数理解析研究所講究録
第 1604 巻 2008 年 42-49

Bargaining Model on the Plane

Vladimir V. Mazalov
Institute of Applied Mathematical Research
Karelian Research Center of Russian Academy of Sciences
Pushkinskaya str. 11, Petrozavodsk 185910, Russia
e-mail: vmazalov@krc.karelia.ru

Julia S. Tokareva
Zabaikalsky State Humanitarian Pedagogical University
named after N.Tchernishevsky
Babushkin str. 129, Chita 672007,Russia
e-mail: jtokareva2@mail.ru

Abstact

Non-coalition non-zero-sum game related with arbitration procedure is considered. Two players (firms) propose some solutions on the plane. The arbtrator has his own solution which is modelling by random variables in the circle on the plane. As a solution of the conflict we use here the final offer arbitration procedure (FOA). The objective is the construction of Nash equilibrium. For two cases of the random solution of the arbitrator we found the equilibrium in pure strategies.

Key words: final offer arbitration, equilibrium, optimal strategies.

AMS Subject Classification: 91A05, 91A80

1 Introduction

Imagine the market where two different firms (players I and II) propose their expected costs for the building construction. The administration have choose one proposal. The first firm can build the building with cost x_1 for the time y_1 and second firm - with cost x_2 and time y_2.

Let the solution of the arbitrator is random variable z distributed in the unit circle with probablility density in polar coordinates $f(r, \theta)$. The objective of the players is to receive the order for the building.

2 Solution of the game

Let the point (x_1, y_1) on the plane corresponds to the offer of the player I and (x_2, y_2) corresponds to the offer II and A is the solution of the arbitrator. The arbitrator prefers a firm which is closer to the point A.

Let us divide the circle for two parts s_1 and s_2. The first part consists of the points of the circle the distance from which to the point (x_1, y_1) is smaler than to the point (x_2, y_2). The second part is opposite case.

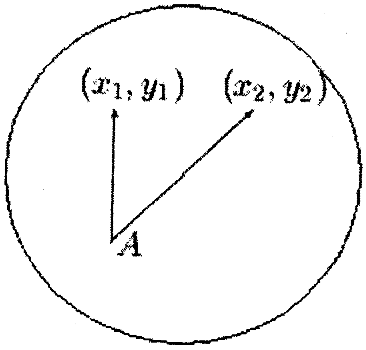

Fig. 1

Then the payoffs of the firms I and II are

$$H_1(r_1, \theta_1, r_2, \theta_2) = h_1(r_1, \theta_1) \cdot \mu(S_1),$$

$$H_2(r_1, \theta_1, r_2, \theta_2) = h_2(r_2, \theta_2) \cdot \mu(S_2) = h_2(r_2, \theta_2) \cdot (1 - \mu(S_1)), \quad (1)$$

where

$\mu(S_i) = \int_s f(r, \theta)\, dr d\theta$ is the probablity measure of S_i,

r_i is radius-vector of the point (x_i, y_i),

θ_i is the angle between Ox and radius-vector r_i,

$h_i(r_i, \theta_i)$ is payoff of the i-th firm if her proposal is closer to the arbitrator solution.

Let us find Nash equilibrium in this game. We find it from the conditions

$$\frac{\partial H_1}{\partial r_1} = \frac{\partial H_2}{\partial r_2} = 0,$$

where

$$\frac{\partial H_i}{\partial r_i} = \frac{\partial h_i(r_i, \theta_i)}{\partial r_i} \cdot \mu(S_i) + \frac{\partial \mu(S_i)}{\partial r_i} \cdot h_i(r_i, \theta_i). \quad (2)$$

Consider two different cases of distribution of the arbitrator.

3 Distribution of the "center" type

Let the distribution density of random variables z in polar coordinates has form:

$$f(r,\theta) = \frac{3(1-r)}{\pi}. \tag{3}$$

Consider the payoffs when the first firm maximises the first coordinate and the second firm - second coordinate.

Then the payoffs of the players are

$$H_1 = x_1(1-\mu(S_2)) = r_1(1-\mu(S_2)),$$
$$H_2 = y_2\mu(S_2) = r_2\mu(S_2). \tag{4}$$

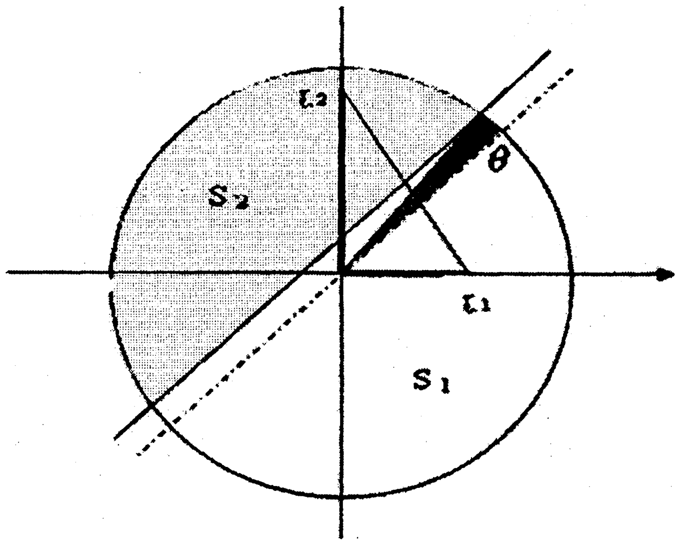

Fig.2

Calculate the measure of S_2. Denote

$$R = \frac{r_2^2 - r_1^2}{2\sqrt{r_2^2 + r_1^2}}.$$

We obtain

$$\mu(S_2) = 2\int\limits_{\arcsin R}^{\pi/2} d\theta \int\limits_{\frac{R}{\sin\theta}}^{1} r\frac{3(1-r)}{\pi}dr = \frac{6}{\pi}\left(\frac{\pi}{12} - \frac{1}{6}\arcsin R + \frac{R^3}{12}\ln\left|1+\sqrt{1-R^2}\right| - \right.$$

$$-\frac{R^3}{12}\ln\left|1-\sqrt{1-R^2}\right|-\frac{R}{3}\sqrt{1-R^2}\Bigg). \tag{5}$$

Consequently, the payoffs of players have the form:

$$H_1 = r_1\left(\frac{1}{2}+\frac{1}{\pi}\arcsin R-\frac{R^3}{2\pi}\ln\left|1+\sqrt{1-R^2}\right|+\right.$$

$$\left.+\frac{R^3}{2\pi}\ln\left|1-\sqrt{1-R^2}\right|+\frac{2R}{\pi}\sqrt{1-R^2}\right),$$

$$H_2 = r_2\left(\frac{1}{2}-\frac{1}{\pi}\arcsin R+\frac{R^3}{2\pi}\ln\left|1+\sqrt{1-R^2}\right|-\right.$$

$$\left.-\frac{R^3}{2\pi}\ln\left|1-\sqrt{1-R^2}\right|-\frac{2R}{\pi}\sqrt{1-R^2}\right). \tag{6}$$

Differentiating the first function in r_1 and second function in r_2 we obtain

$$\frac{\partial H_1}{\partial r_1}=\frac{1}{2}+\frac{1}{\pi}\arcsin R-\frac{R^3}{2\pi}\ln\left|1+\sqrt{1-R^2}\right|+\frac{R^3}{2\pi}\ln\left|1-\sqrt{1-R^2}\right|+$$

$$+\frac{2R}{\pi}\sqrt{1-R^2}+\frac{3r_1}{2\pi}\left(2\sqrt{1-R^2}R'_{r_1}+R^2\cdot R'_{r_1}\cdot\ln\left|1-\sqrt{1-R^2}\right|\right),$$

$$\frac{\partial H_2}{\partial r_2}=\frac{1}{2}-\frac{1}{\pi}\arcsin R+\frac{R^3}{2\pi}\ln\left|1+\sqrt{1-R^2}\right|-\frac{R^3}{2\pi}\ln\left|1-\sqrt{1-R^2}\right|-$$

$$-\frac{2R}{\pi}\sqrt{1-R^2}-\frac{3r_2}{2\pi}\left(2\sqrt{1-R^2}R'_{r_2}+R^2\cdot R'_{r_2}\cdot\ln\left|1-\sqrt{1-R^2}\right|\right), \tag{7}$$

where

$$R'_{r_1}=-\frac{r_1^3+3r_1r_2^2}{2\sqrt{(r_1^2+r_2^2)^3}},$$

$$R'_{r_2}=\frac{r_2^3+3r_2r_1^2}{2\sqrt{(r_1^2+r_2^2)^3}}.$$

Symmetry of the problem yields $r_1=r_2$ hence,

$$R=0,\ \ R'_{r_1}=-\frac{1}{\sqrt{2}}=-R'_{r_2}.$$

It follows

$$\left.\frac{\partial H_1}{\partial r_1}\right|_{r_2=r_1}=\frac{1}{2}-\frac{3r_1}{\sqrt{2}\pi},$$

$$\left.\frac{\partial H_2}{\partial r_2}\right|_{r_1=r_2}=\frac{1}{2}-\frac{3r_2}{\sqrt{2}\pi}. \tag{8}$$

Condition

$$\frac{\partial H_i}{\partial r_i} = 0$$

yields

$$r_i = \frac{\sqrt{2}\pi}{6}.$$

Consequently,

$$\theta_1 = 0,\ x_1 = \frac{\sqrt{2}\pi}{6},\ y_1 = 0,$$

$$\theta_2 = \frac{\pi}{2}\ ,x_2 = 0,\ y_2 = \frac{\sqrt{2}\pi}{6}. \tag{9}$$

Find the second derivative of the function H_1:

$$H''_{1_{r_1r_1}} = \frac{3}{\pi}\left(2\sqrt{1-R^2}\cdot R'_{r_1} + R^2\cdot R'_{r_1}\cdot \ln\left|1-\sqrt{1-R^2}\right|\right) +$$

$$+\frac{3r_1}{2\pi}\left(2R\cdot(R')^2\cdot\ln\left|1-\sqrt{1-R^2}\right| + R^2\cdot R''_{r_1r_1}\cdot\ln\left|1-\sqrt{1-R^2}\right| -\right.$$

$$\left. - R\cdot(R')^2\frac{1-\sqrt{1-R^2}}{\sqrt{1-R^2}}\right). \tag{10}$$

For $r_1 = r_2$ we obtain

$$H''_{1_{r_1r_1}} = \frac{-6}{\pi\sqrt{2}} < 0.$$

Consequently, $r_1 = \frac{\sqrt{2}\pi}{6}$ is the local maximum of H_1. Analogously, $H''_{2_{r_2r_2}} < 0$ and for $r_2 = \frac{\sqrt{2}\pi}{6}$ the function H_2 achieves the maximum.

The expected payoff is equal to

$$H = \frac{\sqrt{2}\pi}{12}.$$

4 Distribution of the "center and seaside" type

Let now the distribution density has form

$$f(r,\theta) = a + \frac{3}{2}\left(\pi^{-1} - a\right)r, \tag{11}$$

where a is some parameter. Consider the case when the first firm is interested to maximise $(y-x)$ and the second - $(x-y)$. Then the payoffs of the players are

$$H_1 = (y_1 - x_1)(1 - \mu(S_2)),$$
$$H_2 = (x_2 - y_2)\mu(S_2). \tag{12}$$

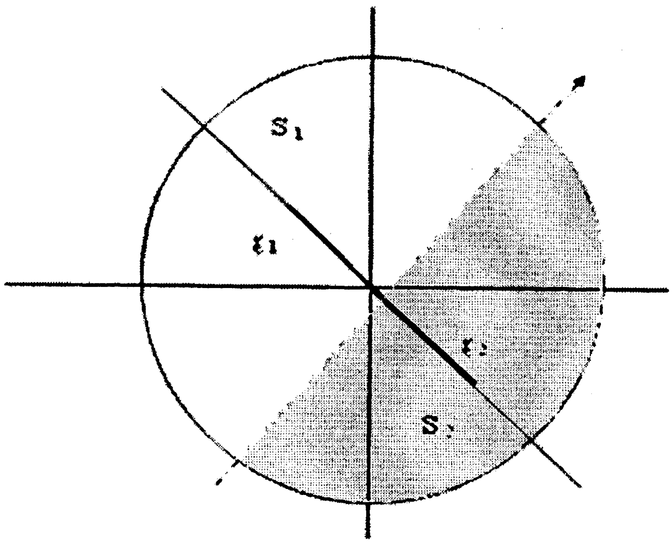

Fig.3

The identities

$$y_1 - x_1 = r_1 \sin\frac{3\pi}{4} - r_1 \cos\frac{3\pi}{4} = r_1\frac{\sqrt{2}}{2} - r_1\frac{\sqrt{2}}{2} = \sqrt{2}r_1,$$
$$x_2 - y_2 = r_2 \cos\left(-\frac{\pi}{4}\right) + r_2 \sin\left(-\frac{\pi}{4}\right) = \sqrt{2}r_2, \tag{13}$$

give the payoffs of the form

$$H_2 = 2\int_0^{\arccos\frac{r_2-r_1}{2}} d\theta \int_{\frac{r_2-r_1}{2\cos\theta}}^{1} dr \cdot r \cdot \left(a + \frac{3}{2}\left(\pi^{-1} - a\right) r\right) \cdot \sqrt{2}r_2 =$$

$$= \sqrt{2}r_2 \left[\frac{1}{\pi}\arccos\frac{r_2 - r_1}{2} - \frac{1}{32}\left(\frac{1}{\pi} - a\right)(r_2 - r_1)^3 \ln\left|\frac{2 + \sqrt{4 - (r_2 - r_1)^2}}{2 - \sqrt{4 - (r_2 - r_1)^2}}\right| + \right.$$
$$\left. + \frac{1}{8}\left(\frac{1}{\pi} - a\right)(r_2 - r_1)\sqrt{4 - (r_2 - r_1)^2}\right],$$

$$H_1 = \sqrt{2} r_1 \left[1 - \frac{1}{\pi} \arccos \frac{r_2 - r_1}{2} + \frac{1}{32} \left(\frac{1}{\pi} - a \right) (r_2 - r_1)^3 \ln \left| \frac{2 + \sqrt{4 - (r_2 - r_1)^2}}{2 - \sqrt{4 - (r_2 - r_1)^2}} \right| - \right.$$

$$\left. - \frac{1}{8} \left(\frac{1}{\pi} - a \right) (r_2 - r_1) \sqrt{4 - (r_2 - r_1)^2} \right]. \qquad (14)$$

Calculate the partial derivative $\frac{\partial H_i}{\partial r_i}$ and let it is equal to zero:

$$\frac{\partial H_1}{\partial r_1} = \sqrt{2} \left[1 - \frac{1}{\pi} \arccos \frac{r_2 - r_1}{2} + \right.$$

$$+ \frac{1}{32} \left(\frac{1}{\pi} - a \right) (r_2 - r_1)^3 \ln \left| \frac{2 + \sqrt{4 - (r_2 - r_1)^2}}{2 - \sqrt{4 - (r_2 - r_1)^2}} \right| -$$

$$\left. - \frac{1}{8} \left(\frac{1}{\pi} - a \right) (r_2 - r_1) \sqrt{4 - (r_2 - r_1)^2} \right] + \sqrt{2} r_1 \left[\frac{-1}{\pi \sqrt{4 - (r_2 - r_1)^2}} + \right.$$

$$\left. + \frac{1}{8} \left(\frac{1}{\pi} - a \right) \sqrt{4 - (r_2 - r_1)^2} - \frac{3}{32} \left(\frac{1}{\pi} - a \right) (r_2 - r_1)^2 \ln \left| \frac{2 + \sqrt{4 - (r_2 - r_1)^2}}{2 - \sqrt{4 - (r_2 - r_1)^2}} \right| \right],$$

$$\frac{\partial H_2}{\partial r_2} = \sqrt{2} \left[\frac{1}{\pi} \arccos \frac{r_2 - r_1}{2} + \frac{1}{32} \left(\frac{1}{\pi} - a \right) (r_2 - r_1)^3 \ln \left| \frac{2 + \sqrt{4 - (r_2 - r_1)^2}}{2 - \sqrt{4 - (r_2 - r_1)^2}} \right| - \right.$$

$$\left. - \frac{1}{8} \left(\frac{1}{\pi} - a \right) (r_2 - r_1) \sqrt{4 - (r_2 - r_1)^2} \right] - \sqrt{2} r_1 \left[\frac{-1}{\pi \sqrt{4 - (r_2 - r_1)^2}} + \right.$$

$$\left. + \frac{1}{8} \left(\frac{1}{\pi} - a \right) \sqrt{4 - (r_2 - r_1)^2} - \frac{3}{32} \left(\frac{1}{\pi} - a \right) (r_2 - r_1)^2 \ln \left| \frac{2 + \sqrt{4 - (r_2 - r_1)^2}}{2 - \sqrt{4 - (r_2 - r_1)^2}} \right| \right].$$

It yields

$$r_i = \frac{2\pi}{1 + \pi a}.$$

From here we obtain

$$\theta_1 = \frac{3\pi}{4}, \; x_1 = -\frac{\sqrt{2}\pi}{1 + \pi a}, \; y_1 = \frac{\sqrt{2}\pi}{1 + \pi a},$$

$$\theta_2 = -\frac{\pi}{4}, \; x_2 = \frac{\sqrt{2}\pi}{1 + \pi a}, \; y_2 = -\frac{\sqrt{2}\pi}{1 + \pi a}. \qquad (16)$$

The expected payoff of the firm is equal to

$$H = \frac{\sqrt{2}\pi}{1 + \pi a}.$$

The condition $r \geq 0$ gives the restriction for a as $a > -\frac{1}{\pi}$. In the table you find the values of r for some a.

Table

No	a	r
1	0	$2\pi \approx 6.28$
2	0.5	$\frac{2\pi}{1+0.5\pi} \approx 2.44$
3	1	$\frac{2\pi}{1+\pi} \approx 1.52$
4	3	$\frac{2\pi}{1+3\pi} \approx 0.6$
5	10	$\frac{2\pi}{1+10\pi} \approx 0.19$

References

[1] Farber H. *An analysis of final-offer arbitration*, Journal of conflict resolution, 24 (1980), 683-705.

[2] Mazalov V.,Sakaguchi M. *Location Game on the Plane*, International Game theory Review, 5 (2003), 13-25.