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The Kodaira dimension of subvarieties of Siegel modular varieties

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1 Introduction

Let t be a positive integer, and $\mathcal{A}_{g,t}$ the moduli space of g-dimensional abelian varieties with polarizations of type $T=(1,\ldots,1,t)$. We write $\widetilde{\mathcal{A}}_{g,t}$ for a smooth compactification of $\mathcal{A}_{g,t}$. It is known that $\widetilde{\mathcal{A}}_{g,t}$ is of general type in the following cases:
(1)(Tai, Freitag, Mumford) $t=1$, $g\geq 7$,
(3)(Gritsenko) $t=2$, $g\geq 13$,
(2)(Tai) $t\neq 1,2$, $g\geq 16$.

We have the same result for subvarieties in $\mathcal{A}_{g,t}$. To be more precise, Freitag, Weissauer and Tsuyumine showed that in the case where $g\geq 10$, $t=1$, all subvarieties of codimension one in $\mathcal{A}_{g,t}$ are of general type. Here they adopted the weakened form of the notion "general type". The purpose of the present paper is to report a similar result for the case where $t\neq 1$. Our main result is the following:

Theorem (Theorem 7) Assume $g\geq 13$. If $\widetilde{\mathcal{A}}_{g,t}$ is of general type, then any irreducible variety in $\mathcal{A}_{g,t}$ is of general type.

Throughout this article, we assume $g\geq 13$.

2 Siegel modular varieties

Let $H_g=\{Z\in M_g(\mathbb{C})\mid {}^tZ=Z,\ \mathrm{Im}\,Z>0\}$ denote the Siegel upper half space. The symplectic group $Sp_{2g}(\mathbb{R})$ acts on H_n by the usual symplectic substitution

$$\gamma\to\gamma Z=(AZ+B)(CZ+D)^{-1},\quad \gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in Sp_{2g}(\mathbb{R}).$$

For a positive integer t, let

$$\Delta_t=\begin{pmatrix}1&&&\\&\ddots&&\\&&1&\\&&&t\end{pmatrix},\quad \Lambda_t=\begin{pmatrix}O&\Delta_t\\-\Delta_t&O\end{pmatrix}.$$

We now define some kinds of modular groups. Define

$$\widetilde{\Gamma}_t=\left\{\gamma\in GL(2g,\mathbb{Z})\mid \gamma\Lambda_t{}^t\gamma=\Lambda_t\right\}.$$

Using $R = \left(\begin{smallmatrix} E & O \\ O & \Delta_t \end{smallmatrix}\right)$, put $\Gamma_t = R^{-1}\widetilde{\Gamma}_t R$. For $L = \mathbb{Z}^{2g} \subset \mathbb{C}^g$, $\langle\, ,\, \rangle : L \times L \to \mathbb{C}$ is defined by $(x, y) \mapsto x\Lambda_t {}^t y$. We denote by $L^\vee$ the dual lattice of L with respect to $\langle\, ,\, \rangle$. Put

$$\Gamma_t^{\mathrm{lev}} = \left\{\gamma \in \Gamma_T \mid M|_{L^\vee/L} = \mathrm{id}|_{L^\vee/L}\right\}.$$

We call Γ_t and Γ_t^{lev} *Siegel modular groups*. When $t = 1$, we write $\Gamma_g = \Gamma_t$. Concerning Γ_t^{lev}, it is well known that Γ_t^{lev} is a subgroup of $Sp_{2g}(\mathbb{Z})$, and that Γ_t^{lev} is a normal subgroup of finite index in Γ_t.

Let Γ be a Siegel modular group occurring in the above. For a function f on H_g and a matrix $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, the *slash-operator* is defined by

$$f|_k\gamma(Z) = \det(CZ + D)^{-k} f(\gamma Z).$$

A holomorphic function f on H_g is a Γ*-modular form of weight* k on H_g if for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$,

$$f|_k\gamma(Z) = f(Z).$$

Define $\mathcal{A}_{g,t} = \Gamma_t \setminus H_g$, $\mathcal{A}_{g,t}^{\mathrm{lev}} = \Gamma_t^{\mathrm{lev}} \setminus H_g$. In particular, write $\mathcal{A}_g = \mathcal{A}_{g,1}$. The quotient spaces $\mathcal{A}_{g,t}$ and $\mathcal{A}_{g,t}^{\mathrm{lev}}$ are the moduli spaces of g-dimensional abelian varieties of the polarization of type T without or with canonical structure, respectively. Let $\mathcal{A}$ be a one of them. From the theory of the toroidal compactification, we can construct a projective variety $\overline{\mathcal{A}}$ such that $\overline{\mathcal{A}} - \mathcal{A}$ has normal crossing and $\overline{\mathcal{A}}$ has only finite quotient singularities. Resolving these singularities, we obtain a projective nonsingular variety $\widetilde{\mathcal{A}}$. We call $\overline{\mathcal{A}}$ and $\widetilde{\mathcal{A}}$ *Siegel modular varieties*. These varieties are central objects in this paper.

Freitag defined in [1] the following weakened form of the notion "general type".

Definition 1 A nonsingular compact irreducible algebraic variety X is *of type* G (*of general type*) if there exist $n = \dim X$ algebraically independent rational functions $f_1, \ldots, f_n$ and a non-zero holomorphic tensor $T \in \Omega^{\otimes d}(X)$ $(d > 0)$ such that tensors $f_1T, \ldots, f_nT$ are holomorphic on X

We adopt this notion for subvarieties of Siegel modular varieties.

3 Construction of certain differential forms

Let $Z = (z_{ij})$. Define

$$e_{ij} = \begin{cases} 1 & (i \neq j) \\ 2 & (i = j) \end{cases}.$$

Using it, put

$$\omega_{ij} = (-1)^{i+j} e_{ij} dz_{11} \wedge dz_{12} \wedge \cdots \wedge \widehat{dz_{ij}} \wedge \cdots \wedge dz_{gg} \quad (1 \leq i \leq j \leq g),$$

where $\widehat{dz_{ij}}$ means that dz_{ij} is omitted. Let $\omega = (\omega_{ij})$. Then for a non-negative integer r, $\omega^{\otimes r}$ stands for the tensor power of ω. The tensor power $\omega^{\otimes r}$ satisfies

$$\gamma \cdot \omega^{\otimes r} = \det(CZ + D)^{-r(g+1)} (CZ + D)^{\otimes r} \omega^{\otimes r} \cdot {}^t(CZ + D)^{\otimes r}$$

for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_{2g}(\mathbb{R})$.

Now let $A = (a_{ij})$ be a matrix of size g, and I, J the ordered sets of r integers in $\{1, \dots, g\}$, where a repeated choice is allowed.

For $I = \{i_1, \dots, i_r\}$, $J = \{j_1, \dots, j_r\}$, define

$$A^{(I,J)} = a_{i_1,j_1} \cdots a_{i_r j_r}.$$

Then the (k, l)-entry of $A^{\otimes r}$ is $A^{(I,J)}$ if

$$k = 1 + \sum_{s=1}^{r} (i_s - 1) g^{s-1}, \quad l = 1 + \sum_{s=1}^{r} (j_s - 1) g^{s-1} \quad (1 \le k, l \le g^r).$$

Put $\operatorname{sgn}(I) = \prod_{i \in I} (-1)^i$. Suppose $m \geq 2(g-1)$. Let η be a complex $m \times (g-1)$ matrix such that ${}^t\eta\eta = 0$ and $\operatorname{rank} \eta = g - 1$. Denote by η_i $(1 \le i \le g)$ the $(g-1) \times g$ matrix such that

$$\eta_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

We take a fixed positive symmetric matrix F of size m with rational coefficients. Define a theta series associated to F by

$$\begin{aligned} &\theta_F^{(I,J)} \begin{bmatrix} u \\ v \end{bmatrix} (Z) \\ &\quad = \operatorname{sgn}(I) \operatorname{sgn}(J) \sum_{G} \prod_{i \in I} \det(\eta_i {}^t(G+u) F^{1/2} \eta) \\ &\quad \times \prod_{j \in J} \det(\eta_j {}^t(G+u) F^{1/2} \eta) \mathbf{e} \left[\operatorname{tr} \left(\frac{1}{2} Z F[G+u] + {}^t(G+u) v \right) \right], \end{aligned}$$

where G runs through all $m \times g$ integral matrices, and u, v are $m \times g$ matrices with rational coefficients.

Let $\Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix} (Z)$ be the square matrix of size g^r whose (k, l)-entry is $\theta_F^{(I,J)} \begin{bmatrix} u \\ v \end{bmatrix} (Z)$, where

$$k = 1 + \sum_{s=1}^{r} (i_s - 1) g^{s-1}, \quad l = 1 + \sum_{s=1}^{r} (j_s - 1) g^{s-1}$$

when $I = \{i_1, \dots, i_r\}$, $J = \{j_1, \dots, j_r\}$. Tsuyumine [9] shows that there exist $l, r' \in \mathbb{N}$ such that for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g(l)$,

$$\begin{aligned} &\left(\Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix} (\gamma Z) \right)^{\otimes r'} \\ &= \det(CZ+D)^{(m/2+2r)r'} \left({}^t(CZ+D)^{-1} \right)^{\otimes rr'} \left(\Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix} (Z) \right)^{\otimes r'} \left((CZ+D)^{-1} \right)^{\otimes rr'}. \end{aligned}$$

Let $\{\gamma_j\}$ be a system of representatives of Γ_g modulo $\Gamma_g(l)$. When $\gamma_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}$, put

$$\Psi(Z) =$$
$$\sum_j \det(C_jZ+D_j)^{-(m/2+2r)r'} \cdot {}^t(C_jZ+D_j)^{\otimes rr'} \left(\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(\gamma_j Z)\right)^{\otimes r'} \left((C_jZ+D_j)^{-1}\right)^{\otimes rr'}.$$

Then for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$,

$$\Psi(\gamma Z) = \det(CZ+D)^{(m/2+2r)r'} \left({}^t(CZ+D)^{-1}\right)^{\otimes rr'} \Psi(Z) \left((CZ+D)^{-1}\right)^{\otimes rr'}.$$

Moreover we construct the symmetrization of $\Psi(Z)$. Let $\{\delta_i\}$ be a system of representatives of Γ_t modulo Γ_t^{lev}. When $\delta_i = \begin{pmatrix} A_i' & B_i' \\ C_i' & D_i' \end{pmatrix}$, put

$$\Phi(Z) = \sum_i \det(C_i'Z+D_i')^{-(m/2+2r)r'} \cdot {}^t(C_i'Z+D_i')^{\otimes rr'} \Psi(\delta_i Z) \left((C_i'Z+D_i')\right)^{\otimes rr'}.$$

Then we have

Proposition 2 *For any* $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t$, *we have*

$$\Phi(\gamma Z) = \det(CZ+D)^{(m/2+2r)r'} \left({}^t(CZ+D)^{-1}\right)^{\otimes rr'} \Phi(Z) \left((CZ+D)^{-1}\right)^{\otimes rr'}.$$

Proposition 3 *Let Z_0 be any fixed point of H_g. Take any non-zero complex symmetric matrix W of size g. Let m be an integer with $m \geq 2(g-1)$. Then for infinitely many r and r', there exists a symmetric matrix $\Phi(Z)$ occurring in the last proposition such that* $\mathrm{tr}(\Phi(Z_0)W^{\otimes rr'}) \neq 0$.

Put
$$\lambda_{m,r,r'} = \mathrm{tr}\left(\Phi(Z)\omega^{\otimes rr'}\right).$$

Combining Proposition 1 with Proposition 2, we conclude

Theorem 4 *Let Z_0 be any fixed point in H_g, and m an integer with $m \geq 2(g-1)$. Then for infinitely many r and r', there exists $\lambda_{m,r,r'}$ such that*
(1) *$\lambda_{m,r,r'}$ does not vanish at $Z = Z_0$.*
(2) *for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t$,*

$$\gamma \cdot \lambda_{m,r,r'} = \det(CZ+D)^{(m/2-r(g-1))r'} \lambda_{m,r,r'}.$$

4 Extensibility of certain differential forms

For a rational boundary component F, let $P(F) \subset Sp_{2g}(\mathbb{R})$ denote the stabilizer of F, $P'(F)$ the center of the unipotent radical of $P(F)$, and $C(F)$ the self-adjoint cone corresponding to F.

Definition 5 Let Γ be a Siegel modular group. A Γ-modular form f *vanishes on the rational boundary component* F *of order at least* l the following condition are satisfied. If we consider the Fourier-Jacobi expansion of f at F

$$f(Z) = \sum_{x \in (P'(F))^{\vee}} a_x^F(u,t)\mathbf{e}[\langle x, z\rangle],$$

then $a_x^F \neq 0$ implies $\min_{y \in P'(F) \cap \overline{C(F)} - \{0\}} (x, y) \geq l$.

Assume that $r(g-1) > m/2$. For any Γ_t-modular form f of weight $(r(g-1) - m/2)r'$, by Theorem 4, $f\lambda_{m,r,r'}$ is a Γ_t-invariant form in $(\Omega_{H_g}^{N-1})^{\otimes rr'}$.

Proposition 6 *If a* Γ_t*-modular form* f *vanishes on all rational corank* 1 *boundary components of order at least* rr'/C_t*, then* $f\lambda_{m,r,r'}$ *extends to* $\widetilde{\mathcal{A}}_{g,t}$*. Here* C_t *stands for* $\min\{1, \sqrt{3}/\sqrt[g]{t^{g-1}}\}$.

5 The main result

Theorem 7 *If* $\widetilde{\mathcal{A}}_{g,t}$ *is of general type, then any irreducible subvariety of codimension one in* $\mathcal{A}_{g,t}$ *is of general type.*

Let us give an outline of the proof to the above theorem. Let D be any irreducible subvariety in $\mathcal{A}_{g,t}$ of codimension 1, and $\pi : H_g \longrightarrow \mathcal{A}_{g,t}$ the canonical map. It should be noted that we can construct a Γ_g-modular form f whose restriction to $\pi^{-1}(D)$ does not vanish ([1], [10]). For such a weight k modular form f, its symmetrization $\mathrm{Sym}(f) = \prod_{\gamma \in \Gamma_t / \Gamma_t^{\mathrm{lev}}} f|_k\gamma$ is a Γ_t-modular form that does not vanish on $\pi^{-1}(D)$. The following diagram is helpful:

$$\begin{array}{ccccc} & & \mathcal{A}_{g,t}^{\mathrm{lev}} & & \\ & \swarrow & & \searrow & \\ \mathcal{A}_g & & & & \mathcal{A}_{g,t} \end{array}$$

If f is a Γ_g-modular form, then we have

$$\frac{(g-1)\mathrm{ord}(\mathrm{Sym}(f))}{\mathrm{weight}(\mathrm{Sym}(f))} = \frac{(g-1)\mathrm{ord}(f)}{\mathrm{weight}(f)}.$$

Here $\mathrm{ord}(\mathrm{Sym}(f))$ and $\mathrm{ord}(f)$ are vanishing orders at rational corank 1 boundary components.

If f is a non-trivial Γ_t-modular form such that $(g-1)\mathrm{ord}(f)/\mathrm{weight}(f) > 1$, then for certain integers a, b, each modular form f' in $f^{ak}M_{bk}(\Gamma_t)$ $(k \geq 1)$ has enough vanishing order at the cusp. If $\mathrm{weight}(f') = (r(g-1) - m/2)r'$, then $f'\lambda_{m,r,r'}$ extends to a section of $(\Omega_{\widetilde{\mathcal{A}}_{g,t}}^{N-1})^{\otimes rr'}$, where $N = g(g+1)/2$.

Generalizing Lemma 2.2 in [2], it is possible to find generators for Γ_t. Using them, we see that $[\Gamma_t, \Gamma_t]$, the commutator subgroup of Γ_t, is Γ_t itself. Moreover, Γ_t satisfies $b_1(\Gamma_t) = 0$, $b_2(\Gamma_t) = 1$. Hence by Theorem 1$'$ in [8], any effective divisor on $\mathcal{A}_{g,t}$ is defined by some Γ_t-modular form. Furthermore, the ring $\bigoplus_{k\geq 0} M_k(\Gamma_t)$ of Γ_t-modular forms is factorial.

There exists a Γ_t-modular form h such that its divisor (h) on H_g is $\pi^{-1}(D)$. By Theorem 4, for infinitely many r, r', $\lambda_{m,r,r'}|\pi^{-1}(D) \neq 0$. We can take such r, r' from the set of multiples of any fixed integer. For a suitable integer k, there exist weight k Γ_t-modular forms $g_1, \ldots, g_N$ such that they do not vanish on D, that each $g_i\lambda_{m,r,r'}$ extends to $\widetilde{\mathcal{A}}_{g,t}$, and that $g_2/g_1, \ldots, g_N/g_1$ are algebraically independent. Therefore D is of general type.

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References

[1] E. Freitag, Holomorphic tensors on subvarieties on the Siegel modular variety. In: *Automorphic Forms of Several Variables, Taniguchi Symposium, Katata, 1983*, 93-113, Birkhäuser, 1984.

[2] V. Gritsenko, Irrationality of the moduli spaces of polarized abelian surfaces. In: *Abelian Varieties. Proc. the Egloffstein Conf. on Abelian Varieties*, de Gruyter, 63-81, 1995.

[3] V. Gritsenko and K. Hulek, Commutator coverings of Siegel threefolds. *Duke Math.* **94** (1998), 509-542.

[4] I. Reiner, Real linear characters of the symplectic modular group. *Proc. Amer. Math. Soc.* **6** (1955), 987-990.

[5] E. Schellhammer, The Kodaira dimension of Siegel modular varieties of genus 3 or higher. *Bollettino U.M.I.* **9** (2006), 749-776.

[6] Y.-S. Tai, On the Kodaira dimension of the moduli spaces of abelian varieties. *Invent. Math.* **68** (1982), 425-439.

[7] Y.-S. Tai, On the Kodaira dimension of moduli spaces of abelian varieties with non-principal polarizations. In: *Abelian Varieties. Proc. the Egloffstein Conf. on Abelian Varieties*, de Gruyter, 293-302, 1995.

[8] S. Tsuyumine, Factorial property of a ring of automorphic forms. *Trans. Amer. Math. Soc.* **296** (1986), 111-123.

[9] S. Tsuyumine, Multi-tensors of differential forms on the Siegel modular variety and on its subvarieties. *Tsukuba J. Math.* **11** (1987), 107-119.

[10] R. Weissauer, Untervarietäten der Siegelschen Modulmannigfaltigkeiten von allgemeinem Typ. *Math. Ann.* **275** (1986), 207-220.