数理解析研究所講究録
第 1599 巻 2008 年 79-85

Fixed point theorem on partial randomness

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Abstract. In this paper, we prove a fixed point theorem on partial randomness in a self-contained manner.

1 Introduction

The fixed point theorem on partial randomness, Theorem 3.1 below, was introduced by Tadaki [11] in the context of a statistical mechanical interpretation of algorithmic information theory, where the theorem is called a fixed point theorem on compression rate. We isolate the fixed point theorem on partial randomness and its proof from [11], and describe them in this paper in a self-contained manner. While Tadaki [11] contains a physical and informal argument in the last of the paper, this paper consists only of purely mathematical arguments. For the variants of the fixed point theorem on partial randomness and their related topics, see [11].

2 Preliminaries

2.1 Basic notation

We start with some notation about numbers and strings which will be used in this paper. $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Z}$ is the set of integers, and $\mathbb{Q}$ is the set of rational numbers. $\mathbb{R}$ is the set of real numbers.

$$\{0,1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, \dots\}$$

is the set of finite binary strings where λ denotes the *empty string*. For any $s \in \{0,1\}^*$, $|s|$ is the *length* of s. A subset S of $\{0,1\}^*$ is called a *prefix-free set* if no string in S is a prefix of another string in S. $\{0,1\}^\infty$ is the set of infinite binary strings, where an infinite binary string is infinite to the right but finite to the left. For any $\alpha \in \{0,1\}^\infty$, α_n is the prefix of α of length n. For any partial function f, the domain of definition of f is denoted by $\operatorname{dom} f$. We write "r.e." instead of "recursively enumerable."

Normally, $o(n)$ denotes any one function $f\colon \mathbb{N}^+ \to \mathbb{R}$ such that $\lim_{n\to\infty} f(n)/n = 0$. On the other hand, $O(1)$ denotes any one function $g\colon \mathbb{N}^+ \to \mathbb{R}$ such that there is $C \in \mathbb{R}$ with the property that $|g(n)| \le C$ for all $n \in \mathbb{N}^+$.

Let T be an arbitrary real number. $T \bmod 1$ denotes $T - \lfloor T \rfloor$, where $\lfloor T \rfloor$ is the greatest integer less than or equal to T. Hence, $T \bmod 1 \in [0,1)$. On the other hand, $\lceil T \rceil$ is the smallest integer greater than or equal to T. We identify a real number T with the infinite binary string α such that $0.\alpha$ is the base-two expansion of $T \bmod 1$ with infinitely many zeros. Thus, T_n denotes the first n bits of the base-two expansion of $T \bmod 1$ with infinitely many zeros. In particular, if $T \in [0,1)$, then T_n denotes the first n bits of the base-two expansion of T with infinitely many zeros.

We say that a real number T is *computable* if there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $|T - f(n)| < 1/n$ for all $n \in \mathbb{N}^+$. We say that T is *right-computable* if there exists a total recursive function $g\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $T \le g(n)$ for all $n \in \mathbb{N}^+$ and $\lim_{n\to\infty} g(n) = T$. We say that T is *left-computable* if $-T$ is right-computable. Then the following theorem holds.

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Theorem 2.1. *Let $T \in \mathbb{R}$.*

(i) T is computable if and only if T is both right-computable and left-computable.

(ii) T is right-computable if and only if the set $\{ (m,n) \in \mathbb{Z} \times \mathbb{N}^+ \mid T < m/n \}$ is r.e. $\square$

See e.g. [7, 13] for the detail of the treatment of the computability of real numbers.

2.2 Algorithmic information theory

In the following we concisely review some definitions and results of algorithmic information theory [3, 4]. A *computer* is a partial recursive function $C\colon \{0,1\}^* \to \{0,1\}^*$ such that $\operatorname{dom} C$ is a prefix-free set. For each computer C and each $s \in \{0,1\}^*$, $H_C(s)$ is defined by $H_C(s) = \min\{\, |p| \mid p \in \{0,1\}^* \ \&\ C(p) = s \,\}$. A computer U is said to be *optimal* if for each computer C there exists a constant $\mathrm{sim}(C)$ with the following property; if $C(p)$ is defined, then there is a p' for which $U(p') = C(p)$ and $|p'| \le |p| + \mathrm{sim}(C)$. It is then shown that there exists an optimal computer. We choose any one optimal computer U as the standard one for use, and define $H(s)$ as $H_U(s)$, which is referred to as the *program-size complexity* of s, the *information content* of s, or the *Kolmogorov complexity* of s [5, 6, 3]. Thus, $H(s)$ has the following property:

$$\forall C : \text{computer} \quad H(s) \le H_C(s) + \mathrm{sim}(C). \tag{1}$$

It can be shown that there is $c \in \mathbb{N}$ such that, for any $s \neq \lambda$,

$$H(s) \le |s| + 2\log_2 |s| + c. \tag{2}$$

Chaitin's halting probability Ω is defined by

$$\Omega = \sum_{p \in \operatorname{dom} U} 2^{-|p|}.$$

Definition 2.2 (weak Chaitin randomness, Chaitin [3, 4]). *For any $\alpha \in \{0,1\}^\infty$, we say that α is weakly Chaitin random if there exists $c \in \mathbb{N}$ such that $n - c \le H(\alpha_n)$ for all $n \in \mathbb{N}^+$.* $\square$

Chaitin [3] showed that Ω is weakly Chaitin random.

2.3 Partial randomness

In the works [9, 10], we generalized the notion of the randomness of an infinite binary string so that *the degree of the randomness*, which is often referred to as *the partial randomness* recently [1, 8, 2], can be characterized by a real number T with $0 \le T \le 1$ as follows.

Definition 2.3 (weak Chaitin T-randomness). *Let $T \in \mathbb{R}$ with $T \ge 0$. For any $\alpha \in \{0,1\}^\infty$, we say that α is weakly Chaitin T-random if there exists $c \in \mathbb{N}$ such that $Tn - c \le H(\alpha_n)$ for all $n \in \mathbb{N}^+$.* $\square$

Definition 2.4 (T-compressibility). *Let $T \in \mathbb{R}$ with $T \ge 0$. For any $\alpha \in \{0,1\}^\infty$, we say that α is T-compressible if $H(\alpha_n) \le Tn + o(n)$, which is equivalent to*

$$\varlimsup_{n\to\infty} \frac{H(\alpha_n)}{n} \le T.$$

$\square$

In the case of $T = 1$, the weak Chaitin T-randomness results in the weak Chaitin randomness. For any $T \in [0,1]$ and any $\alpha \in \{0,1\}^\infty$, if α is weakly Chaitin T-random and T-compressible, then

$$\lim_{n\to\infty} \frac{H(\alpha_n)}{n} = T, \tag{3}$$

and therefore the *compression rate* of α by the program-size complexity H is equal to T. Note, however, that (3) does not necessarily imply that α is weakly Chaitin T-random.

In the works [9, 10], we generalized Chaitin's halting probability Ω to $Z(T)$ by

$$Z(T) = \sum_{p \in \operatorname{dom} U} 2^{-\frac{|p|}{T}} \qquad (T > 0).$$

Thus, $\Omega = Z(1)$. If $0 < T \leq 1$, then $Z(T)$ converges and $0 < Z(T) < 1$, since $Z(T) \leq \Omega < 1$. Note that $Z(T)$ is denoted as Ω^T in [9, 10] and $\Omega(T)$ in [12].

Theorem 2.5 (Tadaki [9, 10]). *Let $T \in \mathbb{R}$.*

(i) If $0 < T \leq 1$ and T is computable, then $Z(T)$ is weakly Chaitin T-random and T-compressible.

(ii) If $1 < T$, then $Z(T)$ diverges to ∞. □

Furthermore, [9, 10] showed that (i) $Z(T)$ is infinitely differentiable as a function of $T \in (0,1)$, and (ii) if T is computable, then the value of each derived function $d^k Z(T)/dT^k$ is weakly Chaitin T-random and T-compressible.

3 Main result

In this section, we prove the following main theorem of this paper.

Theorem 3.1 (fixed point theorem on partial randomness). *For every $T \in (0,1)$, if $Z(T)$ is a computable real number, then the following hold:*

(i) T is right-computable and not left-computable.

(ii) T is weakly Chaitin T-random and T-compressible.

(iii) $\lim_{n\to\infty} H(T_n)/n = T$. □

Theorem 3.1 follows immediately from Theorem 3.3, Theorem 3.5, and Theorem 3.6 below. Theorem 3.1 is just a fixed point theorem on partial randomness, where the computability of the value $Z(T)$ gives a sufficient condition for a real number $T \in (0,1)$ to be a fixed point on partial randomness. Note that $Z(T)$ is a monotonically increasing continuous function on $(0,1)$. In fact, [9, 10] showed that $Z(T)$ is a function of class C^∞ on $(0,1)$. Thus, since computable real numbers are dense in $\mathbb{R}$, we have the following corollary of Theorem 3.1.

Corollary 3.2. *The set $\{T \in (0,1) \mid T$ is weakly Chaitin T-random and T-compressible$\}$ is dense in $[0,1]$.* □

In order to prove Theorem 3.1, for each $k \in \mathbb{N}^+$, we define a function $\widetilde{Z}_k\colon (0,1) \to \mathbb{R}$ by

$$\widetilde{Z}_k(x) = \sum_{i=1}^{k} 2^{-\frac{|p_i|}{x}}.$$

Here, we choose any one recursive enumeration $p_1, p_2, p_3, \dots$ of the r.e. set $\operatorname{dom} U$ as the standard one for use throughout the rest of this paper. It follows that $\lim_{k\to\infty} \widetilde{Z}_k(x) = Z(x)$ for every $x \in (0,1)$, and $\sum_{i=1}^{\infty} 2^{-|p_i|} = \Omega = Z(1)$.

As a first step to prove Theorem 3.1, we prove the following theorem which gives the weak Chaitin T-randomness of T in Theorem 3.1.

Theorem 3.3. *For every $T \in (0,1)$, if $Z(T)$ is a right-computable real number, then T is weakly Chaitin T-random.*

Proof. First, for each $k \in \mathbb{N}^+$, we define a function $\widetilde{W}_k \colon (0,1) \to \mathbb{R}$ by

$$\widetilde{W}_k(x) = \sum_{i=1}^{k} |p_i|\, 2^{-\frac{|p_i|}{x}}.$$

We show that, for each $x \in (0,1)$, $\widetilde{W}_k(x)$ converges as $k \to \infty$. Let x be an arbitrary real number with $x \in (0,1)$. Since $x < 1$, there is $l_0 \in \mathbb{N}^+$ such that

$$\frac{1}{x} - \frac{\log_2 l}{l} \geq 1$$

for all $l \geq l_0$. Then there is $k_0 \in \mathbb{N}^+$ such that $|p_i| \geq l_0$ for all $i > k_0$. Thus, we see that, for each $i > k_0$,

$$|p_i|\, 2^{-\frac{|p_i|}{x}} = 2^{-\left(\frac{1}{x} - \frac{\log_2 |p_i|}{|p_i|}\right)|p_i|} \leq 2^{-|p_i|}.$$

Hence, for each $k > k_0$,

$$\widetilde{W}_k(x) - \widetilde{W}_{k_0}(x) = \sum_{i=k_0+1}^{k} |p_i|\, 2^{-\frac{|p_i|}{x}} \leq \sum_{i=k_0+1}^{k} 2^{-|p_i|} < \Omega.$$

Therefore, since $\{\widetilde{W}_k(x)\}_k$ is an increasing sequence of real numbers bounded to the above, it converges as $k \to \infty$, as desired. For each $x \in (0,1)$, we define a positive real number $W(x)$ as $\lim_{k\to\infty} \widetilde{W}_k(x)$.

On the other hand, since $Z(T)$ is right-computable by the assumption, there exists a total recursive function $g \colon \mathbb{N}^+ \to \mathbb{Q}$ such that $Z(T) \leq g(m)$ for all $m \in \mathbb{N}^+$, and $\lim_{m\to\infty} g(m) = Z(T)$.

We choose any one real number t with $T < t < 1$. Then, for each $i \in \mathbb{N}^+$, using the mean value theorem we see that

$$2^{-\frac{|p_i|}{x}} - 2^{-\frac{|p_i|}{T}} < \frac{\ln 2}{T^2}\, |p_i|\, 2^{-\frac{|p_i|}{t}} (x - T)$$

for all $x \in (T, t)$. We then choose any one $c \in \mathbb{N}$ with $W(t) \ln 2 / T^2 \leq 2^c$. Here, the limit value $W(t)$ exists, since $0 < t < 1$. It follows that

$$\widetilde{Z}_k(x) - \widetilde{Z}_k(T) < 2^c (x - T) \tag{4}$$

for all $k \in \mathbb{N}^+$ and $x \in (T, t)$. We also choose any one $n_0 \in \mathbb{N}^+$ such that $0.T_n + 2^{-n} < t$ for all $n \geq n_0$. Such n_0 exists since $T < t$ and $\lim_{n\to\infty} 0.T_n + 2^{-n} = T$. Since T_n is the first n bits of the base-two expansion of T with infinitely many zeros, we then see that $T < 0.T_n + 2^{-n} < t$ for all $n \geq n_0$.

Now, given T_n with $n \geq n_0$, one can find $k_0, m_0 \in \mathbb{N}^+$ such that

$$g(m_0) < \widetilde{Z}_{k_0}(0.T_n + 2^{-n}).$$

This is possible from $Z(T) < Z(0.T_n + 2^{-n})$, $\lim_{k\to\infty} \widetilde{Z}_k(0.T_n + 2^{-n}) = Z(0.T_n + 2^{-n})$, and the properties of g. It follows from $Z(T) \leq g(m_0)$ and (4) that

$$\sum_{i=k_0+1}^{\infty} 2^{-\frac{|p_i|}{T}} = Z(T) - \widetilde{Z}_{k_0}(T) < \widetilde{Z}_{k_0}(0.T_n + 2^{-n}) - \widetilde{Z}_{k_0}(T) < 2^{c-n}.$$

Hence, for every $i > k_0$, $2^{-\frac{|p_i|}{T}} < 2^{c-n}$ and therefore $Tn - Tc < |p_i|$. Thus, by calculating the set $\{ U(p_i) \mid i \leq k_0 \}$ and picking any one finite binary string which is not in this set, one can then obtain an $s \in \{0,1\}^*$ such that $Tn - Tc < H(s)$.

Hence, there exists a partial recursive function $\Psi\colon \{0,1\}^* \to \{0,1\}^*$ such that $Tn - Tc < H(\Psi(T_n))$ for all $n \geq n_0$. Using (1), there is $c_\Psi \in \mathbb{N}^+$ such that $H(\Psi(T_n)) < H(T_n) + c_\Psi$ for all $n \geq n_0$. Therefore, $Tn - Tc - c_\Psi < H(T_n)$ for all $n \geq n_0$. It follows that T is weakly Chaitin T-random. □

Remark 3.4. *By elaborating Theorem 2.5 (i), we can see that the left-computability of T results in the weak Chaitin T-randomness of $Z(T)$. On the other hand, by Theorem 3.3, the right-computability of $Z(T)$ results in the weak Chaitin T-randomness of T. We can integrate these two extremes into the following form: For every $T \in (0,1]$, there exists $c \in \mathbb{N}^+$ such that, for every $n \in \mathbb{N}^+$,*

$$Tn - c \leq H(T_n, (Z(T))_n), \tag{5}$$

where $H(s,t)$ is defined as $H(< s,t >)$ with any one computable bijection $< s,t >$ from $(s,t) \in \{0,1\}^* \times \{0,1\}^*$ to $\{0,1\}^*$ (see [3] for the detail of the notion of $H(s,t)$). In fact, if T is left-computable, then we can show that $H((Z(T))_n) = H(T_n, (Z(T))_n) + O(1)$, and therefore the inequality (5) results in the weak Chaitin T-randomness of $Z(T)$. On the other hand, if $Z(T)$ is right-computable, then we can show that $H(T_n) = H(T_n, (Z(T))_n) + O(1)$, and therefore the inequality (5) results in the weak Chaitin T-randomness of T.

Note, however, that the inequality (5) is not necessarily tight except for these two extremes, that is, the following inequality does not hold: For every $T \in (0,1]$,

$$H(T_n, (Z(T))_n) \leq Tn + o(n), \tag{6}$$

where $o(n)$ may depend on T in addition to n. To see this, contrarily assume that the inequality (6) holds. Then, by setting T to Chaitin's Ω, we have $H(\Omega_n) \leq H(\Omega_n, (Z(\Omega))_n) + O(1) \leq \Omega n + o(n)$. Since $\Omega < 1$, this contradicts the fact that Ω is weakly Chaitin random. Thus, the inequality (6) does not hold. □

The following Theorem 3.5 and Theorem 3.6 give the T-compressibility of T in Theorem 3.1 together.

Theorem 3.5. *For every $T \in (0,1)$, if $Z(T)$ is a right-computable real number, then T is also a right-computable real number.*

Proof. Since $Z(T)$ is right-computable, there exists a total recursive function $g\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $Z(T) \leq g(m)$ for all $m \in \mathbb{N}^+$, and $\lim_{m\to\infty} g(m) = Z(T)$. Thus, since $Z(x)$ is an increasing function of x, we see that, for every $x \in \mathbb{Q}$ with $0 < x < 1$, $T < x$ if and only if there are $m, k \in \mathbb{N}^+$ such that $g(m) < \widetilde{Z}_k(x)$. It follows from Theorem 2.1 (ii) that T is right-computable. □

The converse of Theorem 3.5 does not hold. To see this, consider an arbitrary computable real number $T \in (0,1)$. Then, obviously T is right-computable. On the other hand, $Z(T)$ is weakly Chaitin T-random by Theorem 2.5 (i), and obviously left-computable. Thus, $Z(T)$ is not right-computable.

Theorem 3.6. *For every $T \in (0,1)$, if $Z(T)$ is a left-computable real number and T is a right-computable real number, then T is T-compressible.*

Proof. For each $i \in \mathbb{N}^+$, using the mean value theorem we see that

$$2^{-\frac{|p_1|}{t}} - 2^{-\frac{|p_1|}{T}} > (\ln 2)\,|p_1|\,2^{-\frac{|p_1|}{T}}(t-T)$$

for all $t \in (T,1)$. We choose any one $c \in \mathbb{N}^+$ such that $(\ln 2)\,|p_1|\,2^{-\frac{|p_1|}{T}} \geq 2^{-c}$. Then, it follows that

$$\widetilde{Z}_k(t) - \widetilde{Z}_k(T) > 2^{-c}(t-T) \tag{7}$$

for all $k \in \mathbb{N}^+$ and $t \in (T,1)$.

Since T is a right-computable real number with $T < 1$, there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $T < f(l) < 1$ for all $l \in \mathbb{N}^+$, and $\lim_{l\to\infty} f(l) = T$. On the other hand, since $Z(T)$ is left-computable, there exists a total recursive function $g\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $g(m) \leq Z(T)$ for all $m \in \mathbb{N}^+$, and $\lim_{m\to\infty} g(m) = Z(T)$. Let β be the infinite binary string such that $0.\beta$ is the base-two expansion of $Z(1)$ (i.e., Chaitin's Ω). Note that β contains infinitely many ones, since $Z(1)$ is weakly Chaitin random.

Given n and $\beta_{\lceil Tn\rceil}$ (i.e., the first $\lceil Tn\rceil$ bits of β), one can find $k_0 \in \mathbb{N}^+$ such that

$$0.\beta_{\lceil Tn\rceil} < \sum_{i=1}^{k_0} 2^{-|p_i|}.$$

This is possible since $0.\beta_{\lceil Tn\rceil} < Z(1)$ and $\lim_{k\to\infty}\sum_{i=1}^{k} 2^{-|p_i|} = Z(1)$. It is then easy to see that

$$\sum_{i=k_0+1}^{\infty} 2^{-|p_i|} = Z(1) - \sum_{i=1}^{k_0} 2^{-|p_i|} < 2^{-\lceil Tn\rceil} \leq 2^{-Tn}.$$

Using the inequality $a^d + b^d \leq (a+b)^d$ for real numbers $a, b > 0$ and $d \geq 1$, it follows that

$$Z(T) - \widetilde{Z}_{k_0}(T) = \sum_{i=k_0+1}^{\infty} 2^{-\frac{|p_i|}{T}} < 2^{-n}. \tag{8}$$

Note that $\widetilde{Z}_{k_0}(T) < \widetilde{Z}_{k_0}(f(l))$ for all $l \in \mathbb{N}^+$, and $\lim_{l\to\infty}\widetilde{Z}_{k_0}(f(l)) = \widetilde{Z}_{k_0}(T)$. Thus, since $\widetilde{Z}_{k_0}(T) < Z(T)$, one can then find $l_0, m_0 \in \mathbb{N}^+$ such that

$$\widetilde{Z}_{k_0}(f(l_0)) < g(m_0).$$

It follows from (8) and (7) that

$$2^{-n} > g(m_0) - \widetilde{Z}_{k_0}(T) > \widetilde{Z}_{k_0}(f(l_0)) - \widetilde{Z}_{k_0}(T) > 2^{-c}(f(l_0) - T).$$

Thus, $0 < f(l_0) - T < 2^{c-n}$. Let t_n be the first n bits of the base-two expansion of the rational number $f(l_0)$ with infinitely many zeros. Then, $|\,f(l_0) - 0.t_n\,| < 2^{-n}$. It follows from $|\,T - 0.T_n\,| < 2^{-n}$ that $|\,0.T_n - 0.t_n\,| < (2^c+2)2^{-n}$. Hence

$$T_n = t_n,\ t_n \pm 1,\ t_n \pm 2,\ \ldots,\ t_n \pm (2^c+1),$$

where T_n and t_n are regarded as a dyadic integer. Thus, there are still $2^{c+1} + 3$ possibilities of T_n, so that one needs only $c+2$ bits more in order to determine T_n.

Thus, there exists a partial recursive function $\Phi\colon \mathbb{N}^+ \times \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ such that

$$\forall n \in \mathbb{N}^+ \quad \exists s \in \{0,1\}^* \quad |s| = c+2 \ \& \ \Phi(n, \beta_{\lceil Tn \rceil}, s) = T_n.$$

It follows from (2) that $H(T_n) \leq |\beta_{\lceil Tn \rceil}| + o(n) \leq Tn + o(n)$, which implies that T is T-compressible. □

Acknowledgments

The author is grateful to Prof. Shigeo Tsujii for the financial supports.

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