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# Communication Reaching Consensus through Robust Messages

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**Abstract.** We present a communication process in multi-agent system. In the model agents have the knowledge model associated with a partition and he/she makes decision by his/her private information with the knowledge model. Each agent sends not exact information on the decision value but approximate information with accuracy to $\varepsilon$. We show that consensus on the decision values for an event among all agents can still be guaranteed in the communication; i.e., all the decision values are equal after long running communication.
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## 1 Introduction

This article presents an extension of the communication system in Krasucki [6] into the communication model under approximate information with accuracy to $\varepsilon$. There are more than two agents and they interact in pairs with private announcement: Each agent has the posterior of an event under his/her private information, and he/she privately announces it to the another agent through robust messages; i.e., there is a possibility losing a bit information when the agent sends his message to the other agent. The recipient revises his/her information structure and recalculate the values of posterior under the approximate information on the posterior. The agent sends the revised posterior to another agent according to a communication graph. The recipient revises his/her posterior and send it to another, and so on. In the circumstances we can show that

**Theorem 1.** *Suppose that all agents have a common prior distribution. Consensus on the limiting values of the posteriors for an event under his/her private information among all agents can still be guaranteed in the communication even when each agent sends not exact information on the posterior value but approximate information on it with accuracy to $\varepsilon$ through robust messages.*

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Recently researchers in such fields as economics, AI, and theoretical computer science have become interested in reasoning of belief and knowledge. There are pragmatic concerns about the relationship between knowledge (belief) and actions. Of most interest to us is the emphasis on situations involving the knowledge (belief) of a group of agents rather than that of a single agent. At the heart of any analysis of such situations as a conversation, a bargaining session or a protocol run by processes is the interaction between agents. An agent in a group must take into account not only events that have occurred in the world but also the knowledge of the other agents in the group.

In some cases we need to consider the situation that the agents has common-knowledge of an event; that is, simultaneously everyone knows the event, everyone knows that everyone knows the event, and so on. This notion also turns out to be a prerequisite for achieving agreement: In fact, Aumann [1] showed the famous agreement theorem; that is, if all posteriors of an event are common-knowledge among the agents then the posteriors must be the same, even when they have different private information. This is precisely what makes it a crucial notion in the analysis of an interacting group of agents.[1]

Because the notion of common-knowledge is defined by the infinite regress of all agents' knowledge as above, common-knowledge is actually so unfeasible a tool in helping us analyse complicated situations involving groups of agents. Thus we would like to remove it from our modelling.

In this regard, Geanakoplos and Polemarchakis [5] investigated a communication process in which two agents announce their posteriors to each other. In the process agents learn and revise their posteriors and they reach consensus without common-knowledge of an event. Furthermore, Krasucki [6] introduced the revision process mentioned in the above. He showed that in the process, consensus on the posteriors can be guaranteed if the communication graph contains no cycle. The result is an extension of the agreement theorem of Aumann [1]. In the communication model of Krasucki [6], the agents sends his/her exact information on the posterior to the another agents. There is no possibility losing information.

All of the information structures in the models of Aumann [1], of Geanakoplos and Polemarchakis [5] and that of Krasucki [6] are given by partition on a state space. Bacharach [2] showed that the information partition model is equivalent to his knowledge operator model with the three axioms about the operators: **T** axiom of knowledge (what is known is true), **4** axiom of transparency (that we know what we do know) and **5** axiom of wisdom (that we know what we do not know.) He pointed out that the assumptions for the partition are problematic in decision making, and hence the model of analysing complicated situations should be also constructed without such strong assumptions.

Matsuhisa and Kamiyama [7] introduced the lattice structure of knowledge for which the requirements such as the three axioms are not imposed, and they succeeded in extending Aumann's theorem to their model. However the extension

[1] C.f.: Fagin et al [4].

of agreement theorem are established under the common-knowledge (or common-belief) assumption.

The purpose of this article is to extend the communication model of Krasucki [6] through robust messages. The emphasis is on that each agent sends not exact information on the posterior value but robust information on it with accuracy to $\varepsilon$. Under the circumstances we show Theorem 1, which is an extension of Krasucki [6], because the result of Krasucki [6] coincides with Theorem 1 when $\varepsilon = 1$.

## 2 The Model

Let $N$ be a set of finitely many *agents* and $i$ denote an agent. A *state-space* is a finitely non-empty set, whose members are called *states*. An *event* is a subset of the state-space. If $\Omega$ is a state-space, we denote by $2^\Omega$ the field of all subsets of it. An event $E$ is said to *occur* in a state $\omega$ if $\omega \in E$.

### 2.1 Information and Knowledge[2]

A *partition information structure* $\langle \Omega, (\Pi_i)_{i\in N}\rangle$ consists of a state space $\Omega$ and a class of the mappings $\Pi_i$ of $\Omega$ into $2^\Omega$ such that

(i) $\{\Pi_i(\omega) \mid \omega \in \Omega\}$ is a partition of $\Omega$;

(ii) $\omega \in \Pi_i(\omega)$ for every $\omega \in \Omega$.

Given our interpretation, an agent $i$ for whom $\Pi_i(\omega) \subseteq E$ knows, in the state $\omega$, that some state in the event $E$ has occurred. In this case we say that in the state $\omega$ the agent $i$ knows $E$.

**Definition 1.** The *knowledge structure* $\langle \Omega, (\Pi_i)_{i\in N}, (K_i)_{i\in N}\rangle$ consists of a partition information structure $\langle \Omega, (\Pi_i)_{i\in N}\rangle$ and a class of $i$'s *knowledge operator* $K_i$ on $2^\Omega$ such that $K_iE$ is the set of states of $\Omega$ in which $i$ knows that $E$ has occurred; that is,

$$K_iE = \{\omega \in \Omega \mid \Pi_i(\omega) \subseteq E\}.$$

The set $\Pi_i(\omega)$ will be interpreted as the set of all the states of nature that $i$ knows to be possible at $\omega$, and $K_iE$ will be interpreted as the set of states of nature for which $i$ knows $E$ to be possible. We will therefore call $\Pi_i$ $i$'s *possibility operator* on $\Omega$ and also will call $\Pi_i(\omega)$ $i$'s *information set* at $\omega$.

We record the properties of $i$'s knowledge operator[3]: For every $E, F$ of $2^\Omega$,

**N** $\quad K_i\Omega = \Omega$ and $K_i\emptyset = \emptyset$ ; $\qquad$ **K** $\quad K_i(E\cap F) = K_iE \cap K_iF$;

[2] C.f.; Fagin et al [4], Binmore [3] for the information structure and the knowledge operator.

[3] According to these properties we can say the structure $\langle \Omega, (K_i)_{i\in N}\rangle$ is a model for the multi-modal logic **S5**.

**T** $K_iF \subseteq F$;  **4** $K_iF \subseteq K_iK_iF$;

**5** $\Omega \setminus K_i(E) \subseteq K_i(\Omega \setminus K_i(E))$.

*Remark 1.* $i$'s possibility operator $\Pi_i$ is uniquely determined by $i$'s knowledge operator $K_i$ satisfying the above five properties, because $\Pi_i(\omega) = \bigcap_{\omega \in K_i E} E$.

### 2.2 Decision function

By $i$'s *decision function* we mean a mapping $f_i$ of $2^\Omega$ into $\mathbf{R}$. It is said to satisfy the *sure thing principle* if it is preserved under disjoint union; that is, for every pair of disjoint events $S$ and $T$ such that if $f_i(S) = f_i(T) = d$ then $f_i(S \cup T) = d$. A decision function $f_i$ is said to be *convex* if for disjoint two events $E, F$, there are positive numbers $\lambda, \delta \in (0,1)$ such that $f_i(E \cup F) = \lambda f_i(E) + \delta f_i(F)$ with $\lambda + \delta = 1$. It is *preserved under difference* if for all events $S$ and $T$ such that $S \subseteq T, f_i(S) = f_i(T) = d$ then we have $f_i(T \setminus S) = d$. All agents have the *common decision function* $f$ if for every $i, j \in N$, $f = f_i = f_j$.

If $f_i$ is intended to be a posterior probability, we assume given a probability measure $\mu$ which is common for all agents and some event $X$. Then $f_i$ is the mapping of the domain of $\mu$ into the closed interval [0, 1] such that $f(E) = \mu(X|E)$, where $\mu(E) \neq 0$. We plainly observe that this $f_i$ satisfies the sure thing principle and is preserved under difference, and the agents have the common decision function $f$.

### 2.3 Protocol [4]

We assume that the agents communicate by sending *messages*. Let $T$ be the time horizontal line $\{0, 1, 2, \cdots t, \cdots\}$. A *protocol* is a mapping $\Pr : T \to N \times N, t \mapsto (s(t), r(t))$ such that $s(t) \neq r(t)$. Here $t$ stands for *time* and $s(t)$ and $r(t)$ are, respectively, the *sender* and the *recipient* of the communication which takes place at time $t$. We consider the protocol as the directed graph whose vertices are the set of all agents $N$ and such that there is an edge (or an arc) from $i$ to $j$ if and only if there are infinitely many $t$ such that $s(t) = i$ and $r(t) = j$.

A protocol is said to be *fair* if the graph is strongly-connected; in words, every agent in this protocol communicates directly or indirectly with every other agent infinitely often. It is said to contain a *cycle* if there are at least agents $i_1, i_2, \ldots, i_k$ with $k \geq 3$ such that for all $m < k$, $i_m$ communicates directly with $i_{m+1}$, and such that $i_k$ communicates directly with $i_1$. The communications is assumed to proceed in *rounds*[5]

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[4] C.f.: Parikh and Krasucki [8]

[5] There exists a time $m$ such that for all $t$, $\Pr(t) = \Pr(t+m)$. The *period* of the protocol is the minimal number of all $m$ such that for every $t$, $\Pr(t+m) = \Pr(t)$.

### 2.4 Communication through robust messages

Let $\varepsilon$ be a real number with $0 \leq \varepsilon < 1$. An *$\varepsilon$-robust communication system* $\pi^{\varepsilon}$ with revisions of agents' decision $(f_i^t)_{(i,t)\in N\times T}$ according to a protocol is a tuple

$$\pi^{\varepsilon} = \langle \Omega, \mathrm{Pr}, (\Pi_i^t)_{i\in N}, (f_i^t)_{(i,t)\in N\times T}\rangle$$

with the following structures: the agents have a common prior $\mu$ on $\Omega$, the protocol Pr among $N$, $\mathrm{Pr}(t) = (s(t), r(t))$, is fair and it satisfies the conditions that $r(t) = s(t+1)$ for every $t$ and that the communications proceed in rounds. The revised information structure $\Pi_i^t$ at time $t$ is the mapping of $\Omega$ into $2^{\Omega}$ for agent $i$. If $i = s(t)$ is a sender at $t$, the *message* sent by $i$ to $j = r(t)$ is $M_i^t$. An $n$-tuple $(f_i)_{i\in N}$ is a revision system of individual conjectures. These structures are inductively defined as follows:

- Set $\Pi_i^0(\omega) = \Pi_i(\omega)$.
- Assume that $\Pi_i^t$ is defined. It yields $i$'s decision $d_i^t(\omega) = f_i(\Pi_i^t(\omega))$. Whence the message $M_i^t : \Omega \to 2^{\Omega}$ sent by the sender $i$ at time $t$ is defined as a robust information:

$$M_i^t(\omega) = \left\{\xi \in \Omega \mid \left|d_i^t(\xi) - d_i^t(\omega)\right| \leq \varepsilon \right\}.$$

  Then:
- The revised partition $\Pi_i^{t+1}$ at time $t+1$ is defined as follows:
  - $\Pi_i^{t+1}(\omega) = \Pi_i^t(\omega) \cap M_{s(t)}^t(\omega)$ if $i = r(t)$;
  - $\Pi_i^{t+1}(\omega) = \Pi_i^t(\omega)$ otherwise,

The specification is that a sender $s(t)$ at time $t$ informs the recipient $r(t)$ his/her decision as approximate information to an accuracy $\varepsilon$. The recipient revises her/his information structure under the information. She/he revises her/his decision according the robust message, and she/he informs her/his the revised decision to the other agent $r(t+1)$.

We denote by $\infty$ a sufficient large $\tau \in T$ such that for all $\omega \in \Omega$, $d_i^{\tau}(\cdot\,;\omega) = d_i^{\tau+1}(\cdot\,;\omega) = d_i^{\tau+2}(\cdot\,;\omega) = \cdots$. Hence we can write $d_i^{\tau}$ by $d_i^{\infty}$.

*Remark 2.* The message $M_{s(t)}^t(\omega)$ is called *exact* if $\epsilon = 0$. In his paper [6] Krasucki treats the 0-robust communication system.

### 2.5 Consensus

We note that the limit $\Pi_i^{\infty}$ exists. [6] We denote $d_i^{\infty}(\omega) = f(\Pi_i^{\infty}(\omega))$ called the *limiting decision* of $f$ at $\omega$ for $i$. We say that *consensus* on the limiting decisions can be guaranteed if $d_i^{\infty}(\omega) = d_j^{\infty}(\omega)$ for each agent $i, j$ and in all the states $\omega$.

[6] Because $\Omega$ is finite, the descending chain $\{\Pi_i^t(\omega) \mid t = 0, 1, 2, \ldots\}$ is finite, and so it must be stationary.

## 3 Proof of Theorem 1

We can observe that Theorem 1 is a corollary of Theorem 2 on noting that the protocol Pr is fair.

**Theorem 2.** *Let $\pi = \langle \Pr, (\Pi_i^t), f \rangle$ be a communication system with $f$ the common decision function. Suppose that the common decision function $f$ is preserved under difference, is convex and satisfies sure thing principle. Then Consensus on the limiting values of the decision function can be guaranteed; i.e., $d_i^\infty(\omega) = d_j^\infty(\omega)$ for every $\omega$ and for all $i, j$.*

**Proof of Theorem 2 (Sketch)** Let us consider the case that $(i, j) = (s(\infty), t(\infty))$. For each state $\omega$ we denote $M_i = M_i^\infty(\omega)$. We can observe the two points: First that $M_i$ can be decomposed into the disjoint union of $\Pi_i^\infty(\xi)$ for $\xi \in M_i$, and secondly that $f(M_i) = \sum_{k=1}^m \lambda_k f(\Pi_j^\infty(\xi_k))$ for some $\lambda_k > 0$ with $\sum_{k=1}^m \lambda_k = 1$. It follows that for all $\omega \in \Omega$, there is some $\xi_\omega \in \Pi_i^\infty(\omega)$ such that $f(M_i) = d_i^\infty(\omega) \leq d_j^\infty(\xi_\omega)$. Next we shall proceed in the general case. Continuing the above system according to the *fair* protocol, we can verify the two facts: For each $\omega \in \Omega$

1. For every $i \neq j$, $d_i^\infty(\omega) \leq d_j^\infty(\xi)$ for some $\xi \in \Omega$; and
2. $d_i^\infty(\omega) \leq d_i^\infty(\xi) \leq d_i^\infty(\zeta) \leq \cdots$ for some $\xi, \zeta, \cdots \in \Omega$.

Hence it follows that $d_i^\infty(\omega) = d_j^\infty(\omega)$ for every $\omega$ and for all $i, j$. □

## 4 Concluding remarks

Our real concern in this article is about relationship between agents' beliefs and their decision making, especially when and how the agents reach consensus about their decisions. We focus on extending the agreeing theorem of Aumann [1] to an$\varepsilon$-robust communication system in the line of Krasucki [6]. We have shown that the nature that consensus can be guaranteed in a communication system is dependent not on exact information that each agent sends through messages, but on the updating his/her knowledge associated with information partition when each agent receives robust information. Another emphasis is on that we require none of the topological assumptions on the communication graph.

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