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Harmonic Univalent Functions with Janowski Starlike Analytic Part

Emel YAVUZ

Abstract

In this paper we define a new subclass of harmonic univalent functions for which analytic part is *Janowski Starlike Function*, and investigate some properties of this type of functions. Also we give a new coefficient inequality for harmonic univalent functions.

1 Introduction

Let Ω be the class of analytic functions $w(z)$ in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$, satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$.

For arbitrary fixed real numbers A and B which satisfy $-1 \leq B < A \leq 1$ we say $p(z)$ belongs to the class $\mathcal{P}(A, B)$ if

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

is analytic in $\mathbb{D}$ and $p(z)$ is given by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}$$

for every z in $\mathbb{D}$ and for some $w(z) \in \Omega$. This class, $\mathcal{P}(A, B)$, was first introduced by W. Janowski [3]. Therefore, we call $p(z)$ in the class $\mathcal{P}(A, B)$ *"Janowski Function"*.

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Let $\mathcal{S}^*(A,B)$ denote the family of functions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

regular in $\mathbb{D}$, and such that $h(z)$ is in $\mathcal{S}^*(A,B)$ if and only if

$$z\frac{h'(z)}{h(z)} = p(z)$$

for some $p(z)$ in $\mathcal{P}(A,B)$ and for every $z \in \mathbb{D}$. Functions in $\mathcal{S}^*(A,B)$ are called the *"Janowski Starlike Functions"* [3].

A continuous complex valued function $f = u + iv$ defined in a simply connected domain $\mathcal{U}$ is said to be *"Harmonic"* in $\mathcal{U}$ if u and v are real harmonic in $\mathcal{U}$. In any simply connected domain $\mathcal{U} \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{U}$. We call h the *"Analytic Part"* and g the *"Co-Analytic Part"* of f.

The *"Jocabian"* of f is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

A necessary and sufficient condition for $f = h + \bar{g}$ is to be locally univalent and sense-preserving in $\mathcal{U}$ such as [2], [4]

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0.$$

This is equivalent to

$$|g'(z)| < |h'(z)|$$

for all $z \in \mathcal{U}$.

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \bar{g}$ that are *"Harmonic Univalent and Sense-Preserving"* in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \,||z| < 1\}$, for which

$$f(0) = h(0) = f_z(0) - 1 = 0.$$

For $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

So, as a result of the sense-preserving property of f, $|b_1| < 1$.

The classical family $\mathcal{S}$ which is analytic, univalent and normalized functions on $\mathbb{D}$ is subclass of $\mathcal{S}_{\mathcal{H}}$ in which $b_n = 0$ for all $n \in \mathbb{N}$.

The function

$$w_1 = \frac{g'}{h'}$$

is called the *"Second Dilatation of $f = h + \bar{g}$"*, and we denote the class of the second dilatation of f by $\mathcal{W}$. Note that $|w_1(z)| < 1$ and $w_1(0) = b_1 \neq 0$ for all z in $\mathbb{D}$.

We consider the transformation $\phi : \mathbb{C} \to \mathbb{C}$, given by

$$\phi(z) = \frac{w_1(z) - w_1(0)}{1 - \overline{w_1(0)}w_1(z)}, \tag{1.2}$$

maps the unit disc $\mathbb{D}$ onto itself, where $w_1(z) \in \mathcal{W}$ for every z in $\mathbb{D}$. It is easy to show that $\phi(z)$ is an analytic function in $\mathbb{D}$, and $|\phi(z)| \leq 1$, and $\phi(0) = 0$ for all $z \in \mathbb{D}$. Hence $\phi(z) \in \Omega$.

Definition 1.1. *Let $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$. We define a new subclass of harmonic univalent functions for which analytic part is Janowski starlike function. We denote by $\mathcal{S}^*_{\mathcal{H}}(A, B)$ the family of all harmonic univalent functions on $\mathbb{D}$ with $h \in \mathcal{S}^*(A, B)$.*

2 Auxiliary Lemmas

Lemma 2.1. *(Schwarz's Lemma [1]) If $\phi(z)$ is analytic for $|z| < 1$ and satisfies the condition $|\phi(z)| \leq 1$, $\phi(0) = 0$ then $|\phi(z)| \leq |z|$ and $|\phi'(0)| \leq 1$. If $|\phi(z)| = z$ for some $z \neq 0$ or if $|\phi'(0)| = 1$, then $\phi(z) = cz$ with a constant c of absolute value* 1.

Lemma 2.2. *[3] If $h(z) \in \mathcal{S}^*(A, B)$, then for $|z| = r$, $0 < r < 1$*

$$C(r; -A, -B) \leq |h'(z)| \leq C(r; A, B), \tag{2.1}$$

where

$$C(r; A, B) = \begin{cases} (1 + Ar)(1 + Br)^{(A-2B)/B}, & if\ B \neq 0, \\ (1 + Ar)e^{Ar}, & if\ B = 0. \end{cases} \tag{2.2}$$

These bounds are sharp, being attained at the point $z = re^{i\varphi}$, $0 \leq \varphi \leq 2\pi$, by

$$h_*(z) = zh_0(z; -A, -B) \tag{2.3}$$

and

$$h^*(z) = zh_0(z; A, B), \tag{2.4}$$

respectively, where

$$h_0(z; A, B) = \begin{cases} (1 + Be^{-i\varphi} z)^{(A-2B)/B}, & \text{for } B \neq 0, \\ e^{-i\varphi} z, & \text{for } B = 0. \end{cases}$$

Lemma 2.3. *Let $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ and $w_1 \in \mathcal{W}$. Then we have*

$$\left| e^{-i\theta} w_1(z) - \frac{\alpha(1-r^2)}{1-\alpha^2 r^2} \right| \leq \frac{r(1-\alpha^2)}{1-\alpha^2 r^2}, \tag{2.5}$$

where first coefficient of g is $b_1 = \alpha e^{i\theta}$, $0 \leq \theta \leq 2\pi$, and $|z| = r < 1$. The equality holds in the inequality (2.5) *only for the function*

$$w_1(z) = e^{i\beta} \frac{e^{i\theta} z + \alpha}{1 + \alpha e^{i\theta} z}, \quad z \in \mathbb{D}. \tag{2.6}$$

Proof. Since $\phi(z)$ which is given by (1.2) satisfies the conditions of Schwarz's lemma then $|\phi(z)| \leq |z| = r < 1$. Hence, we can write

$$|\phi(z)| = \frac{|e^{-i\theta} w_1(z) - \alpha|}{|1 - \alpha e^{-i\theta} w_1(z)|} \leq r \Rightarrow |e^{-i\theta} w_1(z) - \alpha| \leq r|1 - \alpha e^{-i\theta} w_1(z)|$$

for all z in $\mathbb{D}$. By taking $e^{-i\theta} w_1(z) = x + iy$ we get following inequality

$$x^2 + y^2 - 2\frac{\alpha(1-r^2)}{1-\alpha^2 r^2} x + \frac{\alpha^2 - r^2}{1-\alpha^2 r^2} \leq 0.$$

So, $e^{-i\theta} w_1(z)$ maps $|z| = r$ onto the circle, which has a center of $C(r) = \left(\frac{\alpha(1-r^2)}{1-\alpha^2 r^2}, 0\right)$ and radius of $\rho(r) = \frac{r(1-\alpha^2)}{1-\alpha^2 r^2}$. □

Lemma 2.4. *Let $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ and $w_1 \in \mathcal{W}$. Then we have*

$$\frac{|\alpha - r|}{1 - \alpha r} \leq |w_1(z)| \leq \frac{\alpha + r}{1 + \alpha r}, \tag{2.7}$$

for all $|z| = r < 1$ and $|b_1| = \alpha$.

Proof. If we use lemma 2.3, we can obtain the result. □

3 Main Results

Theorem 3.1. *If $f = h + \bar{g} \in \mathcal{S_H}$ be as given in (1.1) and $w_1 \in \mathcal{W}$, then we have*

$$|b_2| < \frac{1}{2} + |a_2|$$

for all z in $\mathbb{D}$.

Proof. Lets consider the function $\phi(z)$ which is given by (1.2). Since $\phi(z)$ satisfies the condition of Schwarz's lemma then $|\phi'(0)| \leq 1$. Hence we can write

$$|\phi'(0)| = \frac{|b_2 - a_2 b_1|}{1 - |b_1|^2} < \frac{1}{2} \tag{3.1}$$

for all $z \in \mathbb{D}$. By using the definition of the second dilatation function $w_1(z)$ in (3.1) we get the desired result, after simple calculations. □

Lemma 3.2. *If $f = h + \bar{g} \in \mathcal{S}^*_{\mathcal{H}}(A, B)$, then we have*

$$C(r; -A, -B)\frac{|\alpha - r|}{1 - \alpha r} \leq |g'(z)| \leq \frac{\alpha + r}{1 + \alpha r} C(r; A, B) \tag{3.2}$$

where $C(r; A, B)$ is given by (2.2). The upper and the lower bounds for $0 < r < 1$ are sharp being attained by functions (2.3) and (2.4), respectively.

Proof. Since the definition of the second dilatation function of f is $w_1(z) = g'(z)/h'(z)$, then we can write

$$|g'(z)| = |w_1(z)||h'(z)| \quad (z \in \mathbb{D}). \tag{3.3}$$

Using (2.1) and (2.7) in (3.3) we obtain desired result. □

Theorem 3.3. *If $f = h + \bar{g} \in \mathcal{S}^*_{\mathcal{H}}(A, B)$, then for $|z| = r$, $0 < r < 1$, we have*

$$\int_0^r (1 - A\rho)(1 - B\rho)^{\frac{A-2B}{B}} \frac{(1-\alpha)(1-\rho)}{(1+\alpha\rho)} d\rho \leq |f(z)| \leq$$
$$\int_0^r (1 + A\rho)(1 + B\rho)^{\frac{A-2B}{B}} \frac{(1+\alpha)(1+\rho)}{(1+\alpha\rho)} d\rho, \quad \text{for } B \neq 0,$$
$$\int_0^r (1 - A\rho)e^{-A\rho} \frac{(1-\alpha)(1-\rho)}{(1+\alpha\rho)} d\rho \leq |f(z)| \leq$$
$$\int_0^r (1 + A\rho)e^{A\rho} \frac{(1+\alpha)(1+\rho)}{(1+\alpha\rho)} d\rho, \quad \text{for } B = 0,$$

where $|b_1| = \alpha$ *and this bound for* $0 < r < 1$ *is sharp being attained by functions* (2.3), (2.4) *and the solution of the differantial equation* $g'(z) = h'(z)\frac{z+\alpha}{1+\alpha z}$.

Proof. For harmonic univalent function $f = h + \bar{g}$ we know that

$$(|h'(z)| - |g'(z)|)|dz| \leq |df(z)| \leq (|h'(z)| + |g'(z)|)|dz|. \tag{3.4}$$

On the other hand, by using (3.3) we obtain

$$|h'(z)| - |g'(z)| = |h'(z)|(1 - |w_1(z)|) \tag{3.5}$$

for all z in $\mathbb{D}$. If we use (2.7) and (2.1) in (3.5) we obtain

$$\frac{(1-\alpha)(1-r)}{(1+\alpha r)} C(r; -A, -B) \leq |h'(z)| - |g'(z)|. \tag{3.6}$$

Furthermore, we have

$$|h'(z)| + |g'(z)| \leq |h'(z)|(1 + |w_1(z)|) \tag{3.7}$$

for all z in $\mathbb{D}$. Again if we use (2.7) and (2.1) in (3.7) we obtain

$$|h'(z)| + |g'(z)| \leq \frac{(1+\alpha)(1+r)}{(1+\alpha r)} C(r; A, B). \tag{3.8}$$

By using (3.6) and (3.8) in (3.4) and integrating this inequality form 0 to r we obtain the desired result. □

Corollary 3.4. *The Heinz's inequality for* $f = h + \bar{g} \in S^*_{\mathcal{H}}(A, B)$ *is*

$$|h'(z)|^2 + |g'(z)|^2 \geq \begin{cases} (1-Br)^{\frac{2A-4B}{B}}(1-Ar)^2\left(1 + \left(\frac{\alpha - r}{1-\alpha r}\right)^2\right), & B \neq 0, \\ e^{-2Ar}(1-Ar)^2\left(1 + \left(\frac{\alpha - r}{1-\alpha r}\right)^2\right), & B = 0, \end{cases}$$

for all $z \in \mathbb{D}$, *and* $|b_1| = \alpha$.

Proof. Since $g'(z) = w_1(z)h'(z)$ for all $z \in \mathbb{D}$, then

$$|h'(z)|^2 + |g'(z)|^2 = |h'(z)|^2(1 + |w_1(z)|^2). \tag{3.9}$$

If we use the inequalities (2.1) and (2.7) in (3.9) we get the result, after simple calculations. □

Theorem 3.5. *If $f = h + \bar{g} \in \mathcal{S}^*_{\mathcal{H}}(A, B)$, then*

$$C^2(r; -A, -B)\frac{(1-r^2)(1-\alpha^2)}{(1+\alpha r)^2} \leq J_f(z) \leq C^2(r; A, B)\left(1 - \frac{|\alpha - r|^2}{(1-\alpha r)^2}\right)$$

for all $z \in \mathbb{D}$, and $|b_1| = \alpha$.

Proof. Using lemma 2.4 and the relations

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2$$

and

$$g'(z) = w(z)h'(z)$$

we obtain the result. □

Note. If we consider the spacial values for A and B as below, we can obtain some subclasses.

- $A = 1,\ B = -1.$
- $A = 1 - 2\alpha\ (0 \leq \alpha < 1),\ B = -1.$
- $A = 1,\ B = \frac{1}{M} - 1\ (M > \frac{1}{2}).$
- $A = \beta,\ B = -\beta\ (0 \leq \beta < 1).$

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Emel Yavuz
Department of Mathematics and Computer Science,
İstanbul Kültür University, 34156 İstanbul, Turkey
e-mail: e.yavuz@iku.edu.tr