ULTRAPRODUCTS OF FINITE ALTERNATING GROUPS

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Abstract. We prove that if $\mathcal{U}$ is a nonprincipal ultrafilter over $\omega$, then the set of normal subgroups of the ultraproduct $\prod_{\mathcal{U}} \text{Alt}(n)$ is linearly ordered by inclusion. We also prove that the number of such ultraproducts up to isomorphism is either $2^{\aleph_0}$ or $2^{2^{\aleph_0}}$, depending on whether or not $CH$ holds.

1. Introduction

If $\mathcal{U}$ is a nonprincipal ultrafilter over $\omega$, then it is easily seen that the ultraproduct $G_{\mathcal{U}} = \prod_{\mathcal{U}} \text{Alt}(n)$ is not a simple group. However, Elek-Szabó [3] have recently shown that $G_{\mathcal{U}}$ has a unique maximal proper normal subgroup. In this paper, extending their analysis, we shall prove that the set $N_{\mathcal{U}}$ of normal subgroups of $G_{\mathcal{U}}$ is linearly ordered by inclusion. As we shall see later, this result is an easy consequence of the fact that the set $E_{\mathcal{U}} = \{ \langle g^{G_{\mathcal{U}}} \rangle \mid 1 \neq g \in G_{\mathcal{U}} \}$ of normal closures of nonidentity elements is linearly ordered by inclusion. More precisely, let $\equiv_{\mathcal{U}}$ be the convex equivalence relation on the linear order $\prod_{\mathcal{U}} \{1, \cdots, n\}$ defined by

$$f_{\mathcal{U}} \equiv_{\mathcal{U}} h_{\mathcal{U}} \text{ iff } 0 < \lim_{\mathcal{U}} \frac{f(n)}{h(n)} < \infty;$$

and let $L_{\mathcal{U}} = (\prod_{\mathcal{U}} \{1, \cdots, n\})/\equiv_{\mathcal{U}}$, equipped with the quotient linear order. Then we shall prove that $(E_{\mathcal{U}}, \subset)$ is isomorphic to $L_{\mathcal{U}}$.

In Section 3, we shall compute the number of ultraproducts $G_{\mathcal{U}} = \prod_{\mathcal{U}} \text{Alt}(n)$ up to isomorphism. Of course, if $CH$ holds, then each such ultraproduct $G_{\mathcal{U}}$ is saturated and hence is determined up to isomorphism by its first order theory; and we shall show that (as expected) there are $2^{\aleph_0}$ many ultraproducts up to elementary equivalence. On the other hand, arguing as in Kramer-Shelah-Tent-Thomas [5], we shall prove that if $CH$ fails, then there exists a family $\{\mathcal{U}_\alpha \mid \alpha < 2^{2^{\aleph_0}} \}$ of nonprincipal ultrafilters over $\omega$ such that the corresponding linear orders $L_{\mathcal{U}_\alpha}$ are

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pairwise nonisomorphic. Hence if $CH$ fails, then there are $2^{2^{\aleph_0}}$ many ultraproducts $G_\mathcal{U}$ up to isomorphism.

Finally, in Section 4, we shall briefly consider the currently open problems of computing the number of universal sofic groups up to isomorphism and elementary equivalence.

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2. The normal subgroups of $G_\mathcal{U}$

Let $\mathcal{U}$ be a nonprincipal ultrafilter over $\omega$ and let $G_\mathcal{U} = \prod_\mathcal{U} \text{Alt}(n)$. In this section, we shall prove the following result.

**Theorem 2.1.** The collection $\mathcal{N}_\mathcal{U}$ of normal subgroups of $G_\mathcal{U}$ is linearly ordered by inclusion.

The following easy observation will enable us to focus our attention on the set $\mathcal{E}_\mathcal{U} = \{ \langle g^{G_\mathcal{U}} \rangle \mid 1 \neq g \in G_\mathcal{U} \}$ of normal closures of nonidentity elements.

**Lemma 2.2.** If $G$ is any group, then the following statements are equivalent.

(a) The set of normal subgroups of $G$ is linearly ordered by inclusion.

(b) The set of normal closures of nonidentity elements of $G$ is linearly ordered by inclusion.

**Proof.** Clearly (a) implies (b). Conversely, assume that (b) holds and let $N$, $M$ be normal subgroups of $G$. If for every $g \in N$, there exists $h \in M$ such that $g \in \langle h^G \rangle$, then clearly $N \leq M$. Otherwise, there exists $g \in N$ such that for every $h \in M$, we have that $\langle g^G \rangle \not\leq \langle h^G \rangle$ and so $\langle h^G \rangle \leq \langle g^G \rangle$, which implies that $M \leq N$. $\square$

For each $\pi \in \text{Alt}(n)$, let $\text{supp}(\pi) = \{ \ell \mid \pi(\ell) \neq \ell \}$. In [3], Elek-Szabó proved that if $g = (\pi_n)_\mathcal{U} \in G_\mathcal{U}$, then

$$\langle g^G_{\mathcal{U}} \rangle = G_\mathcal{U} \quad \text{iff} \quad \lim_\mathcal{U} \frac{|\text{supp}(\pi_n)|}{n} > 0.$$

(This is an immediate consequence of Elek-Szabó [3, Proposition 2.3].) It follows that $M_\mathcal{U} = \{ (\pi_n)_\mathcal{U} \in G_\mathcal{U} \mid \lim_\mathcal{U} \frac{|\text{supp}(\pi_n)|}{n} = 0 \}$ is the unique maximal proper normal subgroup of $G_\mathcal{U}$. This suggests that, in order to understand the normal closure of an element $(\pi_n)_\mathcal{U} \in G_\mathcal{U}$, we should consider the relative growth rate of
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$|\text{supp}(\pi_n)|$. From now on, we adopt the convention that if $(\pi_n)_U \in G_U \setminus 1$, then we always choose $(\pi_n)$ such that $\pi_n \neq 1$ for all $n \in \omega$; and the normal closure of $(\pi_n)_U$ will be denoted by $N((\pi_n)_U)$.

**Definition 2.3.** Let $\leq$ be the quasi-order on $G_U \setminus 1$ defined by

$$(\pi_n)_U \leq (\varphi_n)_U \iff \lim_U \frac{|\text{supp}(\pi_n)|}{|\text{supp}(\varphi_n)|} < \infty.$$ 

**Proposition 2.4.** If $(\pi_n)_U, (\varphi_n)_U \in G_U \setminus 1$ are nonidentity elements, then

$$(\pi_n)_U \in N((\varphi_n)_U) \iff (\pi_n)_U \leq (\varphi_n)_U.$$ 

We shall split the proof of Proposition 2.4 into a sequence of lemmas. We shall begin by proving the easier implication.

**Lemma 2.5.** If $(\pi_n)_U, (\varphi_n)_U \in G_U \setminus 1$ and $(\pi_n)_U \in N((\varphi_n)_U)$, then $(\pi_n)_U \leq (\varphi_n)_U$.

**Proof.** If $(\pi_n)_U \in N((\varphi_n)_U)$, then there exists an integer $k \geq 1$ such that $(\pi_n)_U$ can be expressed as a product of $k$ conjugates of $(\varphi_n)_U^{\pm 1}$. Hence for $U$-a.e. $n \in \mathbb{N}$, the permutation $\pi_n$ can be expressed as a product of $k$ conjugates of $\varphi_n^{\pm 1}$. This implies that $|\text{supp}(\pi_n)| \leq k|\text{supp}(\varphi_n)|$ and so $\lim U |\text{supp}(\pi_n)|/|\text{supp}(\varphi_n)| \leq k$. $\square$

Recall that a permutation $\sigma \in \text{Alt}(m)$ is said to be *exceptional* iff its conjugacy class $\sigma^{\text{Sym}(m)}$ splits into two conjugacy classes in $\text{Alt}(m)$. It is well-known that this occurs iff the cycles of $\sigma$ have distinct odd lengths.

**Lemma 2.6.** If $\sigma \in \text{Alt}(m)$ is a nonexceptional fixed-point-free permutation, then every element of $\text{Alt}(m)$ can be expressed as a product of exactly 4 conjugates of $\sigma$.

**Proof.** This is an immediate consequence of Brenner [2, Theorem 3.05]. $\square$

**Lemma 2.7.** If $(\pi_n)_U, (\varphi_n)_U \in G_U \setminus 1$ and $(\pi_n)_U \leq (\varphi_n)_U$, then $(\pi_n)_U \in N((\varphi_n)_U)$.

**Proof.** Suppose that $(\pi_n)_U \leq (\varphi_n)_U$. As mentioned earlier, Elek-Szabó [3] have proved that if $\lim U \frac{|\text{supp}(\varphi_n)|_n}{n} > 0$, then $N((\varphi_n)_U) = G_U$. Hence we can suppose that $\lim U \frac{|\text{supp}(\varphi_n)|_n}{n} = 0$. Let $\lim U |\text{supp}(\pi_n)|/|\text{supp}(\varphi_n)| \leq k$, where $k \geq 2$ is an integer. Then for $U$-a.e. $n \in \mathbb{N}$, we have that $|\text{supp}(\pi_n)| \leq k|\text{supp}(\varphi_n)| \leq n$. Hence there exists a permutation $\sigma_n \in \text{Alt}(n)$ such that the following conditions are satisfied:
(a) $\sigma_n$ is a product of $k$ conjugates $\psi_1, \cdots, \psi_k$ of $\varphi_n$.
(b) If $1 \leq i < j \leq k$, then $\text{supp}(\psi_i) \cap \text{supp}(\psi_j) = \emptyset$.
(c) $\text{supp}(\pi_n) \subseteq \text{supp}(\sigma_n)$.

Regarding $\sigma_n$ as an element of $\text{Alt}(\text{supp}(\sigma_n))$, we see that $\sigma_n$ is a nonexceptional fixed-point-free permutation. Hence, applying Lemma 2.6, it follows that $\pi_n$ is a product of 4 conjugates of $\sigma_n$ and this implies that $(\pi_n)_\mathcal{U}$ is a product of $4k$ conjugates of $(\varphi_n)_\mathcal{U}$.

Applying Proposition 2.4, it follows that if $(\pi_n)_\mathcal{U}, (\varphi_n)_\mathcal{U} \in G_{\mathcal{U}} \setminus 1$ are nonidentity elements, then

$$N(\pi_n)_\mathcal{U} = N(\varphi_n)_\mathcal{U} \quad \text{iff} \quad 0 < \lim_{\mathcal{U}} \frac{|\text{supp}(\pi_n)|}{|\text{supp}(\varphi_n)|} < \infty;$$

and that $(\mathcal{E}_\mathcal{U}, \subset)$ is isomorphic to the linear order $L_\mathcal{U} = (\prod_{\mathcal{U}}\{1, \cdots, n\})/\equiv_\mathcal{U}$. This completes the proof of Theorem 2.1.

\textbf{Remark 2.8.} Clearly $L_\mathcal{U}$ has a least element; namely, the $\equiv_\mathcal{U}$-class containing the constant functions. If we identify $G_\mathcal{U}$ with its image under the embedding

$$G_\mathcal{U} \to \text{Sym}(\prod_{\mathcal{U}}\{1, \cdots, n\})$$

corresponding to the natural action

$$(\pi_n)_\mathcal{U} \cdot (\ell_n)_\mathcal{U} = (\pi_n(\ell_n))_\mathcal{U}$$

of $G_\mathcal{U}$ on $\prod_{\mathcal{U}}\{1, \cdots, n\}$, then the minimal nontrivial normal subgroup of $G_\mathcal{U}$ is the group $\text{Alt}(\prod_{\mathcal{U}}\{1, \cdots, n\})$ of finite even permutations of $\prod_{\mathcal{U}}\{1, \cdots, n\}$. Hence, by Scott [7, 11.4.7], since

$$\text{Alt}(\prod_{\mathcal{U}}\{1, \cdots, n\}) \leq G_\mathcal{U} \leq \text{Sym}(\prod_{\mathcal{U}}\{1, \cdots, n\}),$$

it follows that $\text{Aut}(G_\mathcal{U})$ is precisely the normalizer of $G_\mathcal{U}$ in $\text{Sym}(\prod_{\mathcal{U}}\{1, \cdots, n\})$. Of course, if $CH$ holds, then the ultraproduct $G_\mathcal{U} = \prod_{\mathcal{U}}\text{Alt}(n)$ is saturated and so $|\text{Aut}(G_\mathcal{U})| = 2^{\mathfrak{c}}$.

\textbf{Question 2.9.} Is it consistent that $\text{Aut}(G_\mathcal{U}) = \prod_{\mathcal{U}}\text{Sym}(n)$?
3. THE NUMBER OF NONISOMORPHIC ULTRAPRODUCTS

In this section, we shall compute the number of ultraproducts $G_{\mathcal{U}} = \prod_{\mathcal{U}} \text{Alt}(n)$ up to isomorphism. If $CH$ holds, then each such ultraprodct is saturated and hence is determined up to isomorphism by its first order theory. So the following result implies that if $CH$ holds, then there exist exactly $2^{\aleph_0}$ ultraproducts $\prod_{\mathcal{U}} \text{Alt}(n)$ up to isomorphism.

**Theorem 3.1.** There exist $2^{\aleph_0}$ many ultraproducts $\prod_{\mathcal{U}} \text{Alt}(n)$ up to elementary equivalence.

**Proof.** For each prime $p \geq 5$, let $D_p = \{n \in \omega \mid n \geq p \text{ and } n \equiv 0, 1, 2 \mod p \}$.

**Claim 3.2.** There exists a first order sentence $\Phi_p$ such that for all $n \geq 7$,

$$n \in D_p \iff \text{Alt}(n) \models \Phi_p.$$

**Proof of Claim 3.2.** Clearly $n \in D_p$ iff $\text{Alt}(n)$ contains an element of order $p$ with at most 2 fixed points. To see that this property is first order definable, first note that an element $\pi \in \text{Alt}(n)$ of order $p \geq 5$ has at most 2 fixed points iff there does not exist a 3-cycle $\sigma \in \text{Alt}(n)$ which commutes with $\pi$. Also note that if $n \geq 7$, then an element $\sigma \in \text{Alt}(n)$ of order 3 is a 3-cycle iff $\sigma\psi\sigma\psi^{-1}$ has order at most 5 for all $\psi \in \text{Alt}(n)$.

Let $\mathbb{P} = \{p \in \omega \mid p \geq 5 \text{ is prime }\}$. Then it is enough to check that for each subset $S \subseteq \mathbb{P}$, the collection $D_S = \{D_p \mid p \in S\} \cup \{\omega \setminus D_p \mid p \in \mathbb{P} \setminus S\}$ has the finite intersection property. So suppose that $p_1, \ldots, p_{\ell} \in S$ and that $q_1, \ldots, q_m \in \mathbb{P} \setminus S$. By the Chinese Remainder Theorem, there exists a positive integer $n \in \omega$ such that

- $n \equiv 0 \mod p_i$ for all $1 \leq i \leq \ell$; and
- $n \equiv 3 \mod q_j$ for all $1 \leq j \leq m$.

Clearly $n \in D_{p_1} \cap \cdots \cap D_{p_{\ell}}, \omega \setminus D_{q_1} \cap \cdots \cap (\omega \setminus D_{q_m})$.

In order to compute the number of nonisomorphic ultraproducts $G_{\mathcal{U}}$ when $CH$ fails, we shall focus our attention on the linearly ordered set $(\mathcal{E}_{\mathcal{U}}, \subset)$ of normal closures of nonidentity elements. Clearly if $\mathcal{U}, \mathcal{B}$ are nonprincipal ultrafilters over $\omega$ and $G_{\mathcal{U}} \cong G_{\mathcal{B}}$, then $(\mathcal{E}_{\mathcal{U}}, \subset) \cong (\mathcal{E}_{\mathcal{B}}, \subset)$. Furthermore, in Section 2, we showed that $(\mathcal{E}_{\mathcal{U}}, \subset)$ is isomorphic to $L_{\mathcal{U}} = (\prod_{\mathcal{U}}\{1, \ldots, n\})/\equiv_{\mathcal{U}}$ and clearly $L_{\mathcal{U}}$ can be
regarded as an initial segment of $(\prod_{\mathcal{U}}\omega)/\equiv \mathcal{U}$. Hence the following result implies if $CH$ fails, then there exist $2^{2^{\aleph_0}}$ ultraproducts $G_{\mathcal{U}}$ up to isomorphism.

**Definition 3.3.** If $L_1, L_2$ are linear orders, then $L_1 \approx_i^* L_2$ iff $L_1$ and $L_2$ have nonempty isomorphic initial segments $I_1, I_2$ with $|I_1|, |I_2| > 1$.

The requirement that $|I_1|, |I_2| > 1$ is needed in Definition 3.3 because of the fact that each linear order $L_{\mathcal{U}} = (\prod_{\mathcal{U}}\{1, \cdots, n\})/\equiv \mathcal{U}$ has a first element; namely, the $\equiv \mathcal{U}$-class containing the constant functions.

**Theorem 3.4.** If $CH$ fails, then there exists a set $\{\mathcal{U}_\alpha | \alpha < 2^{2^{\aleph_0}}\}$ of nonprincipal ultrafilters over $\omega$ such that

$$(\prod_{\mathcal{U}_\alpha} \omega)/\equiv \mathcal{U}_\alpha \not\approx_i^* (\prod_{\mathcal{U}_\beta} \omega)/\equiv \mathcal{U}_\beta$$

for all $\alpha < \beta < 2^{2^{\aleph_0}}$.

**Proof.** The proof of Kramer-Shelah-Tent-Thomas [5, Theorem 3.3] goes through with just one minor change; namely, in the proof of Lemma 4.7, the collection $\{B_{s,t} | s < t \in I\}$, where $B_{s,t} = \{n \in \omega | f_s(n) < f_t(n)\}$, is replaced by $\{B_{s,t,k} | s < t \in I \text{ and } 1 \leq k \in \omega\}$, where $B_{s,t,k} = \{n \in \omega | kf_s(n) < f_t(n)\}$. \qed

4. **Universal sofic groups**

In this final section, we shall briefly consider the currently open problems of computing the number of the universal sofic groups up to isomorphism and elementary equivalence.

Recall that if $\mathcal{U}$ is a nonprincipal ultrafilter over $\omega$ and $G_{\mathcal{U}} = \prod_{\mathcal{U}} \text{Alt}(n)$, then

$$M_{\mathcal{U}} = \left\{ (\pi_n)_{\mathcal{U}} \in G_{\mathcal{U}} | \lim_{\mathcal{U}} \frac{\text{supp}(\pi_n)}{n} = 0 \right\}$$

is the unique maximal proper normal subgroup of $G_{\mathcal{U}}$. Let $S_{\mathcal{U}} = G_{\mathcal{U}}/M_{\mathcal{U}}$. Then by Elek-Szabó [3], if $\Gamma$ is a finitely generated group, then the following statements are equivalent:

- $\Gamma$ is a sofic group.
- $\Gamma$ embeds into $S_{\mathcal{U}}$ for some (equivalently every) nonprincipal ultrafilter $\mathcal{U}$. 


For this reason, $S_{\mathcal{U}}$ is said to be a universal sofic group. (A clear account of the basic theory of sofic groups can be found in Pestov [6]. It is an important open problem whether every finitely generated group is sofic.)

It is natural to conjecture that the number of universal sofic groups up to isomorphism is either $2^{\aleph_0}$ or $2^{2^{\aleph_0}}$, depending on whether or not $CH$ holds. However, it is currently not even known whether it is consistent that there exist two nonisomorphic universal sofic groups.

**Question 4.1.** Compute the number of universal sofic groups up to isomorphism.

In Section 3, simple arithmetic considerations enabled us to construct $2^{\aleph_0}$ non-elementarily equivalent ultraproducts $\prod_{\mathcal{U}} \text{Alt}(n)$. However, factoring by the maximal proper normal subgroup $M_{\mathcal{U}}$ appears to eliminate all the arithmetic aspects of the group $S_{\mathcal{U}}$. For example, Glebsky-Rivera [4] have recently shown that if $g \in S_{\mathcal{U}}$ and if $p$ is any prime, then there exists $h \in S_{\mathcal{U}}$ such that $h^p = g$.

**Question 4.2.** Are all universal sofic groups $S_{\mathcal{U}}$ elementarily equivalent?

**REFERENCES**


