2. *Suppose that κ is huge and μ is a regular cardinal $< \kappa$. Then there is a forcing extension in which $\kappa = \mu^+$ and κ carries a saturated filter.*

Proof. Let $j : V \to M$ be a huge embedding with critical point κ and $\lambda = j(\kappa)$. Define

$$P = \prod^{\mu}\{S(\alpha,\kappa) : \alpha \in [\mu,\kappa) \cap \mathrm{Reg}\}.$$

It is easy to see that $P \subset V_\kappa$ is μ-closed and has size κ.

Claim. *P has κ-cc.*

Proof. Let $A \in [P]^\kappa$. We need to find distinct $s, t \in A$ such that $s(\alpha)$ and $t(\alpha)$ agree on $\operatorname{dom} s(\alpha) \cap \operatorname{dom} t(\alpha)$ for every $\alpha \in \operatorname{dom} s \cap \operatorname{dom} t$.

Since κ is inaccessible and $|\operatorname{dom} s| < \mu < \kappa$ for $s \in A$, there is $B \in [A]^\kappa$ such that $\{\operatorname{dom} s : s \in B\}$ forms a Δ-system. Let r be the root. Since $|r| < \mu$, there is $\beta < \kappa$ with $r \subset \beta$. For $s \in B$ and $\alpha \in r$ let $\operatorname{dom} s(\alpha) = \delta^s_\alpha \times d^s_\alpha$. Then $\delta^s_\alpha < \alpha < \beta$ and $|d^s_\alpha| \leq \alpha < \beta$. Since κ is inaccessible, there are $C \in [B]^\kappa$ and $\langle \delta_\alpha : \alpha \in r\rangle$ such that $\langle \delta^s_\alpha : \alpha \in r\rangle = \langle \delta_\alpha : \alpha \in r\rangle$ for every $s \in C$ and $\{\bigcup_{\alpha\in r}\{\alpha\} \times d^s_\alpha : s \in C\}$ forms a Δ-system. Let $\bigcup_{\alpha\in r}\{\alpha\} \times d_\alpha$ be the root. Since $|r| < \mu$ and $d_\alpha \in [\kappa]^{<\beta}$ for $\alpha \in r$, there is $\gamma < \kappa$ with $\bigcup_{\alpha\in r} d_\alpha \subset \gamma$. Then $s(\alpha)``(\delta_\alpha \times d_\alpha) \subset \gamma$ for every $\alpha \in r$. Since κ is inaccessible, there is $D \in [C]^\kappa$ such that $s \mapsto \langle s(\alpha)|(\delta_\alpha \times d_\alpha) : \alpha \in r\rangle$ is constant on D. Now it is easy to check that any two elements of D are as desired. □

Since $S(\mu,\kappa)$ can be completely embedded into P, P forces $\kappa = \mu^+$. Thus $P * \dot{S}(\kappa,\lambda)$ forces $\lambda = \kappa^+ = \mu^{++}$.

Claim. *$P * \dot{S}(\kappa,\lambda)$ forces that κ carries a saturated filter.*

Proof. Since ${}^\lambda M \subset M$, we have

$$j(P) = \prod^{\mu}\{S(\alpha,\lambda) : \alpha \in [\mu,\lambda) \cap \mathrm{Reg}\}.$$

Define $\varphi : j(P) \to P \times S(\kappa,\lambda)$ by

$$\varphi(t) = (\langle t(\alpha)|(\alpha\times\kappa) : \alpha \in \operatorname{dom} t \cap \kappa\rangle, t(\kappa)).$$

(It is understood that $t(\kappa) = \emptyset$ if $\kappa \notin \operatorname{dom} t$.) It is easy to check that φ is a total projection. By Lemma 3 there is a total projection $\pi : P \times S(\kappa,\lambda) \to P * \dot{S}(\kappa,\lambda)$ that is the identity on the first coordinate.

Let $\bar{G} \subset j(P)$ be V-generic. Since φ is a projection, φ"$\bar{G}$ generates a V-generic filter in $P \times S(\kappa, \lambda)$ that has the form $G \times H$. Since π is a projection, π"$(G \times H)$ generates a V-generic filter in $P * \dot{S}(\kappa, \lambda)$. Since π is the identity on the first coordinate, the generated filter has the form $G * K$. Note that j"$G = G \subset \bar{G}$ by $P \subset V_\kappa$. Hence we can extend j to $j : V[G] \to M[\bar{G}]$ in $V[\bar{G}]$. Since $j(P)$ has λ-cc and ${}^\lambda M \subset M$ in V, we have ${}^\lambda M[\bar{G}] \subset M[\bar{G}]$ in $V[\bar{G}]$.

The rest of the proof is essentially the same as that of Kunen, so we just give an outline. In $V[G]$ let $\{\dot{X}_\xi : \xi < \lambda\}$ list the set of $S(\kappa, \lambda)$-names for subsets of κ. In $V[\bar{G}]$ let

$$s^* = \bigcup j\text{“}K.$$

The standard arguments show that $s^* \in S(\lambda, j(\lambda))^{M[\bar{G}]}$. Since ${}^\lambda M[\bar{G}] \subset M[\bar{G}]$ in $V[\bar{G}]$, we get a desending sequence $\{s_\xi : \xi < \lambda\} \subset S(\lambda, j(\lambda))^{M[\bar{G}]}$ such that $s_0 \leq s^*$ and each s_ξ decides $\kappa \in j(\dot{X}_\xi)$. Then in $V[\bar{G}]$

$$U = \{(\dot{X}_\xi)_K : s_\xi \Vdash \kappa \in j(\dot{X}_\xi)\}$$

is a $V[G][K]$-ultrafilter on κ. Since $P * \dot{S}(\kappa, \lambda)$ can be completely embedded into $B(j(P))$ and $\bar{G}$ is arbitrary, there is a $j(P)/(G * K)$-name $\dot{U}$ such that in $V[G][K]$

$$j(P)/(G * K) \Vdash \dot{U} \text{ is a } V[G][K]\text{-ultrafilter on } \kappa.$$

Now in $V[G][K]$

$$F = \{X \subset \kappa : j(P)/(G * K) \Vdash X \in \dot{U}\}$$

is a filter on κ. It is easy to check that $X \mapsto \sum\{p : p \Vdash X \in \dot{U}\}$ defines a complete embedding from F^+ into $B(j(P)/(G * K))$. Since $j(P)/(G * K)$ has λ-cc, F is saturated. □

This completes the proof. □

Remark 3. Suppose that κ is huge with target λ. Then it is easy to see that λ is a Woodin cardinal. Refining a result of [4], Todorčević showed that if λ is Woodin, the Levy collapse $\text{Col}(\omega_1, \lambda)$ produces a saturated filter on ω_1 (see [1]). In contrast $\text{Col}(\omega_2, \lambda)$ does not necessarily produce a saturated filter on ω_2 (see [3]). On the other hand the Todorčević result implies that the iteration $\text{Col}(\omega, \kappa) * \dot{\text{Col}}(\kappa, \lambda)$ produces a saturated filter on ω_1. It appears unknown, however, whether the iteration $\text{Col}(\omega_1, \kappa) * \dot{\text{Col}}(\kappa, \lambda)$ produces a saturated filter on ω_2.

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