数理解析研究所講究録
第 1628 巻 2009 年 58-62

π_1- and π_2-theories of operators

Shunsaku Nii*

In this talk, a topological index theory which can be seen as π_2-*theory of operators* is introduced. This terminology is inspired by the one π_1-*theory of operators* by Sanson [6] referring to infinite dimensional Maslov index theory. This viewpoint begins by seeing the classical theory of Strum-Liouville operators as π_1-*theory of* $\mathbb{S}^1$.

1 The theory of Strum-Liouville operators: a π_1-*theory of* $\mathbb{S}^1$

Consider the eigenvalue problem of a Strum-Liouville operator:

$$-p'' + f(x)p = \lambda p, \qquad on \quad I = [-1, 1] \ or \ \mathbb{R}$$

This equation is written as a system of first order equations:

$$\begin{cases} p' = q \\ q' = (f(x) - \lambda)\, p. \end{cases}$$

In the polar coordinate $p = r\cos\theta,\ q = r\sin\theta$, this system becomes:

$$\begin{cases} r' = (1 - \lambda + f(x))\, r \sin\theta\cos\theta \\ \theta' = 1 + (\lambda - f(x) - 1)\sin^2\theta \end{cases}$$

Because the right hand side of the θ equation is monotone in λ, the number of $\theta(I)$ winds $\mathbb{S}^1$ increase as λ does. Therefore for each eigenvalue λ, $\theta(I) \in \pi_1(\mathbb{S}^1)$ identifies the eigenfunction. That is, the eigenfunctions are ordered by the number of humps.

*Faculty of Mathematics, Kyushu University

2 Maslov index: a π_1-*theory of matrices*

The first natural generalization of the above theory is what is called (Keller-)Maslov(-Arnol'd) index for the eigenvalue problem of a Schrö dinger operator:

$$-p'' + M(x)p = \lambda p, \quad p \in \mathbb{R}^n, \qquad on \quad I = [-1,1] \text{ or } \mathbb{R}$$

This system is equivalent to the following Hamiltonian system:

$$\begin{cases} p' = \dfrac{\partial H}{\partial q} \\ q' = -\dfrac{\partial H}{\partial p} \end{cases},$$

where the Hamiltonian is given by $H(p,q) = \frac{1}{2}\left\{|q|^2 + {}^t p\,(\lambda I - M(x))\,p\right\}$.

Because a Hamiltonian system preserves the symplectic structure, this system induces a flow on the Lagrangian Graßmannian manifold $\Lambda(n) = Sp(n)\,(\mathbb{R}^n \times \{0\})$, where $Sp(n)$ is the symplectic group.

Fact $\pi_1(\Lambda(n)) \cong \mathbb{Z}$

Therefore $(p,q)(I) \in \pi_1(\Lambda(n))$ characterizes the eigenfunctions. ($(p,q)(x)$ is not necessarily monotone.) This is what is called Maslov index.

3 Infinite dimensional Maslov index: a π_1-*theory of operators*

There are several infinite dimensional generalization of Maslov index. One by Swanson [6] is among the earliests.

Let $E = H \times H^*$ for a Hilbert space H and its dual H^*. Define a symplectic structure on E by $\omega((e,\alpha),(f,\beta)) = \alpha\cdot f - \beta\cdot e$ Then the Fredholm Lagrangian Graßmannian manifold $\mathcal{F}\Lambda_H$ is defined by $\mathcal{F}\Lambda_H = Sp_C(E)H$, where $Sp_C(E) := \{\, id + \text{compact} \mid \text{preserves } \omega\} \subset GL(E)$

Fact $\pi_1(\mathcal{F}\Lambda_H) \cong \mathbb{Z}$

Swanson applied this fact for deformations of elliptic operators, and call his theory as π_1-*theory of operators* contrasting it to Fredholm index (π_0-*theory of operators*) which distinguishes connected components of operators.

Recently, Deng [2] reformulated this theory on $E = H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ for a star-shaped domain Ω and applied to the boundary value problem of an elliptic operator.

4 The Stability index: a π_2-*theory of matrices*

In spite of early development of the π_1-theory for selfadjoint operators, any analogous theory for non-selfadjoint operators has not appeared until recently. The obstacles were that the eigenvalues are not real and the systems are no longer Hamiltonian. The first step for this direction seems to be the Stability index theory by Alexander-Gardner-Jones [1, 5] explained below.

Consider the eigenvalue problem for a not-selfadjoint operator:

$$-p'' + M(x)p' + N(x)p = \lambda p, \quad p \in \mathbb{C}^n, \qquad \text{on} \quad I.$$

This system is equivalent to the following system on $\mathbb{C}^{2n}$:

$$\begin{cases} p' = q \\ q' = (N(x) - \lambda I)\, p + M(x) q\ . \end{cases}$$

This time, the system induces a flow on the complex Graßmannian manifold $G_n(\mathbb{C}^{2n}) = GL(2n)\,(\mathbb{C}^n \times \{0\})$. For a disc $D \subset \mathbb{C}$, this flow induces a map

$$\Phi \colon \mathbb{S}^2 \cong (D \times \partial I) \cup (\partial D \times I) \to G_n(\mathbb{C}^{2n})$$

Fact $\pi_2\left(G_n\left(\mathbb{C}^{2n}\right)\right) \cong \mathbb{Z}$

Then Alexander-Gardner-Jones proved sort of π_2-*theory of matrices.*

Theorem (Alexander-Gardner-Joned [1],Gardner-Jones [5])
$\Phi\left(\mathbb{S}^2\right) \in \pi_2\left(G_n\left(\mathbb{C}^{2n}\right)\right)$ *represents the number of eigenvalues in D including the multiplicity.*

This theory is sometimes referred to as Alexander-Gardner-Jones bundle theory, as it is formulated by the terminology of line bundles and the Chern class.

5 The infinite dimensional Stability index: a π_2-*theory of operators*

It is natural to think about infinite dimensional generalization of the Stability index from the viewpoints both in pure mathematics and in application. One such example is the following eigenvalue problem:

$$\begin{cases} u_{xx} + \Delta_y u + \beta(y)u_x + f(x,y)u = \lambda u, & (x,y) \in \mathbb{R} \times \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on} \quad \mathbb{R} \times \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^m$ is a bounded domain. This equation can be written as an ordinary differential equation in x-variable on an appropriate Hilbert space H_Ω.

Here is a difficulty: $GL(H)$ is contractible for an infinite dimensional Hilbert space H. This means that a naive generalization of the Stability index becomes trivial and does not detect any information.

Fortunately, we can exploit compactness of the problem: Let $GL_C(H) := \{ id + \text{compact} \mid \text{invertible} \} \subset GL(H)$ and fix a polarization $H = H_- \oplus H_+$, then the Fredholm Graßmannian manifold $F(H_+)$ is the orbit of H_+ under the action of $GL_C(H)$ *i.e.* $F(H_+) = GL_C(H)H_+$.

Under this setting, the problem induces a system on $F(H_+)$.

Remark *In this case, the system does not generate a flow, as the problem is ill-posed.*

Then, for a disc $D \subset \mathbb{C}$, this system induces a map

$$\Phi \colon \mathbb{S}^2 \cong (D \times \partial I) \cup (\partial D \times I) \to F(H_+),$$

and we have the following theorem.

Theorem (Deng-N. [3])
$\Phi(\mathbb{S}^2) \in \pi_2(F(H_+))$ *represents the number of eigenvalues in D including the multiplicity.*

We also have a similar result for an elliptic operator posed on a bounded domain [4]

These results can be called π_2-*theory of operators* in the Swanson's expression.

References

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