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Location of the asymptotic profile for one-dimensional chemotaxis system

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1 Introduction

We consider the Cauchy problem for a one-dimensional model system of chemotaxis

$$(P)\qquad \begin{cases} u_t = au_{xx} - (uv_x)_x, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}^1 \\ v_t = bv_{xx} - v + u, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}^1 \\ (u,v)(0,x) = (u_0, v_0)(x), & x \in \mathbf{R}^1 \end{cases} \qquad (a, b > 0 : \text{constants}).$$

Our interest is in the asymptotic profile of solutions (u, v) as $t \to \infty$ when bounded solutions exist in the sense that

$$\sup_{t>0}(\|u(t,\cdot)\|_{L^q} + \|v(t,\cdot)\|_{L^q}) < +\infty \ (q = 1, \infty). \tag{1.1}$$

By Nagai, Shukuinn and Umesako [2] and Nagai and Yamada [3], it has been showed that the bounded solution to (P) in $\mathbf{R}^N$ $(N \geq 1)$ with $a = b = 1$ satisfies

$$\sup_{t>2} d(t;p)\|(u - M_0G, v - M_0G)(t,\cdot)\|_{L^p} < +\infty, \quad M_0 = \int_{\mathbf{R}^N} u_0(x)\,dx \tag{1.2}$$

with $d(t;p) = \begin{cases} t^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{2}}(\log t)^{-1} & (N=1) \\ t^{\frac{N}{2}(1-\frac{1}{p})+\frac{1}{2}} & (N \geq 2), \end{cases}$ where $G(t,x) = (4\pi t)^{-1/2}\exp(-|x|^2/4t)$.
Kato [1] has recently improved (1.2) for $N = 1$ as that the "logarithmic tail" in $d(t;p)$ can be deleted even for $a, b > 0$, not necessarily $a = b = 1$. More precisely, the second term of the asymptotics is given. If $W(t,x)$ is defined by the solution to

$$\begin{aligned} &W_t = W_{xx} - \frac{M_0^2}{2a}(G^2(a+t,x))_{xx}, \\ &W(0,x) = -\left(\int_{\mathbf{R}^1} xu_0(x)\,dx + \int_0^\infty \int_{\mathbf{R}^1} (uv_x)(t,x)\,dx\,dt\right)\frac{d}{dx}\delta(x), \end{aligned} \tag{1.3}$$

then it satisfies

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{2}}\|u(t,\cdot) - M_0G(at,\cdot) - W(at,\cdot)\|_{L^p} = 0 \tag{1.4}$$

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with $\|W(t,\cdot)\|_{L^p} \le CM_0^2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$ and $\|W(t,\cdot)\|_{L^\infty} \ge CM_0^2(1+t)^{-1}$, $t \ge 2$. The same estimate on v also holds. In the result, the logarithmic tail in (1.2) is deleted.

Here and after, let $a = 1$, $b > 0$ without loss of generality.

In this note we want to discuss the profile of solutions from the following point of view. The results above mentioned, of course, show that $M_0G(t,x)$ is an asymptotic profile of both u and v. However, we take the location of the profile into consideration. For example, when discrete statistical data are distributed by the Gauss distribution, the data are approximated by

$$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (\mu : \text{ mean}, \ \sigma : \text{ standard deviation}).$$

Here, the choice of both μ and σ is important. Suggested by this, we propose an asymptotic profile with the location μ_∞

$$M_0G(t+1, x-\mu_\infty), \quad \mu_\infty = \frac{1}{M_0}\left\{\int_{-\infty}^{\infty} xu_0(x)\,dx + \int_0^\infty\int_{-\infty}^{\infty}(uv_x)(t,x)\,dx\,dt\right\}. \tag{1.5}$$

Then we have the following theorem.

Theorem 1 *Let $N = 1$, and suppose that $u_0, v_0, v_{0x} \in L^1 \cap \mathcal{B}$ with*

$$(1+|x|^2)u_0(x) \in L^1_x \quad \text{with} \quad M_0 = \int_{\mathbf{R}^1} u_0(x)\,dx \neq 0. \tag{1.6}$$

Then the bounded solution (u, v) to (P) satisfies for $1 \le p \le \infty$ and $t \ge 0$

$$\begin{aligned} &\|u(t,\cdot) - M_0G(t+1, \cdot - \mu_\infty) + W_1(t,\cdot;\mu_\infty)\|_{L^p} \\ &\le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}(\log(2+t))^2, \end{aligned} \tag{1.7}$$

where μ_∞ by (1.5) is well-defined and the second term W_1 of asymptotics is given by

$$W_1(t,x;\mu_\infty) = \int_0^t\int_{-\infty}^{\infty} G(t-s, \cdot - y)M_0^2(GG_x)_x(s+1, y-\mu_\infty)\,dy\,ds. \tag{1.8}$$

The same estimate on v as (1.7) also holds. Moreover, W_1 is estimated from above and below:

$$\begin{aligned} &\|W_1(t,\cdot;\mu_\infty)\|_{L^p} \le CM_0^2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad t \ge 0, \\ &\|W_1(t,\cdot;\mu_\infty)\|_{L^p} \ge C^{-1}M_0^2 t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad t \ge t_0 > 0. \end{aligned} \tag{1.9}$$

In Theorem 1.1 we apply (1.8)-(1.9) to (1.7) and have the following behaviors from above and below.

Corollary 1 *Under the assumptions in Theorem 1.1, for $1 \le p \le \infty$ there hold that*

$$\|u(t,\cdot) - M_0G(t+1, \cdot - \mu_\infty), v(t,\cdot) - M_0G(t+1,\cdot - \mu_\infty)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

for $t \ge 0$, and that, for $t \ge t_1 > 0$

$$\|u(t,\cdot) - M_0G(t+1, \cdot - \mu_\infty), v(t,\cdot) - M_0G(t+1,\cdot - \mu_\infty)\|_{L^p} \ge C^{-1}M_0^2 t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$

2 Location of the profile

Integrating $(\mathrm{P})_1$(first equation of (P)) over $(0,t)\times\mathbf{R}^1$, we have

$$\int_{-\infty}^{\infty} u(t,x)\,dx = \int_{-\infty}^{\infty} u_0(x)\,dx = M_0. \tag{2.1}$$

For v, by integration of $(\mathrm{P})_2$,

$$\int_{-\infty}^{\infty} v(t,x)\,dx = e^{-t}\int_{-\infty}^{\infty} v_0(x)\,dx + M_0(1-e^{-t}) \to M_0\ (t\to\infty). \tag{2.2}$$

Hence, taking the location into consideration, we define the profile by

$$\phi(t,x) := M_0 G(t+1, x-\mu(t)), \tag{2.3}$$

and choose $\mu(t)$ as $\int_{-\infty}^{\infty}\int_{-\infty}^{x}(u-\phi)(t,y)\,dy\,dx = 0$. Since ϕ satisfies

$$\partial_t\phi = \phi_{xx} - \frac{d\mu}{dt}(t)\cdot\phi_x(t,x), \tag{2.4}$$

$u-\phi$ does

$$\partial_t(u-\phi) = (u-\phi)_{xx} + \mu'(t)\phi_x - (uv_x)_x. \tag{2.5}$$

By (2.1) we can integrate (2.5) in x twice to get

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{x}(u-\phi)(t,y)\,dy\,dx = M_0\mu'(t) - \int_{-\infty}^{\infty}(uv_x)(t,x)\,dx, \tag{2.6}$$

and hence

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{x}(u-\phi)(t,y)\,dy\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{x}(u_0(y)-\phi(0,y))\,dy\,dx + M_0(\mu(t)-\mu(0)) - \int_0^t\int_{-\infty}^{\infty}(uv_x)(s,x)\,dx\,ds \\
&= -\int_{-\infty}^{\infty} xu_0(x)\,dx + M_0\mu(t) - \int_0^t\int_{-\infty}^{\infty}(uv_x)(s,x)\,dxds,
\end{aligned} \tag{2.7}$$

because

$$\int_{-\infty}^{\infty} x\phi(0,x)\,dx = \int_{-\infty}^{\infty} x\cdot M_0G(1,x-\mu(0))\,dx = M_0\mu_0,\ \ \mu_0=\mu(0).$$

We now define $\mu(t)$ by

$$\mu(t) = \frac{1}{M_0}\left\{\int_{-\infty}^{\infty} xu_0(x)\,dx + \int_0^t\int_{-\infty}^{\infty}(uv_x)(s,x)\,dx\,ds\right\}. \tag{2.8}$$

Therefore, we can define

$$U(t,x) := \int_{-\infty}^{x}\int_{-\infty}^{y}(u-\phi)(t,z)\,dz\,dy \ \text{ or }\ u=\phi+U_{xx}, \tag{2.9}$$

which satisfies

$$
\text{(2.10)}\qquad \begin{cases} U_t = U_{xx} + \int_{-\infty}^{x} [\mu'(t)\phi(t,y) - (uv_x)(t,y)]\,dy \\ U(0,x) := U_0(x) = \int_{-\infty}^{x}\int_{-\infty}^{y} (u_0(z) - M_0 G(1, z-\mu_0))\,dz\,dy. \end{cases}
$$

To show Theorem 1.1, we need to estimate

$$
\text{(2.11)}\qquad \begin{aligned} (u-\phi)(t,x) &= \int_{-\infty}^{\infty} G_{xx}(t, x-y) U_0(y)\,dy \\ &+ \int_0^t \int_{-\infty}^{\infty} G_{xx}(t-s, x-y) \int_{-\infty}^{y} [\mu'(s)\phi(s,z) - (uv_x)(s,z)]\,dz\,dy\,ds. \end{aligned}
$$

Here we note that $U_0 \in L^1 \cap \mathcal{B}$ by (1.6) and that

$$
\text{(2.12)}\qquad \int_{-\infty}^{\infty} [\mu'(t)\phi(t,z) - (uv_x)(t,z)]\,dz = 0.
$$

3 Proof of Theorem 1.1

We only sketch the proof, whose details are given in [4]. Known estimates on the solution (u, v) to (P) in Nagai and Yamada [3] and Kato [1] are the followings.

Lemma 3.1 *For $1 \le p \le \infty$ and $t \ge 0$, the bounded solution (u, v) to (P) satisfies*

$$
\text{(3.1)}\qquad \|u(t,\cdot) - M_0 G(t+1,\cdot), v(t,\cdot) - M_0 G(t+1,\cdot)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}},
$$

$$
\text{(3.2)}\qquad \|v_x(t,\cdot) - M_0 G_x(t+1,\cdot)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}\log(2+t),
$$

$$
\text{(3.3)}\qquad \|(u-v)(t,\cdot)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}\log(2+t).
$$

By (3.1)-(3.3) we have the properties of $\mu(t)$.

Lemma 3.2 *The location $\mu(t)$ by (2.8) satisfies for $t \ge 0$*

$$
\text{(3.4)}\qquad |\mu'(t)| \le C(1+t)^{-\frac{3}{2}}\log(2+t),
$$

which implies that $\mu(\infty) = \mu_\infty$ is well-defined, and

$$
\text{(3.5)}\qquad |\mu(t) - \mu_\infty| \le C(1+t)^{-\frac{1}{2}}\log(2+t).
$$

Proof. By (3.1)-(3.2), (3.4) follows from

$$\begin{aligned}|\mu'(t)| &\leq \tfrac{1}{M_0}\left(\|(u-M_0G)(t)\|_{L^1}\|v_x(t)\|_{L^\infty}+\|M_0G(t)\|_{L^1}\|(v_x-M_0G_x)(t)\|_{L^\infty}\right)\\ &\leq C(1+t)^{-\frac{3}{2}}\log(2+t).\end{aligned}$$

Hence (3.5) follows easily. □

By the mean value theorem we have

$$\|\phi(t,\cdot)-M_0G(t+1,\cdot-\mu_\infty)\|_{L^p}\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}\log(2+t)$$

and

$$\|W_1(t,\cdot;\mu_\infty)-W_1(t,\cdot;0)\|_{L^p}\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}. \tag{3.6}$$

Hence, to show (1.7), it is enough to prove the following proposition, which is a main estimate in this note. The same estimate on v is derived by (3.3).

Proposition 3.1 *Under the conditions in Theorem 1.1 it holds*

$$\|u(t,\cdot)-\phi(t,\cdot)+W_1(t,\cdot;0\|_{L^p}\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}(\log(2+t))^2. \tag{3.7}$$

Proof. By (2.11) and (1.8)

$$\begin{aligned}&(u-\phi)(t,x)+W_1(t,x;0)\\ &=\int G_{xx}(t,x-y)U_0(y)\,dy+\int_0^t\int G_{xx}(t-s,x-y)\times\\ &\qquad\times\int_{-\infty}^y[\mu'(s)\phi(s,z)-(uv_x)(s,z)+M_0^2(GG_x)(s+1,z)]\,dz\,dy\,ds\\ &=: I_0+I_1.\end{aligned}$$

By $U_0\in L^1\cap\mathcal{B}$ it is easy to see that

$$\|I_0\|_{L^p}\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}$$

for $t\geq 0$. For $0\leq t\leq 1$, $\|I_1\|_{L^p}\leq C$ easily. For $t\geq 1$ we set

$$I_1=\int_0^{t/2}+\int_{t/2}^t=:I_{11}+I_{12}.$$

By (1.6) we note that

$$\|u(t,\cdot)\|_{L^{1,1}}\leq C(1+t)^{\frac{1}{2}},\quad \|u(t,\cdot)-M_0G(t+1,\cdot)\|_{L^{1,1}}\leq C, \tag{3.8}$$

where $L^{p,m} = \{f \in L^p;\ \|f\|_{L^{p,m}} := \|(1+|\cdot|)^m f\|_{L^p} < +\infty\}$ (These are shown by applying the method in [2]). Therefore, by (2.12) and (3.8)

$$
\begin{aligned}
&\|I_{11}\|_{L^p} \\
&\le C\int_0^{t/2} (t-s)^{-\frac{1}{2}(1-\frac{1}{p})-1} \Big\| \int_{-\infty}^{x} \{\mu'(s)\phi(s,z) \\
&\qquad\qquad -[(u-M_0G)v_x + M_0G(v_x - M_0G_x)](s,z)\}\, dz \Big\|_{L^1_x} ds \\
&\le Ct^{-\frac{1}{2}(1-\frac{1}{p})-1} \int_0^{t/2} [|\mu'(s)|\|G(s,\cdot-\mu(s))\|_{L^{1,1}} + \|(u-M_0G)(s)\|_{L^{1,1}}\|v_x(s)\|_{L^\infty} \\
&\qquad\qquad +\|G(s)\|_{L^{1,1}}\|(v_x - M_0G_x)(s)\|_{L^\infty}]\, ds \\
&\le Ct^{-\frac{1}{2}(1-\frac{1}{p})-1} \int_0^{t/2} [(1+s)^{-\frac{3}{2}}\log(2+s)\cdot(1+s)^{\frac{1}{2}} \\
&\qquad\qquad +(1+s)^{-1} + (1+s)^{\frac{1}{2}}(1+s)^{-\frac{3}{2}}\log(2+s)]\, ds \\
&\le Ct^{-\frac{1}{2}(1-\frac{1}{p})-1}(\log(2+t))^2.
\end{aligned}
$$

Here, we have denoted $\|u(s,y) - M_0G(s+1,y)\|_{L^p_y} = \|(u-M_0G)(s)\|_{L^p}$ etc. for simplicity. For I_{12}, by the integral by parts,

$$
\begin{aligned}
I_{12} &= \int_{t/2}^{t}\int G(t-s,x-y)\mu'(s)M_0G_x(s+1,y-\mu(s))\, dy\, ds \\
&\quad + \int_{t/2}^{t}\int G_x(t-s,x-y)[(u-M_0G)v_x + M_0G(v_x - M_0G_x)](s,y)\, dy\, ds \\
&=: I_{12}^1 + I_{12}^2,
\end{aligned}
$$

Each part is estimated as follow:

$$
\begin{aligned}
\|I_{12}^1\|_{L^p} &\le C\int_{t/2}^{t} (t-s)^{-\frac{1}{2}(1-\frac{1}{p})}|\mu'(s)|\|G_x(s)\|_{L^1}\, ds \\
&\le C\int_{t/2}^{t} (t-s)^{-\frac{1}{2}(1-\frac{1}{p})}(1+s)^{-\frac{3}{2}}\log(2+s)\cdot(1+s)^{-\frac{1}{2}}\, ds \\
&\le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}\log(2+t),
\end{aligned}
$$

$$
\begin{aligned}
\|I_{12}^2\|_{L^1} &\le C\int_{t/2}^{t} \|G_x(t-s)\|_{L^1}\Big(\|(u-M_0G)(s)\|_{L^1}\|v_x(s)\|_{L^\infty} \\
&\qquad +\|G(s)\|_{L^1}\|(v_x - M_0G_x)(s)\|_{L^\infty}\Big)\, ds \\
&\le C\int_{t/2}^{t} (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{3}{2}}\log(2+s)\, ds \\
&\le C(1+t)^{-1}\log(2+t)
\end{aligned}
$$

and

$$
\begin{aligned}
\|I_{12}^2\|_{L^\infty} &\le C\int_{t/2}^{t} \|G_x(t-s)\|_{L^2}\Big(\|(u-M_0G)(s)\|_{L^2}\|v_x(s)\|_{L^\infty} \\
&\qquad +\|G(s)\|_{L^2}\|(v_x - M_0G_x)(s)\|_{L^\infty}\Big)\, ds \\
&\le C\int_{t/2}^{t} (t-s)^{-\frac{1}{4}}(1+s)^{-\frac{7}{4}}\log(2+s)\, ds \\
&\le C(1+t)^{-1}\log(2+t).
\end{aligned}
$$

Combining all estimates, we obtain (3.6). □

Completion of the proof of Theorem 1.1. We show (1.9). By an elementary calculation

$$\int_{-\infty}^{\infty} G(t-s, x-y)G^2(s+1, y)\,dy = \frac{G(t-\frac{s-1}{2}, x)}{\sqrt{8\pi(s+1)}}.$$

Hence, when $\mu_\infty = 0$,

$$W_1(t, x; 0) = \frac{M_0^2}{2}\int_0^t \frac{G_{xx}(t-\frac{s-1}{2}, x)}{\sqrt{8\pi(s+1)}}ds. \tag{3.9}$$

Similar representation to (3.9) is found in [1]. We craim, for $t \ge t_0 > 0$,

$$\int_0^{\sqrt{(t+1)/2}} \left| \int_0^t \frac{G_{xx}(t-\frac{s-1}{2}, x)}{\sqrt{s+1}}ds \right| dx \ge c(t+1)^{-\frac{1}{2}}, \tag{3.10}$$

and, when $0 \le x \le \sqrt{(t+1)/2}$,

$$\left| \int_0^t \frac{G_{xx}(t-\frac{s-1}{2}, x)}{\sqrt{s+1}}ds \right| \ge c(t+1)^{-1}. \tag{3.11}$$

In fact, since $G_x(t, x) = -\frac{x}{2t}G(t, x)$ and $G_{xx}(t, x) = \frac{1}{2t}(\frac{x^2}{2t} - 1)G(t, x)$,

$$-G_{xx}(t - \frac{s-1}{2}, x) \ge \frac{G(t-\frac{s-1}{2}, x)}{4(t-\frac{s-1}{2})} > 0, \quad \text{for } 0 \le x \le \sqrt{(t+1)/2}.$$

Hence,

$$\begin{aligned}
&\text{the left-hand side in (3.10)}\\
\ge\ & c\int_0^{\sqrt{(t+1)/2}} \int_0^t \frac{-G_{xx}(t-\frac{s-1}{2}, x)}{\sqrt{s+1}}ds\,dx \ = \ c\int_0^t \frac{-G_x(t-\frac{s-1}{2}, \sqrt{\frac{t+1}{2}})}{\sqrt{s+1}}ds\\
\ge\ & c(t+1)^{\frac{1}{2}} \int_0^t (s+1)^{-\frac{1}{2}}(t - \frac{s-1}{2})^{-\frac{3}{2}}ds\\
\ge\ & c(t+1)^{-\frac{1}{2}}, \quad t \ge t_0,
\end{aligned}$$

and

$$\begin{aligned}
&\text{the left-hand side in (3.11)}\\
\ge\ & c\int_0^t \frac{G(t-\frac{s-1}{2}, x)}{\sqrt{s+1}(t-\frac{s-1}{2})}ds \ge c\int_0^t \frac{G(t-\frac{s-1}{2}, \sqrt{\frac{t+1}{2}})}{\sqrt{s+1}(t-\frac{s-1}{2})}ds\\
\ge\ & c\int_0^t (s+1)^{-\frac{1}{2}}(t - \frac{s-1}{2})^{-\frac{3}{2}}ds\\
\ge\ & c(t+1)^{-1}, \quad t \ge t_0.
\end{aligned}$$

By (3.10) and (3.11), for $1 \leq p < \infty$,

$$\begin{aligned} \|W_1(t,\cdot;0)\|_{L^p} &\geq \left(\int_0^{\sqrt{(t+1)/2}} \left| \int_0^t \frac{G_{xx}(t-\frac{s-1}{2},x)}{\sqrt{s+1}} ds \right|^p dx \right)^{\frac{1}{p}} \\ &\geq c \left(\int_0^{\sqrt{(t+1)/2}} (t+1)^{-(p-1)} \int_0^t \frac{-G_{xx}(t-\frac{s-1}{2},x)}{\sqrt{s+1}} ds\, dx \right)^{\frac{1}{p}} \\ &\geq c(t+1)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad t \geq t_0. \end{aligned}$$

When $p = \infty$, it is easy to show

$$\|W_1(t,\cdot\,;0)\|_{L^\infty} \geq |W_1(t,0;0)| \geq c(t+1)^{-1}, \quad t \geq t_0.$$

When $\mu_\infty \neq 0$, $W_1(t,x;\mu_\infty) = W_1(t,x;0) + (W_1(t,x;\mu_\infty) - W_1(t,x;0))$ and $W_1(t,x;\mu_\infty) - W_1(t,x;0)$ decays faster by (3.6). Hence the estimate from below in (1.9) holds. The estimate from above is obtained easier by (3.9), which completes the proof of Theorem 1.1.

References

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