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ON THE BERGMAN FUNCTION FOR SEMIPOSITIVE HOLOMORPHIC LINE BUNDLES

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1. Introduction

Let M be a compact complex manifold of dimension n and let $(L, h) \to M$ be a Hermitian holomorphic line bundle. Let g be a Kähler metric on M and ω_g the Kähler form associated to g. For each $m \in \mathbb{N}$, h naturally induces a Hermitian metric h_m on $L^m := L \otimes \cdots \otimes L$ (m times). Let $\{S_0^m, \ldots, S_{d_m-1}^m\}$ be an orthonormal basis of the space $H^0(M, L^m)$ of all holomorphic global sections of L^m with respect to the inner product $(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x))dV_g$, where $dV_g = \frac{1}{n!}\omega_g^n$ is the volume form of g. The Bergman function of $H^0(M, L^m)$ is the function $B_m : M \to \mathbb{R}$ defined by

$$B_m(x) = \sum_{j=0}^{d_m-1} h_m(S_j^m(x), S_j^m(x)).$$

First, let us consider the case that L is an ample line bundle. Here we take a Hermitian metric h on L whose Ricci form is equal to ω_g. Tian [9], Zelditch [10] and Catlin [3] give the following asymptotic expansion of the Bergman function of $H^0(M, L^m)$: for arbitrary $N \in \mathbb{N}$, $k \in \mathbb{Z}_+$,

$$\left\| B_m(x) - m^n \left(a_0 + \frac{a_1(x)}{m} + \cdots + \frac{a_N(x)}{m^N} \right) \right\|_{C^k} < \frac{C_{N,k}}{m^{N+1-n}}, \tag{1.1}$$

where $C_{N,k}$ is a positive constant depending only on k, N and the manifold M and the norm is the C^k-norm. Moreover, Lu [7] shows that each coefficient a_j is a polynomial of the curvature and its covariant derivatives at x and this polynomial is obtained by finitely many steps of algebraic operations. More explicitly, he expresses the coefficients ($a_0 = 1$), a_1, a_2, a_3 by using the Ricci curvature and the scalar curvature of g and so on.

The above expansion (1.1) has many important application to many mathematical area, for example, complex geometry and differential geometry, but the analysis to compute (1.1) itself is also very interesting. As we know, there are three kinds of method to compute the asymptotic expansion. The first method, due to Zelditch [10] and Catlin [3], is based on the asymptotic formula of the Szegö kernel for strictly pseudoconvex domains given by Boutet de Monvel-Sjöstrand [2]. Note that

[2] extends an earlier result of C. Fefferman [5]. In fact, since the Bergman function B_m appears in the coefficients of the Fourier expansion of the Szegö kernel on the unit circle bundle over M, the asymptotic formula of the Szegö kernel in [2] induces the expansion (1.1). The second method has been developed by Tian [9], Ruan [8] and Lu [7]. Tian [9] constructs nice L^2-holomorphic sections of L^m, which are called as *global peak sections*, by the standard $\bar{\partial}$-estimate of Hörmander [6]. For a fixed point x on M, these sections can be approximated by monomials near x and their behavior can be completely controlled around x (see Section 2.2). In [7], after the Bergman function is expressed by using these sections, through many ingenious arguments, the problem is reduced to the asymptotic analysis of the integral:

$$\int_U f(z)e^{-m\varphi(z)}dV_g(z) \tag{1.2}$$

where f, φ are smooth near the origin and U is a small neighborhood of the origin. In this case, φ is approximated by $|z_1|^2+\cdots+|z_n|^2$ near the origin. As a result, many interesting analysis of the integral (1.2) implies the asymptotic expansion (1.1). This method is also valuable for the explicit computation of the coefficients. The third method was found by Berman-Berndtsson-Sjöstrand [1]. By using simple Fourier integral operators and the standard L^2-estimate of [6], they give a direct proof of the existence of the asymptotic expansion (1.1).

Now, without the assumption of the positivity of (L, h), there is no analogous result on the asymptotics of the Bergman function at present. In this note, we announce a result for the asymptotic expansion of the Bergman function in a special case of semipositive (L, h), but our result can be considered as a generalization of (1.1) in the positive case. Roughly speaking, our asymptotic expansion is expressed by a Puiseux type expansion.

Let us explain our method to compute the asymptotic expansion. Our analysis is based on the above second method, which was developed in [9],[8],[7]. Unfortunately, Tian's argument about the peak sections does not work in the degenerate case, so we deal with the only case that $H^0(M, L^m)$ has global peak sections. Note that it holds, if L is ample. In a similar argument to [8], [7], the problem is reduced to the analysis of the integral (1.2). Here, φ is approximated by some quasihomogeneous polynomial. In our case, the computation of the integral (1.2) is much more complicated and interesting. In particular, the concept *Newton polyhedra* plays an important role in the analysis in the degenerate case. Owing to Newton polyhedra, we can understand the structure of the asymptotic behavior of (1.2) clearly. Finally, we obtain an asymptotic expansion of the Bergman function of Puiseux type as in Theorem 2.2, below. Details will appear in [4] near future.

2. Main result

2.1. **Newton diagram.** The concept "Newton diagram" is very useful for our analysis. $\mathbb{Z}_+ = \mathbb{Z} \cap \{x \geq 0\}, \mathbb{R}_+ = \mathbb{R} \cap \{x \geq 0\}$.

Let F be a C^∞-smooth function in a neighborhood of the origin in $\mathbb{C}^n$ with $F(0) = 0$. Let

$$\sum_{I,J \in \mathbb{Z}_+^n} C_{IJ} z^I \bar{z}^J = \sum_{I,J \in \mathbb{Z}_+^n} C_{i_1,\dots,i_n,j_1,\dots,j_n} z_1^{i_1} \cdots z_n^{i_n} \bar{z}_1^{j_1} \cdots \bar{z}_n^{j_n}$$

be the Taylor series of F at the origin. Then the *support* of F is $S_F = \{I + J \in \mathbb{Z}_+^n ; C_{IJ} \neq 0\}$ and the *Newton polyhedron* of F is

$$\Gamma_+(F) = \text{the convex hull of the set } \{I + J + \mathbb{R}_+^n ; I + J \in S_F\}.$$

The *Newton diagram* $\Gamma(F)$ of F is the union of the compact faces of the Newton polyhedron $\Gamma_+(F)$. The *principal part of* F is

$$F_0(z) = \sum_{I+J \in \Gamma(F)} C_{IJ} z^I \bar{z}^J.$$

F is said to be *convenient*, if $\Gamma(F)$ intersects all the coordinate axes.

2.2. **Global peak sections.** Let x_0 be a point on M and $z = (z_1, \dots, z_n)$ a local coordinate at x_0 with $z(x_0) = 0$. We say that $H^0(M, L^m)$ *has global peak sections with respect to* x_0, z, if the following holds:

There exists a neighborhood U of x_0 as follows. For any $P = (p_1, \dots, p_n) \in \mathbb{Z}^n$ and any integer $p' > |P| = p_1 + \cdots + p_n$, there exists a positive integer m_0 such that for $m \geq m_0$, there is a holomorphic global section $S_{P,m}$ in $H^0(M, L^m)$ satisfying

$$\int_M \|S_{P,m}\|_{h_m}^2 dV_g = 1, \quad \int_{M \setminus U} \|S_{P,m}\|_{h_m}^2 dV_g = O\left(\frac{1}{m^{p'}}\right)$$

and $S_{P,m}$ can be represented as

$$S_{P,m} = \lambda_P z^P \chi_U(z) e_L^m + u_{P,m},$$

where e_L^m is a local holomorphic frame of L around x_0, such that

$$u_{P,m} = O\left(|z|^{2p'}\right) \quad \text{on } U,$$

and

$$\int_M \|u_{P,m}\|_{h_m}^2 dV_g = O\left(\frac{1}{m^{2p'}}\right),$$

where χ_U is the characteristic function for U. Moreover

$$\lambda_P^{-2} = \int_U \left|z^P\right|^2 a^m dV_g.$$

Here a is a local representation of h.

Remark 2.1. Tian [9] essentially shows that if (L, h) is positive, then $H^0(M, L^m)$ has global peak sections. Actually, he shows this property in the case of $\{z; |z| < (\log m)/\sqrt{m}\}$ instead of U. But in the case of U, one can prove the existence of peak sections much easiler. We also remark that if L is ample, then $H^0(M, L^m)$ has always global peak sections.

2.3. **Asymptotic expansion.** The following is our main result.

Theorem 2.2. *Let M be a compact complex manifold of dimension n, $(L, h) \to M$ a Hermitian holomorphic line bundle and x_0 a point on M. Suppose that there exists a local coordinate $z = (z_1, \ldots, z_n)$ at x_0 such that $H^0(M, L^m)$ has global peak sections with respect to x_0, z and such that the Hermitian metric h can be represented by the function $a(z)$ near x_0 satisfying that the principal part $\varphi_0(z)$ of $-\log a(z)$ has the following propertirs:*

(i) *$\varphi_0(z)$ is convenient and is positive away from 0,*
(ii) *$\varphi_0(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) = \varphi_0(z)$ for any $\theta_j \in \mathbb{R}$,*
(iii) *$\varphi_0(t^{\frac{1}{2k_1}}z_1, \ldots, t^{\frac{1}{2k_n}}z_n) = t\varphi_0(z)$ for any $t > 0$ where $k_1, \ldots, k_n$ are natural numbers.*

Then the Bergman function B_m of $H^0(M, L^m)$ admits the following asymptotic expansion at x_0: for arbitrary $N \in \mathbb{N}$,

$$\left| B_m(x_0) - m^{\sum_{j=1}^n 1/k_j}\left(a_0 + \frac{a_1}{m^{1/k}} + \cdots + \frac{a_N}{m^{N/k}}\right)\right| < \frac{C_N}{m^{(N+1)/k - \sum_{j=1}^n 1/k_j}},$$

where k is the least common multiple of $\{k_1, \ldots, k_n\}$, $a_0, a_1, \ldots, a_N$ are real numbers, with $a_0 > 0$, and C_N is a positive constant depending only on N and the manifold M.

Remark 2.3. The property (iii) means that the Newton diagram of φ_0 has only one face. A simple example of φ_0 is $|z_1|^{2k_1} + \cdots + |z_n|^{2k_n}$.

Remark 2.4. If L is positive, then we can take the local coordinate such that $\varphi_0(z) = |z_1|^2 + \cdots + |z_n|^2$. By Remark 2.1, our result can be considered as a generalization of the asymptotic expansion (1.1) in the Introduction.

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