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n-th Derivatives of Some Functions in terms of N-Fractional Calculus

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Abstract

In this article, N-fractional calculus and $n(\in Z^+)$-th derivatives of functions

$$f(z)=\frac{1}{(\sqrt{z-b}-c)^2-d} \qquad ((\sqrt{z-b}-c)^2-d\neq 0)$$

are discussed. That is, n-th derivatives of the function,

$$(f(z))_n=(-1)^n(z-b)^{-1-n}\sum_{m,k=0}^{\infty}\frac{[1]_m[2+2m]_k[\frac{k}{2}+1+m]_n}{m!k!}S^kT^m$$

where

$$S=\frac{c}{\sqrt{z-b}},\quad T=\frac{d}{z-b}\quad (|S|<1,\quad |T|<1).$$

is reported for example.

1 Definition of N-Fractional Calculus

In order to treat the derivatives of arbitrary order, we descrive the definition of fractional calculus and some basical theorems and identities.

(I) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D=\{D_-,D_+\}$, $C=\{C_-,C_+\}$, C_- be a curve along the cut joining two points z and $-\infty+iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty+iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = {}_C(f)_\nu \\ &= \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^{\nu+1}} \quad (\nu \notin Z^-), \end{aligned} \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \quad (m \in Z^+), \tag{2}$$

where

$$-\pi \le arg(\zeta - z) \le \pi \; for \; C_-, \quad 0 \le arg(\zeta - z) \le 2\pi \;\; for \; C_+,$$

$$\zeta \ne z, \quad z \in C, \quad \nu \in R, \quad \Gamma; \; Gamma \; function,$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν , some fundamental properties have reported. ([3], [5])

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}}\right) \quad (\nu \notin Z^-), \quad (Refer \; to [1]) \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \ne |f_\nu| \le \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. Fractional calculus operator group $\{N^\nu\}$ is an Action product group which has continuous index ν for the set of F .

Theorem C. Let

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} \quad (\nu \in R). \tag{7}$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^{\alpha} + N^{\beta} = N^{\gamma} \quad (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) \tag{8}$$

holds. ([5])

(III) We have following results for some elementary functions. ([1])

(i)

$$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(log(z-c))_{\alpha} = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_{\alpha} := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

2 Preliminary

(I) The following theorem is reported by K. Nishimoto [12].

Theorem D. We have

(i)

$$(((z-b)^{\beta}-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_k\Gamma(\beta k-\alpha\beta+\gamma)}{k!\Gamma(\beta k-\alpha\beta)}(\frac{c}{(z-b)^{\beta}})^k \tag{1}$$

$$(|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}| < \infty),$$

and

(ii)

$$(((z-b)^{\beta}-c)^{\alpha})_n=(-1)^n(z-b)^{\alpha\beta-n}\sum_{k=0}^{\infty}\frac{[-\alpha]_k[\beta k-\alpha\beta]_n}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{2}$$

$$\left(n\in Z_0^+,\ \left|\frac{c}{(z-b)^{\beta}}\right|<1\right),$$

where

$$[\lambda]_k=\lambda(\lambda+1)\cdots(\lambda+k-1)=\Gamma(\lambda+k)/\Gamma(\lambda)\quad with\quad [\lambda]_0=1,$$

(Pochhammer's Notation).

(II) The following theorem is reported by K. Nishimoto already[13].

Theorem E. We have

(i)

$$((((z-b)^{\beta}-c)^{\alpha})-d)^{\delta})_{\gamma}=e^{-i\pi\gamma}(z-b)^{\alpha\beta\delta-\gamma}$$
$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{m!k!\Gamma(\beta k-\alpha\beta(\delta-m))}\left(\frac{c}{(z-b)^{\beta}}\right)^k\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m, \tag{3}$$

$$\left(\left|\frac{\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{\Gamma(\beta k-\alpha\beta(\delta-m))}\right|<\infty\right),$$

and

(ii)

$$((((z-b)^{\beta}-c)^{\alpha})-d)^{\delta})_n=(-1)^n(z-b)^{\alpha\beta\delta-n}$$
$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k[\beta k-\alpha\beta(\delta-m)]_n}{m!k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m, \tag{4}$$

$$(n\in Z_0^+)$$

where

$$((z-b)^{\beta}-c)^{\alpha}-d\neq 0,\quad \left|\frac{c}{(z-b)^{\beta}}\right|<1,\quad \left|\frac{d}{(z-b)^{\alpha\beta}}\right|<1).$$

We apply these theorems to obtain some theorems for the function $\frac{1}{(\sqrt{z-b}-c)^2-d}$.

3 N-Fractional Calculus of Functions $\frac{1}{(\sqrt{z-b}-c)^2-d}$

Theorem 1. We have

(i)

$$\left(\frac{1}{(\sqrt{z-b}-c)^2-d}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{-1-\gamma}$$
$$\times \sum_{m,k=0}^{\infty} \frac{[1]_m[2+2m]_k\Gamma(\frac{k}{2}+1+m+\gamma)}{m!k!\Gamma(\frac{k}{2}+1+m)}\left(\frac{c}{\sqrt{z-b}}\right)^k\left(\frac{d}{z-b}\right)^m \quad (1)$$

$$(|\Gamma(\frac{k}{2}+1+m+\gamma)|<\infty)$$

and

(ii)

$$\left(\frac{1}{(\sqrt{z-b}-c)^2-d}\right)_n = (-1)^n(z-b)^{-1-n}$$
$$\times \sum_{m.k=0}^{\infty} \frac{[1]_m[2+2m]_k[\frac{k}{2}+1+m]_n}{m!k!}\left(\frac{c}{\sqrt{z-b}}\right)^k\left(\frac{d}{z-b}\right)^m \quad (2)$$

$$(n \in Z_0^+)$$

$$\textit{where} \quad (\sqrt{z-b}-c)^2-d \neq 0, \quad \left|\frac{c}{\sqrt{z-b}}\right|<1, \quad \left|\frac{d}{z-b}\right|<1.$$

and $\sum_{m,k=0} := \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}$.

Proof of (i). We operate N^γ to the both sides of following relation,

$$\frac{1}{(\sqrt{z-b}-c)^2-d} = \left(((z-b)^{1/2}-c)^2-d\right)^{-1} \quad (3)$$

and by setting $\alpha=2,\ \beta=1/2,\ \delta=-1$ in Theorem E (i) we obtain (1), under the conditions stated before.

Proof of (ii). We have the result by setting $\gamma=n$ in the equation (1).

Furthermore by setting $\alpha=0$ in Theorem 1, we have the following corollary immediately.

Corollary 1. We have

(i)

$$\left(\frac{1}{z-2c\sqrt{z-b}-b+c^2}\right)_\gamma = e^{-i\pi\gamma}(z-b)^{-1-\gamma}\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(\frac{k}{2}+1+\gamma)}{k!\Gamma(\frac{k}{2}+1)}\left(\frac{c}{\sqrt{z-b}}\right)^k \tag{4}$$

$$(|\Gamma(\frac{k}{2}+1+\gamma)| \leq \infty)$$

and

(ii)

$$\left(\frac{1}{z-2c\sqrt{z-b}-b+c^2}\right)_n = (-1)^n(z-b)^{-1-n}\sum_{k=0}^{\infty}\frac{[2]_k[\frac{k}{2}+1]}{k!}\left(\frac{c}{\sqrt{z-b}}\right)^k \tag{5}$$

$$(n \in Z_0^+),$$

$$where \ \sqrt{z-b}-c \neq 0, \quad |\frac{c}{\sqrt{z-b}}| \leq 1.$$

Theorem 2. We have the following identities,

(i)

$$(z-b)^{-1/2}\sum_{m,k=0}^{\infty}\frac{[1]_m[2+2m]_k\Gamma(\frac{k}{2}+1+m+\gamma)}{m!k!\Gamma(\frac{k}{2}+1+m)}\left(\frac{c}{\sqrt{z-b}}\right)^k\left(\frac{d}{z-b}\right)^m$$

$$= \frac{1}{2\sqrt{d}}\sum_{k=0}^{\infty}\frac{[1]_k\Gamma(\frac{k}{2}+\frac{1}{2}+\gamma)}{k!\Gamma(\frac{k}{2}+\frac{1}{2})}\left\{\left(\frac{c+\sqrt{d}}{\sqrt{z-b}}\right)^k - \left(\frac{c-\sqrt{d}}{\sqrt{z-b}}\right)^k\right\} \tag{6}$$

$$(|\Gamma(\frac{k}{2}+1+m+\gamma)| < \infty) \quad (|\Gamma(\frac{k}{2}+\frac{1}{2}\gamma| < \infty)$$

and

(ii) for $n \in Z_0^+$,

$$(z-b)^{-1/2}\sum_{m,k=0}^{\infty}\frac{[1]_m[2+2m]_k[\frac{k}{2}+1+m]_n}{m!k!}\left(\frac{c}{\sqrt{z-b}}\right)^k\left(\frac{d}{z-b}\right)^m$$

$$= \frac{1}{2\sqrt{d}}\sum_{k=0}^{\infty}\frac{[1]_k[\frac{k}{2}+\frac{1}{2}]_n}{k!}\left\{\left(\frac{c+\sqrt{d}}{\sqrt{z-b}}\right)^k - \left(\frac{c-\sqrt{d}}{\sqrt{z-b}}\right)^k\right\} \tag{7}$$

where $(\sqrt{z-b}-c)^2-d \neq 0, \quad d \neq 0,$

$$|\frac{c}{\sqrt{z-b}}| \neq 1, |\frac{d}{z-b}| \neq 1, \ |\frac{c+\sqrt{d}}{\sqrt{z-b}}| \neq 1, \ |\frac{c-\sqrt{d}}{\sqrt{z-b}}| \neq 1.$$

Proof of (i). We have the following relation,

$$\frac{1}{(\sqrt{z-b}-c)^2-d}=\frac{1}{2\sqrt{d}}\left(\frac{1}{\sqrt{z-b}-c-\sqrt{d}}-\frac{1}{\sqrt{z-b}-c+\sqrt{d}}\right). \quad (8)$$

From Theorem D,(i), we have

$$\left(\frac{1}{\sqrt{z-b}-p}\right)_\gamma=\left(((z-b)^{1/2}-p)^{-1}\right)_\gamma \quad (9)$$

$$=e^{-i\pi\gamma}(z-b)^{-1/2-\gamma}\sum_{k=0}^{\infty}\frac{[1]_k\Gamma(\frac{k}{2}+\frac{1}{2}+\gamma)}{k!\Gamma(\frac{k}{2}+\frac{1}{2})}\left(\frac{p}{\sqrt{z-b}}\right)^k. \quad (10)$$

$$(|\Gamma(2m+2+\gamma-k)|<\infty)$$

Therefore, setting $p=c+\sqrt{d}$ or $c-\sqrt{d}$, we have

$$\left(\frac{1}{(\sqrt{z-b}-c)^2-d}\right)_\gamma=\frac{1}{2\sqrt{d}}\left((\frac{1}{\sqrt{z-b}-c-\sqrt{d}})_\gamma-(\frac{1}{\sqrt{z-b}-c+\sqrt{d}})_\gamma\right)$$

$$=e^{-i\pi\gamma}(z-b)^{-1/2-\gamma}\sum_{k=0}^{\infty}\frac{[1]_k\Gamma(\frac{k}{2}+\frac{1}{2}+\gamma)}{k!\Gamma(|frack2+\frac{1}{2})}\{\left(\frac{c+\sqrt{d}}{\sqrt{z-b}}\right)^k-\left(\frac{c-\sqrt{d}}{\sqrt{z-b}}\right)^k\}. \quad (11)$$

Proof of (ii). Set $\gamma=n$ in (6).

4 Semi Derivatives and Integrals

[I] We have

(i)

$$\left(\frac{1}{(\sqrt{z-b}-c)^2-d}\right)_{1/2}=-i(z-b)^{-3/2}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[1]_m[2+2m]_k\Gamma(\frac{k}{2}+\frac{3}{2}+m)}{m!k!\Gamma(\frac{k}{2}+1+m)}(\frac{c}{\sqrt{z-b}})^k(\frac{d}{z-b})^m \quad (1)$$

(semi derivative)

and

(ii)

$$\left(\frac{1}{(\sqrt{z-b}-c)^2-d}\right)_{-1/2}=i(z-b)^{-1/2}$$
$$\times\sum_{m,k=0}^{\infty}\frac{[1]_m[2+2m]_k\Gamma(\frac{k}{2}+\frac{1}{2}+m)}{m!k!\Gamma(\frac{k}{2}+1+m)}(\frac{c}{\sqrt{z-b}})^k(\frac{d}{z-b})^m \quad (2)$$

(semi integral)

where

$$(\sqrt{z-b}-c)^2-d\neq 0,\quad |\frac{c}{\sqrt{z-b}}|<1,\quad |\frac{d}{z-b}|<1,$$

from Theorem 1 by setting $\gamma=1/2$ and $-1/2$, respectively.

[II] We have

(i)

$$\left(\frac{1}{z-2c\sqrt{z-b}-b+c^2}\right)_{1/2}=-i(z-b)^{-3/2}$$
$$\times\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(\frac{k}{2}+\frac{3}{2})}{k!\Gamma(\frac{k}{2}+1)}(\frac{c}{\sqrt{z-b}})^k \quad (3)$$

(semi derivative)

and

(ii)

$$\left(\frac{1}{z-2c\sqrt{z-b}-b+c^2}\right)_{-1/2}=i(z-b)^{-1/2}$$
$$\times\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(\frac{k}{2}+\frac{1}{2})}{k!\Gamma(\frac{k}{2}+1)}(\frac{c}{\sqrt{z-b}})^k \quad (4)$$

(semi integral)

where

$$|\frac{c}{\sqrt{z-b}}|<1,$$

from Corollary 1 by setting $\gamma=1/2$ and $-1/2$, respectively.

5 Some Special Cases

When the order of differentiation is some integer, our results coincide the classical calculus. So, we illustrate some examples in cases of $n = 0, 1$.

(I) When $n = 0$, from Theorem 1.(ii), we have the followings,

$$\left(\frac{1}{(\sqrt{z-b}-c)^2-d}\right)_0 = (z-b)^{-1} \times \sum_{m,k=0}^{\infty} \frac{[1]_m[2+2m]_k[\frac{k}{2}+1+m]_0}{m!k!} S^k T^m \tag{1}$$

Here we set

$$S = \frac{c}{\sqrt{z-b}}, \quad T = \frac{d}{z-b}.$$

Indeed we have

$$R.H.S.\ of\ (1) = (z-b)^{-1} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m \sum_{k=0}^{\infty} \frac{[2+2m]_k}{k!} S^k \tag{2}$$

$$= (z-b)^{-1} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m (1-S)^{-2-2m} \tag{3}$$

$$= \frac{1}{z-b} \frac{1}{(1-S)^2} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} (\frac{T}{(1-S)^2})^m \tag{4}$$

$$= \frac{1}{z-b} \frac{1}{(1-S)^2} \left(1 - \frac{T}{(1-S)^2}\right)^{-1} \tag{5}$$

$$= \frac{1}{z-b} \frac{1}{(1-S)^2 - T} = \frac{1}{(\sqrt{z-b}-c)^2-d}. \tag{6}$$

(II) When $n = 1$, our result is written as follows.

$$\left(\frac{1}{(\sqrt{z-b}-c)^2-d}\right)_1 = -(z-b)^{-2} \times \sum_{m,k=0}^{\infty} \frac{[1]_m[2+2m]_k[\frac{k}{2}+1+m]_1}{m!k!} S^k T^m. \tag{7}$$

Indeed we have

$$R.H.S.\ of\ (7) = -(z-b)^{-2} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m \times \sum_{k=0}^{\infty} \frac{[2+2m]_k(\frac{k}{2}+1+m)}{k!} S^k. \tag{8}$$

Now we take notice that

$$\sum_{k=0}^{\infty}\frac{[2+2m]_k(\frac{k}{2})}{k!}S^k=\frac{1}{2}\sum_{k=1}^{\infty}\frac{[2+2m]_k}{(k-1)!}S^k \tag{9}$$

$$=\frac{1}{2}S\sum_{k=0}^{\infty}\frac{[2+2m]_{k+1}}{k!}S^k \tag{10}$$

$$=\frac{1}{2}S(2+2m)\sum_{k=0}^{\infty}\frac{[3+2m]_k}{k!}S^k \tag{11}$$

$$=S(1+m)(1-S)^{-3-2m} \tag{12}$$

and

$$\sum_{k=0}^{\infty}\frac{[2+2m]_k(1+m)}{k!}S^k=(1+m)\sum_{k=0}^{\infty}\frac{[2+2m]_k}{k!}S^k$$
$$=(1+m)(1-S)^{-2-2m}. \tag{13}$$

Therefore we obtain

$$R.H.S.\ of\ (7)=-(z-b)^{-2}(1-S)^{-3}\sum_{m=0}^{\infty}\frac{[1]_m}{m!}(1+m)\left(\frac{T}{(1-S)^2}\right)^m \tag{14}$$

$$=-(z-b)^{-2}(1-S)^{-3}\{\sum_{m=0}^{\infty}\frac{[1]_m}{m!}\left(\frac{T}{(1-S)^2}\right)^m+\sum_{m=0}^{\infty}\frac{[1]_m}{m!}\left(\frac{T}{(1-S)^2}\right)^m\}$$

$$=-(z-b)^{-2}(1-S)^{-3}\{\left(1-\frac{T}{(1-S)^2}\right)^{-1}+\frac{T}{(1-S)^2}\left(1-\frac{T}{(1-S)^2}\right)^{-2}\}$$

$$=-(z-b)^{-2}(1-S)^{-3}\frac{(1-S)^4}{((1-S)^2-T)^2}$$

$$=-(z-b)^{-2}\frac{1-S}{((1-S)^2-T)^2}$$

$$=-\frac{1}{\sqrt{z-b}}\frac{\sqrt{z-b}-c}{((\sqrt{z-b}-c)^2-d)^2}. \tag{15}$$

This result coincides with the one obtained from the classical calculus

$$\frac{d}{dz}\left(\frac{1}{(\sqrt{z-b}-c)^2-d}\right)$$

(III) The cases of $n=2$ and 3 are somewhat complicated , we will report those cases at anothor time.

Note. In this section we use the following identities.

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} S^k = (1-S)^{-\lambda}, \quad (|S|<1) \tag{16}$$

$$[\lambda]_{k+1} = \lambda[\lambda+1]_k \tag{17}$$

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k k}{k!} S^k = \sum_{k=1}^{\infty} \frac{[\lambda]_k}{(k-1)!} S^k = S \sum_{k=0}^{\infty} \frac{[\lambda]_{k+1}}{k!} S^k \tag{18}$$

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