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Asymptotic Stability of Combination of Viscous Contact Wave with Rarefaction Waves for 1-D Compressible Navier-Stokes System

Feimin HUANG[a], Jing LI[a,b], Akitaka MATSUMURA[b*]
a. Institute of Applied Mathematics, AMSS, Academia Sinica, Beijing 100080, P. R. China
b.Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, 560-0043 Toyonaka, Osaka, Japan

1 Introduction

We consider the one-dimensional compressible Navier-Stokes system in Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu \left(\frac{u_x}{v}\right)_x, \\ \left(e + \frac{u^2}{2}\right)_t + (pu)_x = \left(\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v}\right)_x \end{cases} \tag{1.1}$$

for $x \in \mathbb{R} = (-\infty, +\infty), t > 0$, where $v(x,t) > 0, u(x,t), \theta(x,t) > 0, e(x,t) > 0$ and $p(x,t)$ are the specific volume, fluid velocity, internal energy, absolute temperature, and pressure respectively, while the positive constants μ and κ denote the viscosity and heat conduction coefficients respectively. Here we study the ideal polytropic fluids so that p and e are given by the state equations

$$p = \frac{R\theta}{v} = Av^{-\gamma}e^{\frac{\gamma-1}{R}s}, \quad e = \frac{R\theta}{\gamma-1} + const.,$$

where s is the entropy, $\gamma > 1$ is the adiabatic exponent and A, R are both positive constants. We concern the Cauchy problem to the system (1.1) supplemented with the following initial and far field conditions:

$$\begin{cases} (v,u,\theta)(x,0) = (v_0,u_0,\theta_0)(x), & x \in \mathbb{R}, \\ (v,u,\theta)(\pm\infty,t) = (v_\pm,u_\pm,\theta_\pm), & t > 0, \end{cases} \tag{1.2}$$

where $v_\pm(>0), u_\pm$ and $\theta_\pm(>0)$ are given constants, and we assume $\inf_{\mathbb{R}} v_0 > 0, \inf_{\mathbb{R}} \theta_0 > 0$, and $(v_0,u_0,\theta_0)(\pm\infty) = (v_\pm,u_\pm,\theta_\pm)$ as compatibility conditions.

We are interested in the global solutions in time of the Cauchy problem (1.1)(1.2) and their large time behaviors in the relations with the spatial asymptotic states $(v_\pm, u_\pm, \theta_\pm)$. It has been known that these asymptotic behaviors are well characterized by those of the

*Email adresses: fhuang@amt.ac.cn(F. Huang), ajingli@gmail.com(J. Li), akitaka@math.sci.osaka-u.ac.jp (A. Matsumura)

solutions of the corresponding Riemann problem for the hyperbolic part of (1) (Euler system):

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \frac{u^2}{2}\right)_t + (pu)_x = 0 \end{cases} \tag{1.3}$$

with the initial Riemann data

$$(v, u, \theta)(x, 0) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0. \end{cases} \tag{1.4}$$

The system of conservation laws (1.3) has three distinct real eigenvalues for positive v and θ

$$\lambda_1 = -\sqrt{\gamma p/v} < 0, \quad \lambda_2 = 0, \quad \lambda_3 = -\lambda_1 > 0$$

which implies the first and third characteristic fields are genuinely nonlinear and the second field is linearly degenerate. Then it is known that the basic Riemann solutions of the problem (1.3)(1.4) are dilation invariant solutions: shock waves, rarefaction waves, contact discontinuities, and the linear combinations of these basic waves ([20]). Since the inviscid system (1.3) is an idealization when the dissipative effects are neglected, it is of great importance to study the large-time asymptotic behavior of solutions of the corresponding viscous system (1.1) toward the viscous versions of these basic waves. In fact, in the cases the Riemann solution consists of shock waves or rarefaction waves, there have been intensive studies completed in the theory of viscous conservation laws since 1985 when the asymptotic stability of the viscous shock waves is studied by a L^2 elementary energy method in Goodman [2] and Matsumura-Nishihara [16]. The viscous shock profiles and viscous rarefaction waves have been shown to be asymptotically stable for quite general perturbation for the compressible Navier-Stokes system (1.1) and more general systems of viscous strictly hyperbolic conservation laws([9, 10, 12–14, 17–19, 21, 22]). In particular, the asymptotics toward the rarefaction waves for (1.1) is established in Kawashima-Matsumura-Nishihara [10] by using an energy form associated with the physical total energy and a monotone property of rarefaction waves.

On the other hand, in the case the Riemann solution consists of contact discontinuity, the problem is more subtle and the progress has been less satisfactory, except for the studies in [4, 5, 7, 15, 23]. The stability of a viscous version of contact discontinuity ("viscous contact wave") for systems of viscous conservation laws was first studied by Xin ([23]) who obtained the metastability of a weak contact discontinuity for the compressible Euler system with uniform viscosity, that is, the solution tends toward a viscous contact wave which approximates the contact discontinuity locally in time as zero viscosity limit, but not uniformly in time. Later Liu-Xin in [15] showed the metastability of contact discontinuities for a class of general systems of nonlinear conservation laws with uniform viscosity, and obtained pointwise asymptotic behavior towards viscous contact waves by approximate fundamental solutions, which also leads to the asymptotic stability of the viscous contact wave in L^p norms for all $p \geq 1$. However, the theory in [15, 23] does not apply to the compressible Navier-Stokes system (1.1) since the viscosity matrix in (1.1) is only semi-positive definite. For a free-boundary value problem for (1.1) on the half line, the asymptotic stability of a viscous contact wave is proved by an elementary energy method by Huang-Matsumura-Shi in [4]. However, this approach neither can be applied to Cauchy problem case of (1) since the analysis in [4] depends crucially on a Poincaré-type inequality, which cannot be true for Cauchy problem. Recently, Huang-Matsumra-Xin [5] and Huang-Xin-Yang [7] have made some progresses, that is, they succeeded in using the anti-derivative method, motivated by the theory of viscous shock waves([9]), to obtain not only the stability of the viscous contact wave but also the convergence rate. However, their method is again not available for the stability of the combination wave of viscous contact wave with rarefaction waves, since the anti-derivative method has never worked for the stability of the rarefaction waves.

More recently, Huang-Matsumura-Xin [6] succeeded in obtaining a new estimate on heat equations which can be applied to the study of the stability of the combination of viscous contact wave with viscous shock profiles. In this paper, motivated by the arguments in [6] and [10], we show the asymptotic stability of the linear combination wave of viscous contact wave and the rarefaction waves for the Cauchy problem of the compressible Navier-Stokes system (1.1)(1.2) provided the strength of the combination wave is suitably small.

To state our main results, we first recall the viscous contact wave (V, U, Θ) for the compressible Navier-Stokes system (1.1) defined in [7]. For the Riemann problem (1.3)(1.4), it is known that the contact discontinuity solution takes the form

$$(\tilde{V}, \tilde{U}, \tilde{\Theta})(x,t) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, t > 0, \\ (v_+, u_+, \theta_+), & x > 0, t > 0, \end{cases} \tag{1.5}$$

provided that

$$u_- = u_+, \quad p_- \triangleq \frac{R\theta_-}{v_-} = p_+ \triangleq \frac{R\theta_+}{v_+}. \tag{1.6}$$

In the setting of the compressible Navier-Stokes system (1.1), the corresponding wave (V, U, Θ) to the contact discontinuity $(\tilde{V}, \tilde{U}, \tilde{\Theta})$ becomes smooth and behaves as a diffusion wave due to the dissipation effect. We call this wave "viscous contact wave". The viscous contact wave (V, U, Θ) can be constructed as follows. Since the pressure for the profile (V, U, Θ) is expected to be almost constant corresponding to the arguments in [5], i.e., we set

$$\frac{R\Theta}{V} = p_+,$$

which indicates the leading part of the energy equation $(1.1)_3$ is

$$\frac{R}{\gamma - 1}\Theta_t + p_+ U_x = \kappa \left(\frac{\Theta_x}{V}\right)_x. \tag{1.7}$$

The equations (1.7) and $(1.1)_1$ lead to a nonlinear diffusion equation,

$$\Theta_t = a\left(\frac{\Theta_x}{\Theta}\right)_x, \quad \Theta(\pm\infty, t) = \theta_\pm, \quad a = \frac{\kappa p_+(\gamma-1)}{\gamma R^2} > 0, \tag{1.8}$$

which has a unique self similarity solution $\Theta(x,t) = \Theta(\xi), \xi = \frac{x}{\sqrt{1+t}}$ due to [1,3]. Furthermore, on one hand, $\Theta(\xi)$ is a monotone function, increasing if $\theta_+ > \theta_-$ and decreasing if $\theta_+ < \theta_-$; On the other hand, there exists some positive constant $\bar{\delta}$, such that for $\delta = |\theta_+ - \theta_-| \leq \bar{\delta}$, Θ satisfies

$$(1+t)|\Theta_{xx}| + (1+t)^{1/2}|\Theta_x| + |\Theta - \theta_\pm| \leq c_1 \delta e^{-\frac{c_2 x^2}{1+t}} \text{ as } |x| \to \infty, \tag{1.9}$$

where c_1 and c_2 are both positive constants depending only on θ_- and $\bar{\delta}$. Once Θ is determined, the contact wave profile $(V, U, \Theta)(x,t)$ is then defined as follows:

$$V = \frac{R}{p_+}\Theta, \quad U = u_- + \frac{\kappa(\gamma-1)}{\gamma R}\frac{\Theta_x}{\Theta}, \quad \Theta = \Theta. \tag{1.10}$$

It is straightforward to check that (V, U, Θ) satisfies

$$\|V-\tilde{V}, U-\tilde{U}, \Theta-\tilde{\Theta}\|_{L^p}=O\left(\kappa^{1/(2p)}\right)(1+t)^{1/(2p)}, p \geq 1,$$

which means the nonlinear diffusion wave $(V, U, \Theta)(x, t)$ approximates the contact discontinuity $(\tilde{V}, \tilde{U}, \tilde{\Theta})(x, t)$ to the Euler system (1.3) in L^p norm, $p \geq 1$ on any finite time interval as the heat conductivity coefficient κ tends to zero. More importantly, the contact wave (V, U, Θ) solves the compressible Navier-Stokes system (1.1) time asymptotically, ie.,

$$\begin{cases} V_t - U_x = 0, \\ U_t + (\frac{R\Theta}{V})_x = \mu\left(\frac{U_x}{V}\right)_x + R_1, \\ \left(e(V,\Theta)+\frac{U^2}{2}\right)_t + (p(V,\Theta)U)_x = \left(\kappa\frac{\Theta_x}{V}+\mu\frac{UU_x}{V}\right)_x + R_2, \end{cases}$$

where

$$\begin{aligned} R_1 &= \frac{\kappa(\gamma-1)}{\gamma R}\left((\ln\Theta)_{xt} - \mu\left(\frac{p_+}{R\Theta}(\ln\Theta)_{xx}\right)_x\right) \\ &= O(\delta)(1+t)^{-3/2}e^{-\frac{c_2x^2}{1+t}} \text{ as } |x| \to \infty, \\ R_2 &= \left(\frac{\kappa(\gamma-1)}{\gamma R}\right)^2\left((\ln\Theta)_x(\ln\Theta)_{xt} - \mu\left(\frac{p_+}{R\Theta}(\ln\Theta)_x(\ln\Theta)_{xx}\right)_x\right) \\ &= O(\delta)(1+t)^{-2}e^{-\frac{c_2x^2}{1+t}} \text{ as } |x| \to \infty. \end{aligned}$$

We are now in a position to state our main results. Let

$$(\phi, \psi, \zeta)(x,t) = (v-V, u-U, \theta-\Theta)(x,t).$$

For interval $I \subset [0,\infty)$, we define a function space

$$X(I) \subset Y(I) \triangleq C\left(I; H^1(\mathbb{R})\right)$$

as

$$X(I) = \left\{(\phi,\psi,\zeta) \in Y(I)\,|\phi_x \in L^2\left(I; L^2(\mathbb{R})\right), (\psi_x, \zeta_x) \in L^2\left(I; H^1(\mathbb{R})\right)\right\}.$$

Our first main result is as follows:

Theorem 1 *For any given (v_-, u_-, θ_-), suppose that (v_+, u_+, θ_+) satisfies (1.6). Let (V, U, Θ) be the viscous contact wave defined in (1.10) with strength $\delta = |\theta_+ - \theta_-| \leq \overline{\delta}$. Then there exist positive constants δ_0 and ε_0, such that if $\delta < \delta_0$ and the initial data (v_0, u_0, θ_0) satisfies*

$$\|(v_0(\cdot) - V(\cdot,0), u_0(\cdot) - u_-, \theta_0(\cdot) - \Theta(\cdot,0))\|_{H^1} \leq \varepsilon_0,$$

then the Cauchy problem (1.1)(1.2) admits a unique global solution (v, u, θ) satisfying $(v-V, u-U, \theta-\Theta) \in X([0,\infty))$ and

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|(v-V, u-u_-, \theta-\Theta)(x,t)| = 0. \tag{1.11}$$

Remark 1 *Theorem 1 doesn't give the convergence rate of (1.11) as in [5, 7]. However, compare to the anti-derivative method used in [5, 7], our energy method is elementary and much simpler, and more importantly, it can be used to investigate the asymptotic behavior of the solutions toward the combination of viscous contact wave with rarefaction ones (see Theorem 2 below).*

When the relation (1.6) fails, the basic theory of hyperbolic system of conservation laws (e.g., see [20]) implies that for any given constant state (v_-,u_-,θ_-) with $v_->0,\theta_->0$ and $u_-\in\mathbb{R}$, there exists a suitable neighborhood $\Omega(v_-,u_-,\theta_-)$ of (v_-,u_-,θ_-) such that for any $(v_+,u_+,\theta_+)\in\Omega(v_-,u_-,\theta_-)$ the Riemann problem of the Euler sysytem (1.3)(1.4) has a unique solution. In this paper, we only consider the stability of the superposition of viscous contact wave and rarefaction ones. In this situation, we assume that

$$(v_+,u_+,\theta_+)\in R_1CR_3(v_-,u_-,\theta_-)\subset\Omega(v_-,u_-,\theta_-)$$

where

$$R_1CR_3(v_-,u_-,\theta_-)\triangleq\Big\{(u,v,\theta)\in\Omega(v_-,u_-,\theta_-)\Big|\,s\neq s_-,$$
$$u\geq u_--\int_{v_-}^{e^{\frac{\gamma-1}{R\gamma}(s_--s)}v}\lambda_-(\eta,s_-)d\eta,\ u\geq u_--\int_{e^{\frac{\gamma-1}{R\gamma}(s-s_-)}v_-}^{v}\lambda_+(\eta,s)d\eta\Big\}$$

with

$$s=\frac{R}{\gamma-1}\ln\frac{R\theta}{A}+R\ln v,\qquad s_\pm=\frac{R}{\gamma-1}\ln\frac{R\theta_\pm}{A}+R\ln v_\pm,$$

and

$$\lambda_\pm(v,s)=\pm\sqrt{A\gamma v^{-\gamma-1}e^{(\gamma-1)s/R}}.$$

It is well-known ([20]) that there exists some suitably small $\delta_1>0$ such that for

$$(v_+,u_+,\theta_+)\in R_1CR_3(v_-,u_-,\theta_-),\quad |\theta_--\theta_+|\leq\delta_1,\tag{1.12}$$

there exist a positive constant $C=C(\theta_-,\delta_1)$ and a unique pair of points (v_-^m,u^m,θ_-^m) and (v_+^m,u^m,θ_+^m) in $\Omega(v_-,u_-,\theta_-)$ satisfying

$$\frac{R\theta_-^m}{v_-^m}=\frac{R\theta_+^m}{v_+^m}\triangleq p^m,$$

and

$$|v_\pm^m-v_\pm|+|u^m-u_\pm|+|\theta_\pm^m-\theta_\pm|\leq C|\theta_--\theta_+|;$$

Moreover, the points (v_-^m,u^m,θ_-^m) and (v_+^m,u^m,θ_+^m) belong to the 1-rarefaction wave curve $R_-(v_-,u_-,\theta_-)$ and the 3-rarefaction wave curve $R_+(v_+,u_+,\theta_+)$ respectively, where

$$R_\pm(v_\pm,u_\pm,\theta_\pm)=\left\{(v,u,\theta)\,\middle|\,s=s_\pm,u=u_\pm-\int_{v_\pm}^{v}\lambda_\pm(\eta,s_\pm)d\eta,v>v_\pm\right\}.$$

The 1-rarefaction wave $(v_-^r,u_-^r,\theta_-^r)(\frac{x}{t})$ (resp. the 3-rarefaction wave $(v_+^r,u_+^r,\theta_+^r)(\frac{x}{t})$) connecting (v_-,u_-,θ_-) and (v_-^m,u^m,θ_-^m) (resp. (v_+^m,u^m,θ_+^m) and (v_+,u_+,θ_+)) is the weak solution of the Riemann problem of the Euler system (1.3) with the following initial Riemann data

$$(v_\pm^r,u_\pm^r,\theta_\pm^r)(x,0)=\begin{cases}(v_\pm^m,u^m,\theta_\pm^m), & \pm x<0,\\(v_\pm,u_\pm,\theta_\pm), & \pm x>0.\end{cases}\tag{1.13}$$

Since the rarefaction waves $(v_\pm^r, u_\pm^r, \theta_\pm^r)$ are not smooth enough solutions, it is convenient to construct smooth approximate ones of them. Motivated by [17], the smooth solutions of Euler system (1.3), $(V_\pm^r, U_\pm^r, \Theta_\pm^r)$, which approximate $(v_\pm^r, u_\pm^r, \theta_\pm^r)$, are given by

$$\begin{cases} \lambda_\pm(V_\pm^r(x,t), s_\pm) = w_\pm(x,t), \\ U_\pm^r = u_\pm - \displaystyle\int_{v_\pm}^{V_\pm^r(x,t)} \lambda_\pm(\eta, s_\pm)d\eta, \\ \Theta_\pm^r = \theta_\pm(v_\pm)^{\gamma-1}(V_\pm^r)^{1-\gamma}, \end{cases} \tag{1.14}$$

where w_- (resp. w_+) is the solution of the initial problem for the typical Burgers equation:

$$\begin{cases} w_t + ww_x = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty). \\ w(x,0) = (w_r + w_l)/2 + ((w_r - w_l)/2)\tanh x, \end{cases} \tag{1.15}$$

with $w_l = \lambda_-(v_-, s_-), w_r = \lambda_+(v_-^m, s_-)$ (resp. $w_l = \lambda_+(v_+^m, s_+), w_r = \lambda_+(v_+, s_+)$).

Let $(V^{cd}, U^{cd}, \Theta^{cd})(x,t)$ be the viscous contact wave constructed in (1.10) and (1.8) with $(v_\pm, u_\pm, \theta_\pm)$ replaced by $(v_\mp^m, 0, \theta_\mp^m)$ respectively. We define

$$\begin{pmatrix} V \\ U \\ \Theta \end{pmatrix}(x,t) = \begin{pmatrix} V^{cd} + V_-^r + V_+^r \\ U^{cd} + U_-^r + U_+^r \\ \Theta^{cd} + \Theta_-^r + \Theta_+^r \end{pmatrix}(x,t) - \begin{pmatrix} v_-^m + v_+^m \\ u^m \\ \theta_-^m + \theta_+^m \end{pmatrix}, \tag{1.16}$$

and

$$(\phi, \psi, \zeta)(x,t) = (v - V, u - U, \theta - \Theta)(x,t).$$

Our second main result is as follows:

Theorem 2 *For any given (v_-, u_-, θ_-), suppose that (1.12) holds for some small $\delta_1 > 0$. Let (V, U, Θ) be as in (1.16) with strength $\delta = |\theta_+ - \theta_-| \le \bar\delta$. Then there exist positive constants $\delta_0(\le \min\{\delta_1, \bar\delta\})$ and ε_0, such that if $\delta < \delta_0$ and the initial data (v_0, u_0, θ_0) satisfies*

$$\|(v_0(\cdot) - V(\cdot,0), u_0(\cdot) - U(\cdot,0), \theta_0(\cdot) - \Theta(\cdot,0))\|_{H^1} \le \varepsilon_0,$$

then the Cauchy problem (1.1)(1.2) admits a unique global solution (v,u,θ) satisfying $(v - V, u - U, \theta - \Theta) \in X([0,\infty))$ and

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \begin{pmatrix} |(v - v_-^r - V^{cd} - v_+^r + v_-^m + v_+^m)(x,t)| \\ |(u - u_-^r - u_+^r + u^m)(x,t)| \\ |(\theta - \theta_-^r - \Theta^{cd} - \theta_+^r + \theta_-^m + \theta_+^m)(x,t)| \end{pmatrix} = 0, \tag{1.17}$$

where the $(v_-^r, u_-^r, \theta_-^r)(x,t)$ and $(v_+^r, u_+^r, \theta_+^r)(x,t)$ are the 1-rarefaction wave and 3-rarefaction one uniquely determined by (1.3)(1.13) respectively.

We now make some comments on the analysis of this paper. To show our ideas clearly, we first consider the stability of the wave only consisting of viscous contact one. As mentioned above, the elementary energy estimate of [4] cannot be applied directly to study the asymptotic behavior of viscous contact waves for solutions to Cauchy problems of (1.1) due to their analysis depending crucially on the availability of a Poincaré-type inequality, which cannot be true for Cauchy problems. To overcome this difficulty, motivated by the arguments in [6], we first derive an elementary inequality concerning the estimate of the term

$$\int\int h^2\Theta_x^2 dxdt,$$

which can be controlled essentially by some estimates of h_t (see (2.3) for details). Secondly, by using the special structure of the viscous contact wave and the compressible Navier-Stokes system (1.1), we translate the third equation of (1.1) to another form whose advantage is that we can use the first equation of (1.1) and the new form of $(1.1)_3$, together with the Sobolev inequality to get the desired estimates. Finally, for the stability of the combination of viscous contact wave with rarefaction waves, combining with the arguments in [10] to use the monotonicity of the rarefaction waves $U_\pm^r$ *w.r.t.* x, we can modify our method slightly to overcome the difficulties caused by the rarefaction waves.

Notations. Throughout this article, several positive generic constants are denoted by C, c without confusions. For functional spaces, $H^l(\mathbb{R})$ denotes the $l-$th order Sobolev space with its norm

$$\|f\|_l = \sum_{j=0}^{l} \|\partial_x^j f\|, \quad \text{where } \|\cdot\| \triangleq \|\cdot\|_{L^2(\mathbb{R})}.$$

2 Key Lemma

For $\alpha > 0$, let

$$\omega = \exp\left\{-\frac{\alpha x^2}{1+t}\right\}, \quad g = (1+t)^{-1/2} \int_{-\infty}^{x} \omega dy. \tag{2.1}$$

It is easy to check that

$$4\alpha g_t = g_{xx} = (1+t)^{-1/2}\omega_x, \quad \|g(\cdot,t)\|_{L^\infty} = \sqrt{\pi}\alpha^{-1/2}. \tag{2.2}$$

Lemma 1 *For $0 < T \le +\infty$, suppose that $h(x,t)$ satisfies*

$$h \in L^\infty(0,T;L^2(\mathbb{R})), \quad h_x \in L^2(0,T;L^2(\mathbb{R})), \quad h_t \in L^2(0,T;H^{-1}(\mathbb{R})).$$

Then the following estimate holds for any $t \in (0,T]$,

$$\begin{aligned} &\int_0^t (1+s)^{-1} \int_{\mathbb{R}} h^2\omega^2 dxds \\ &\le 4\pi\|h(0)\|^2 + 4\pi\alpha^{-1}\int_0^t \int_{\mathbb{R}} h_x^2 dxds + 8\alpha \int_0^t \langle h_t, hg^2\rangle_{H^{-1}\times H^1} ds. \end{aligned} \tag{2.3}$$

Proof. By standard arguments, we get

$$\begin{aligned} &2\int_0^t \langle h_t, hg^2\rangle_{H^{-1}\times H^1} ds \\ &= \int_{\mathbb{R}} h^2 g^2 dx - \int_{\mathbb{R}} h^2(x,0)g^2(x,0)dx - \frac{1}{2\alpha}\int_0^t (1+s)^{-1/2}\int_{\mathbb{R}} h^2 g\omega_x dxds \\ &= \int_{\mathbb{R}} h^2 g^2 dx - \int_{\mathbb{R}} h^2(x,0)g^2(x,0)dx + \frac{1}{2\alpha}\int_0^t (1+s)^{-1}\int_{\mathbb{R}} h^2\omega^2 dxds \\ &\quad + \frac{1}{\alpha}\int_0^t (1+s)^{-1/2}\int_{\mathbb{R}} hh_x g\omega dxds \\ &\ge -\frac{\pi}{\alpha}\int_{\mathbb{R}} h^2(x,0)dx + \frac{1}{4\alpha}\int_0^t (1+s)^{-1}\int_{\mathbb{R}} h^2\omega^2 dxds - \frac{\pi}{\alpha^2}\int_0^t\int_{\mathbb{R}} h_x^2 dxds, \end{aligned}$$

which yields (2.3) directly.

3 Sketch Proof of Theorem 1

We will establish the following main a priori estimate.

Proposition 1 (A priori estimate) *There exist positive constants $\delta_0 \le \min\{\delta_1, \bar{\delta}, 1\}$, $\varepsilon_0 \le 1$ and C, such that for $T > 0$ and $(\phi, \psi, \zeta) \in X([0,T])$ satisfying*

$$N(T) = \sup_{0\le t\le T} \|(\phi,\psi,\zeta)(t)\|_1^2 < \varepsilon_0, \quad |\theta_- - \theta_+| = \delta < \delta_0, \tag{3.1}$$

it follows the estimate

$$\begin{aligned}
&\|(\phi,\psi,\zeta)(t)\|^2 + \int_0^t \|(\psi_x,\zeta_x)(s)\|^2 ds \\
&\le C\left(\delta^{1/2} + \|(\phi_0,\psi_0,\zeta_0)\|^2 + \delta^{1/2}\int_0^t \|\phi_x(s)\|^2 ds\right).
\end{aligned}$$

Proof. We rewrite the Cauchy problem (1.1)(1.2) as

$$\begin{cases}
\phi_t - \psi_x = 0, \\
\psi_t + \left(\frac{R\zeta - p_+\phi}{v}\right)_x = \mu\left(\frac{\psi_x}{v}\right)_x + F, \\
\frac{R}{\gamma-1}\zeta_t + pu_x - p_+U_x = \kappa\left(\frac{\Theta_x+\zeta_x}{V+\phi} - \frac{\Theta_x}{V}\right)_x + G, \\
(\phi,\psi,\zeta)(\pm\infty, t) = 0, \\
(\phi,\psi,\zeta)(x,0) = (\phi_0,\psi_0,\zeta_0)
\end{cases} \tag{3.2}$$

where

$$F = -U_t + \mu\left(v^{-1}U_x\right)_x, G = \mu\frac{(U_x+\psi_x)^2}{v}.$$

Multiplying $(3.2)_1$ by $-R\Theta(v^{-1} - V^{-1})$, $(3.2)_2$ by ψ and $(3.2)_3$ by $\zeta\theta^{-1}$, then adding the resulting equations together, we have

$$\begin{aligned}
&\|(\phi,\psi,\zeta)(t)\|^2 + \int_0^t \|(\psi_x,\zeta_x)(s)\|^2 ds \\
&\le C\|\,(\phi_0,\psi_0,\zeta_0)\,\|^2 + C\int_0^t \|F\|_{L^1}^{4/3} ds \\
&\quad + C\int_0^t\int_{\mathbb{R}} (\zeta^2+\phi^2)\left(|\Theta_{xx}| + \Theta_x^2\right) dxds \\
&\le C\|\,(\phi_0,\psi_0,\zeta_0)\,\|^2 + C\delta^{4/3} + C\delta^{4/3}\int_0^t \|\phi_x\|^2 ds \\
&\quad + C\delta\int_0^t (1+s)^{-1}\int_{\mathbb{R}} (\zeta^2+\phi^2)\, e^{-c_2x^2/(1+s)} dxds.
\end{aligned} \tag{3.3}$$

We claim that

$$\begin{aligned}
&\int_0^t (1+s)^{-1}\int_{\mathbb{R}} (\phi^2+\psi^2+\zeta^2)\,\omega^2 dxds \\
&\le C + C\int_0^t \left(\|\phi_x\|^2 + \|\psi_x\|^2 + \|\zeta_x\|^2\right) ds.
\end{aligned} \tag{3.4}$$

Proposition 1 thus follows directly from (3.3) and (3.4) by choosing δ suitably small.

Now we are supposed to prove (3.4). Multiplying $(3.2)_2$ by $(R\zeta-p_+\phi)v(1+t)^{-1}\int_{-\infty}^x \omega^2 dy$, integrating the resulting equation over $\mathbb{R}$ leads to

$$\begin{aligned}
&\int_0^t \frac{1}{1+s}\int_{\mathbb{R}} (R\zeta - p_+\phi)^2\,\omega^2 dxds + \int_0^t \frac{1}{1+s}\int_{\mathbb{R}} \psi^2\omega^2 dxds\\
&\quad \le C + C\int_0^t \left(\|\phi_x\|^2 + \|\psi_x\|^2 + \|\zeta_x\|^2\right) ds\\
&\quad\quad + C\delta \int_0^t \frac{1}{1+s}\int_{\mathbb{R}} \left(\phi^2+\zeta^2\right)\omega^2 dxds. \qquad (3.5)
\end{aligned}$$

With (3.5) at hand, we are supposed to use Lemma 1 to get

$$\begin{aligned}
&\int_0^t \frac{1}{1+s}\int_{\mathbb{R}} (R\zeta + (\gamma-1)p_+\phi)^2\,\omega^2 dxds\\
&\quad \le C + \frac{C}{\eta}\int_0^t \left(\|\phi_x\|^2 + \|\psi_x\|^2 + \|\zeta_x\|^2\right) ds\\
&\quad\quad + C(\delta+\eta) \int_0^t \frac{1}{1+s}\int_{\mathbb{R}} \left(\phi^2+\zeta^2\right)\omega^2 dxds, \qquad (3.6)
\end{aligned}$$

which together with (3.5) thus implies (3.4) by taking first η then δ suitably small.

To get (3.6), we take $h = R\zeta + (\gamma-1)p_+\phi$ in Lemma 1 and use

$$\frac{h_t}{\gamma-1} = -\frac{R\zeta - p_+\phi}{v}(\psi_x + U_x) + \kappa\left(\frac{V\zeta_x - \phi\Theta_x}{Vv}\right)_x + G, \qquad (3.7)$$

to derive

$$\begin{aligned}
&\langle h_t, hg^2\rangle_{H^{-1}\times H^1}\\
&= -(\gamma-1)\int_{\mathbb{R}} \frac{R\zeta - p_+\phi}{v} h\psi_x g^2 dx - (\gamma-1)\int_{\mathbb{R}} \frac{R\zeta - p_+\phi}{v} hU_x g^2 dx\\
&\quad -\kappa(\gamma-1)\int_{\mathbb{R}} \frac{V\zeta_x - \phi\Theta_x}{Vv}\left(hg^2\right)_x dx + (\gamma-1)\int_{\mathbb{R}} Ghg^2 dx\\
&\le C + \frac{C}{\eta}\left(\|\phi_x\|^2 + \|\psi_x\|^2 + \|\zeta_x\|^2\right) + \frac{C(\delta+\eta)}{1+s}\int_{\mathbb{R}} \left(\phi^2+\zeta^2\right)\omega^2 dx\\
&\quad -(\gamma-1)J, \qquad (3.8)
\end{aligned}$$

with

$$\begin{aligned}
2J &= 2\int_{\mathbb{R}} v^{-1}(h^2 - \gamma p_+ h\phi)\psi_x g^2 dx\\
&= 2\int_{\mathbb{R}} v^{-1}(h^2 - \gamma p_+ h\phi)\phi_t g^2 dx\\
&= \int_{\mathbb{R}} \left(2v^{-1}h^2 g^2\phi_t - \gamma p_+ v^{-1} hg^2 (\phi^2)_t\right) dx\\
&= \left(\int_{\mathbb{R}} v^{-1} hg^2\phi\,(2h - \gamma p_+\phi)\, dx\right)_t - \int_{\mathbb{R}} v^{-1}g^2\phi(4h - \gamma p_+\phi)h_t dx\\
&\quad -\frac{1}{2\alpha\sqrt{1+t}}\int_{\mathbb{R}} v^{-1} hg\phi(2h-\gamma p_+\phi)\omega_x dx\\
&\quad +\int_{\mathbb{R}} v^{-2}v_t g^2 h\phi(2h - \gamma p_+\phi) dx. \qquad (3.9)
\end{aligned}$$

Estimate (3.6) thus holds due to (3.7).

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