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Homeomorphism and diffeomorphism groups of non-compact manifolds with the Whitney topology

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In this report we discuss local and global topological types of homeomorphism and diffeomorphism groups of non-compact manifolds with the Whitney topology [3] (cf. [1, 2, 4]). For this purpose we need some criterion which implies that a topological group is (locally) homeomorphic to the box power $\square^\omega l_2$ of the Hilbert space l_2 and the small box power $\boxdot^\omega l_2$ ($\approx \mathbb{R}^\infty \times \ell_2$). This problem is treated in Section 1, and the applications to homeomorphism and diffeomorphism groups are discussed in Section 2.

1. Local topological type of topological groups

1.1. LF-spaces and box products.

Suppose G is a separable completely metrizable ANR topological group. It is known that G is a Lie group (so that G is an n-manifold) if G is locally compact and that G is an l_2-manifold if G is non-locally compact (A. Gleason [9], D. Montgomery – L. Zippin [22], T. Dobrowolski – H. Toruńczyk [7]).

A *Fréchet space* is a completely metrizable locally convex topological linear space and an LF-space is the direct limit of increasing sequence of Fréchet spaces in the category of locally convex topological linear spaces. The simplest example is $\mathbb{R}^\infty$, which is the direct limit of the tower of Euclidean spaces

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots .$$

P. Mankiewicz [20] showed that a separable LF-space is homeomorphic to either l_2, $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times l_2$. The space $l_2 \times \mathbb{R}^\infty$ is homeomorphic to $\boxdot^\omega l_2$ and the latter is a subspace of the box power $\square^\omega l_2$.

The box product $\Box_{n\in\mathbb{N}}X_n$ of a sequence of topological spaces $(X_n)_{n\in\mathbb{N}}$ is the product $\prod_{n\in\mathbb{N}} X_n$ endowed with the box topology. This topology generated by the base consisting of subsets $\prod_{n\in\mathbb{N}} U_n$, where U_n is an open set in X_n. The small box product $\boxdot_{n\in\mathbb{N}}X_n$ of a sequence of pointed spaces $(X_n, *_n)_{n\in\mathbb{N}}$ is the subspace of $\Box_{n\in\mathbb{N}}X_n$ defined by

$$\boxdot_{n\in\mathbb{N}}X_n = \left\{(x_n)_{n\in\mathbb{N}} \in \Box_{n\in\mathbb{N}}X_n : \exists\, m \in \mathbb{N} \text{ s.t. } x_n = *_n \ (n \geq m)\right\}.$$

For simplicity, the pair $(\Box_{n\in\mathbb{N}}X_n, \boxdot_{n\in\mathbb{N}}X_n)$ is denoted by the symbol $(\Box, \boxdot)_{n\in\mathbb{N}}X_n$. Note that the small box product $\boxdot_{n\in\omega}L_n$ of non-trivial separable Hilbert spaces is homeomorphic to $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times l_2$ (the first case occurs iff all L_n are finite-dimensional).

In this section we seek some conditions under which a topological group is (locally) homeomorphic to $\boxdot^\omega l_2$ and $\Box^\omega l_2$.

1.2. Topological groups (locally) homeomorphic to $\boxdot^\omega l_2$.

Suppose G is a topological group with the unit element $e \in G$. A tower of closed subgroups in G is a sequence $(G_n)_{n\in\mathbb{N}}$ of closed subgroups of G such that

$$G_1 \subset G_2 \subset G_3 \subset \cdots \quad \text{and} \quad G = \bigcup_{n\in\mathbb{N}} G_n.$$

This tower yields the small box product $\boxdot_{n\in\mathbb{N}}G_n$ with the unit element $\boldsymbol{e} = (e, e, \dots)$ and the multiplication map

$$p : \boxdot_{n\in\mathbb{N}}G_n \to G, \quad p(x_0, \dots, x_n, e, e, \cdots) = x_0\, x_1 \cdots x_n.$$

This map p is continuous.

Definition 1.1. We say that G carries the box topology with respect to $(G_n)_{n\in\mathbb{N}}$ if the map $p : \boxdot_{n\in\mathbb{N}}G_n \to G$ is open.

The box topology is related to the direct limit in the category of topological groups. Recall that G is the direct limit of the tower $(G_n)_{n\in\mathbb{N}}$ in G in the category of topological groups iff any group homomorphism $h : G \to H$ to an arbitrary topological group H is continuous provided its restrictions $h|_{G_n}$ are continuous for all $n \in \mathbb{N}$. It is seen that if G carries the box topology with respect to $(G_n)_{n\in\mathbb{N}}$, then (i) G is the direct limit of $(G_n)_{n\in\mathbb{N}}$ in the category of topological groups and (ii) any compact subset $K \subset G$ lies in some subgroup G_n.

Each closed subgroup H of G induces the left coset space $G/H = \{xH : x \in G\}$ (endowed with the quotient topology) and the quotient map $\pi : G \to G/H$, $\pi(x) = \overline{x} \equiv xH$. Let $(G/H)_0$ denote the connected component of $\overline{e}$ in G/H.

Definition 1.2. We say that H is (locally) topologically complemented ((L)TC) in G if the map $\pi : G \to G/H$ has a (local) section at $\overline{e}$, If H is LTC in G, then the map $\pi : G \to G/H$ is a principal H-bundle, and if H is TC in G, then this bundle is trivial.

Definition 1.3. A tower $(G_n)_{n\in\mathbb{N}}$ of closed subgroups of G is said to be (locally) topologically complemented ((L)TC) if each G_n is (L)TC in G_{n+1}.

The following is our main theorem in this subsection.

Theorem 1.1. Suppose that

($*_1$) G carries the box topology with respect to a LTC tower $(G_n)_{n\in\mathbb{N}}$,

($*_2$) $G_n/G_{n-1} \approx_\ell E_n$ $(n \in \mathbb{N})$,

where $(E_n)_{n\in\mathbb{N}}$ is a sequence of non-trivial Hilbert spaces and $G_0 = \{e\}$.

Let H and H_n $(n \in \mathbb{N})$ be the identity component of G and G_n $(n \in \mathbb{N})$ respectively. Then, the following hold:

(1) $G \approx_\ell \boxdot_{n\in\mathbb{N}} E_n$.

(2) H is an open normal subgroup of G.
Hence, G/H carries the discrete topology and $G \approx H \times G/H$.

(3) (i) If each H_n is contractible, then $H \approx \boxdot_{n\in\mathbb{N}} E_n$.
(ii) If $\left(G_n/G_{n-1}\right)_0$ is contractible for every $n \geq 2$, then $H \approx H_1 \times \boxdot_{n\geq 2} E_n$.

This theorem follows from the next key lemma.

Lemma 1.1. If G carries the box topology with respect to a (L)TC tower $(G_n)_{n\in\mathbb{N}}$ in G, then $G \approx_{(\ell)} \boxdot_{n\in\mathbb{N}} G_n/G_{n-1}$ (where $G_0 = \{e\}$).

1.3. **Topological groups (locally) homeomorphic to $\square^\omega l_2$.**

Throughout this subsection we assume that M is a locally compact and σ-compact space. Let $\mathcal{H}(M)$ denote the group of homeomorphisms of M endowed with the Whitney topology. The topological group $\mathcal{H}(M)$ has a canonical continuous action on M and admits infinite products of elements with discrete supports. This geometric property distinguishes this group from other abstract topological groups and relates it to the box topology. In this subsection we discuss a condition under which transformation groups are locally homeomorphic to the box product $\square^\omega l_2$.

A transformation group G on M means a topological group G acting on M continuously and effectively. The group $\mathcal{H}(M)$ is a typical example. Each $g \in G$ induces a canonical homeomorphism of M, which is denoted by the symbol $\hat{g}$. Let G_0 denote the connected component of the unit element e in G and let $G_c = \{g \in G : \operatorname{supp}\hat{g} \text{ is compact}\}$. For any subsets K, N of M we obtain the following subgroups of G:

$$G_K = \{g \in G : \hat{g}|_K = \mathrm{id}_K\}, \quad G(N) = G_{M\setminus N}, \quad G_K(N) = G_K \cap G(N), \quad G_{K,c} = G_K \cap G_c.$$

For a subgroup H of G we have a natural projection $\pi : G \to G/H$. The symbols H_0 and $(G/H)_0$ denote the connected components of e in H and $\overline{e}$ in G/H respectively.

For subsets $K \subset L \subset N$ of M, consider the space of G-embeddings

$$\mathcal{E}_K^G(L,N) = \{\hat{g}|_L : L \to M \mid g \in G_K(N)\}.$$

This set is endowed with the quotient topology induced by the restriction map

$$r : G_K(N) \to \mathcal{E}_K^G(L,N), \quad r(g) = \hat{g}|_L.$$

Let $\mathcal{E}_K^G(L,N)_0$ denote the connected component of the inclusion $i_L : L \subset M$ in $\mathcal{E}_K^G(L,N)$. The subgroup $G_K(N)$ acts on $\mathcal{E}^G(L,M)$ continuously and transitively and we have the commutative triangle:

$$\begin{tikzcd} & G_K(N) \arrow[dl, "\pi"'] \arrow[dr, "r"] & \\ G_K(N)/G_L(N) \arrow[rr, "\varphi", "\approx"'] & & \mathcal{E}_K^G(L,N) \end{tikzcd} \qquad \varphi(\overline{g}) = \hat{g}|_L = g \cdot i_L.$$

The symbol K is omitted if $K = \emptyset$.

Definition 1.4. We say that a triple (N, L, K) has the local section property for G (LSP$_G$) if the restriction map $r : G_K(N) \to \mathcal{E}_K^G(L,M)$ has a local section at the inclusion i_L.

Every discrete family $\mathcal{L} = \{L_i\}_{i\in\mathbb{N}}$ in M induces maps

(i) $\lambda_{\mathcal{L}} : \square_{i\in\mathbb{N}} G(L_i) \to \mathcal{H}(M)$ defined by

$$\lambda_{\mathcal{L}}((g_i)_{i\in\mathbb{N}})|_{L_j} = \widehat{g_j}|_{L_j} \ (j \in \mathbb{N}) \quad \text{and} \quad \lambda_{\mathcal{L}}((g_i)_{i\in\mathbb{N}}) = \mathrm{id} \ \text{ on } \ M \setminus \textstyle\bigcup_{i\in\mathbb{N}} L_i$$

(ii) $r_{\mathcal{L}} : G \to \square_{i\in\mathbb{N}} \mathcal{E}^G(L_i, M) : \quad r_{\mathcal{L}}(g) = (\hat{g}|_{L_i})_{i\in\mathbb{N}}$.

Definition 1.5. Suppose $\mathcal{F}$ is a collection of subsets of M. We say that a transformation group G on M has a strong topology with respect to $\mathcal{F}$ if it satisfies the following conditions:

(i) The injection $G \to \mathcal{H}(M) : g \mapsto \hat{g}$ is continuous.

(ii) For any discrete family $\mathcal{L} = \{L_i\}_{i\in\mathbb{N}}$ in M, (a) $\mathrm{im}\, \lambda_{\mathcal{L}} \subset G$ and (b) the map $\lambda_{\mathcal{L}} : \square_{i\in\mathbb{N}} G(L_i) \to G\big(\bigcup_{i\in\mathbb{N}} L_i\big)$ is an open embedding.

(iii) For any discrete family $\mathcal{L} = \{L_i\}_{i\in\mathbb{N}}$ with $L_i \in \mathcal{F}$ $(i \in \mathbb{N})$, the map $r_{\mathcal{L}} : G \to \square_{i\in\mathbb{N}} \mathcal{E}^G(L_i, M)$ is continuous.

It is shown that if the injection $G \to \mathcal{H}(M)$ is continuous, then $G_0 \subset G_c$ and every compact subspace $\mathcal{K} \subset G_c$ is contained in $G(K)$ for some compact subset $K \subset M$.

There exists a sequence $(M_i)_{i\in\mathbb{N}}$ of compact regular closed subsets of M such that $M_i \subset \mathrm{int}_M M_{i+1}$ $(i \in \mathbb{N})$ and $M = \bigcup_{i\in\mathbb{N}} M_i$. It induces the tower $(G(M_i))_{i\in\mathbb{N}}$ of closed subgroups of G_c and the multiplication map

$$p : \square_{i\in\mathbb{N}} G(M_i) \to G_c, \quad p(h_1, \ldots, h_n) = h_1 \cdots h_n.$$

Let $K_i = M \setminus \mathrm{int}_M M_i$ ($i \in \omega$) and $L_i = M_i \setminus \mathrm{int}_M M_{i-1}$ ($i \in \mathbb{N}$), where $M_0 = \emptyset$. There exists a sequence $(N_i)_{i\in\mathbb{N}}$ of compact subsets of M such that $L_i \subset \mathrm{int}_M N_i$ and $N_i \cap N_j \neq \emptyset$ iff $|i-j| \leq 1$. We call each of the sequences $(M_i)_{i\in\mathbb{N}}$, $(M_i, L_i, N_i)_{i\in\mathbb{N}}$ and $(M_i, K_i, L_i, N_i)_{i\in\mathbb{N}}$ an *exhausting sequence* for M.

From now on, we assume that G is a transformation group on M with a strong topology with respect to a collection $\mathcal{F}$ of subsets of M.

Lemma 1.2. Suppose $(M_i, L_i, N_i)_{i\in\mathbb{N}}$ is an exhausting sequence for M. If (N_{2i}, L_{2i}) has LSP_G and $L_{2i} \in \mathcal{F}$ for each $i \in \mathbb{N}$, then the following holds:

(1) $(G, G_c) \approx_\ell (\square, \boxdot)_{i\in\mathbb{N}}\mathcal{E}^G(L_{2i}, M) \times (\square, \boxdot)_{i\in\mathbb{N}} G(L_{2i-1})$,

(2) The map $p : \boxdot_{i\in\mathbb{N}} G(M_i) \to G_c$ has a local section. Hence, the group G_c carries the box topology with respect to the tower $(G(M_i))_{i\in\mathbb{N}}$.

We can combine Lemma 1.2 and Theorem 1.1 to obtain the next practical criterion. For any exhausting sequence $(M_i)_{i\in\mathbb{N}}$ of M, we set

$$M_i = \emptyset \quad (i \leq 0) \qquad \text{and} \qquad M_i^j = M_j \setminus \mathrm{int}_M M_i \quad (i < j).$$

Theorem 1.2. Suppose $(M_i)_{i\in\mathbb{N}}$ is an exhausting sequence for M such that

$(\star_0)$ $M_i^j \in \mathcal{F}$ $(\star_1)$ (M_{i-1}^{j-1}, M_i^j) has LSP_G and $(\star_2)$ $G(M_i^j) \approx_\ell \ell_2$ for any $0 \leq i < j$.

Then, the following hold:

(1) $(G, G_c) \approx_\ell (\square, \boxdot)^\omega \ell_2$.

(2) The group G_c carries the box topology with respect to the tower $(G(M_i))_{i\in\mathbb{N}}$. The tower $(G(M_i))_{i\in\mathbb{N}}$ is LTC.

(3) G_0 is an open normal subgroup of G_c.
Hence, G_c/G_0 carries the discrete topology and $G_c \approx G_0 \times G_c/G_0$.
   (i) If each $G(M_i)_0$ is contractible, then $G_0 \approx \boxdot^\omega \ell_2$.
   (ii) If $\mathcal{E}^G_{K_i}(K_{i-1}, M)_0$ is contractible for every $i \geq 2$, then $G_0 \approx G(M_1)_0 \times \boxdot^\omega \ell_2$.

2. Homeomorphism and diffeomorphism groups of non-compact manifolds with the Whitney topology

In this section we apply Theorem 1.2 to study (local) topological types of homeomorphism groups and diffeomorphism groups of non-compact manifolds endowed with the Whitney topology.

2.1. Homeomorphism groups of non-compact n-manifolds.

Suppose M is a separable n-manifold possibly with boundary. When M is compact, the group $\mathcal{H}(M)$ is known to be locally contractible [6, 8]. We can extend this result to the non-compact case.

Proposition 2.1. The group $\mathcal{H}_c(M)$ is locally contractible for any separable n-manifold M possibly with boundary.

In [2] it is shown that $(\mathcal{H}(\mathbb{R}), \mathcal{H}_c(\mathbb{R})) \approx (\square, \boxdot)^\omega l_2$. Below we deduce a complete classification of the topological types of the groups $\mathcal{H}_c(M)$ and $\mathcal{H}_0(M)$ for a non-compact connected 2-manifold M.

Suppose M is a connected 2-manifold possibly with boundary. Since $\mathcal{H}_0(M) \subset \mathcal{H}_c(M)$, we obtain the mapping class group $\mathcal{M}_c(M) = \mathcal{H}_c(M)/\mathcal{H}_0(M)$ with the quotient topology. First we recall the topological classification of $\mathcal{H}_0(M)$ in the compact case. (R. Luke – W.K. Mason [19], M. Hamstrom [12], cf. [29]).

Theorem 2.1. Suppose M is a compact connected 2-manifold.

(1) $\mathcal{H}(M)$ is an l_2-manifold and $\mathcal{H}_0(M)$ is an open normal subgroup of $\mathcal{H}(M, K)$.

(2) $\mathcal{H}_0(M) \approx$
$$\begin{array}{ll} \mathrm{SO}(3) \times l_2 & \text{if } M \approx \mathbb{S}^2 \text{ or } \mathbb{P}, \\ \mathbb{T} \times l_2 & \text{if } M \approx \mathbb{T}, \\ \mathbb{S}^1 \times l_2 & \text{if } M \approx \mathbb{D}, \mathbb{A}, \mathbb{M} \text{ or } \mathbb{K}, \\ l_2 & \text{in all other cases.} \end{array}$$

Here, $\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{D}$, $\mathbb{A}$, $\mathbb{P}$, $\mathbb{M}$, $\mathbb{K}$ and $\mathbb{T}$ stand for the circle, the sphere, the disk, the annulus, the projective plane, the Möbius band, the Klein bottle and the torus, respectively.

The following is the main theorem of this subsection, which describes the topological classification in the non-compact case. Consider the following condition for non-compact connected 2-manifolds M:

(*) $M \approx X \backslash K$, where $X = \mathbb{A}$, $\mathbb{D}$ or $\mathbb{M}$, and K is a non-empty compact subset of a boundary circle of X.

Theorem 2.2. Suppose M is a non-compact connected 2-manifold.

(1) $(\mathcal{H}(M), \mathcal{H}_c(M)) \approx_\ell (\square^\omega l_2, \boxdot^\omega l_2)$. Hence $\mathcal{H}_c(M)$ is an $(\mathbb{R}^\infty \times l_2)$-manifold.

(2) $\mathcal{H}_0(M)$ is an open normal subgroup of $\mathcal{H}_c(M)$.
Thus, $\mathcal{M}_c(M)$ has the discrete topology and $\mathcal{H}_c(M) \approx \mathcal{H}_0(M) \times \mathcal{M}_c(M)$.

(3) $\mathcal{H}_0(M) \approx \mathbb{R}^\infty \times l_2$ and $\mathcal{H}_c(M) \approx \begin{cases} \mathbb{R}^\infty \times l_2 & \text{in the case } (*), \\ \mathbb{R}^\infty \times l_2 \times \mathbb{Z} & \text{in all other cases.} \end{cases}$

Theorem 2.2 follows from Theorem 1.2 together with Theroem 2.3 and Lemma 2.1 below. We say that an embedding $f : L \to M$ is (i) proper if $f^{-1}(\partial M) = L \cap \partial M$ and (ii) extendable if $f = h|_L$ for some $h \in \mathcal{H}(M)$. For subsets $K \subset L \subset M$, let $\mathcal{E}_K(L, M)$ denote the space of embeddings $f : L \to M$ with $f|_K = \mathrm{id}_K$ endowed with the compact-open topology, and let $\mathcal{E}_K^\star(L, M) \subset \mathcal{E}_K^*(L, M)$ denote the subspaces of $\mathcal{E}_K(L, M)$ consisting of extendable embeddings and proper embeddings respectively. The next theorem represents an extension property of embeddings (or a bundle theorem for the homeomorphism groups of 2-manifolds) and the ℓ_2-manifold property of the embedding spaces ([19], [28, 29]).

Theorem 2.3. Suppose M is a 2-manifold and $K \subset L$ are two subpolyhedra of M such that $\mathrm{cl}_M(L \setminus K)$ is compact.

(1) (i) For any closed subset C of M with $C \cap \mathrm{cl}_M(L \setminus K) = \emptyset$, the restriction map

$$r : \mathcal{H}(M, K) \longrightarrow \mathcal{E}_K^*(L, M), \quad r(h) = h|_L$$

has a local section $s : \mathcal{U} \longrightarrow \mathcal{H}_0(M, K \cup C) \subset \mathcal{H}(M, K)$ at i_L.

(ii) The restriction map $r : \mathcal{H}(M, K) \longrightarrow \mathcal{E}_K^*(L, M)$ is a principal $\mathcal{H}(M, L)$-bundle. Thus, $\mathcal{H}(M, K)/\mathcal{H}(M, L) \approx \mathcal{E}_K^*(L, M)$ and $\mathcal{H}(M, L)$ is LTC in $\mathcal{H}(M, K)$.

(2) (i) $\mathcal{E}_K^*(L, M)$ is open in $\mathcal{E}_K^*(L, M)$.

(ii) $\mathcal{E}_K^*(L, M)$ and $\mathcal{E}_K^*(L, M)$ are l_2-manifolds if $\dim(L \setminus K) \geq 1$.

Lemma 2.1. Suppose M is a connected non-compact 2-manifold M possibly with boundary. Then, the following conditions are equivalent:

(1) $\mathcal{M}_c(M)$ is trivial.
(2) $\mathcal{M}_c(M)$ is a torsion group.
(3) M satisfies the condition $(*)$.

Remark 2.1. It is interesting to compare the Whitney topology with the compact-open topology on the homeomorphism group $\mathcal{H}(M)$ of a non-compact connected 2-manifold M. Let $\mathcal{H}_0(M)_{co}$ denote the identity component of the group $\mathcal{H}(M)$ endowed with the compact-open topology. In [29] we have shown that

$$\mathcal{H}_0(M)_{co} \approx \begin{cases} \mathbb{S}^1 \times l_2 & \text{if } M \approx \mathbb{R}^2,\ \mathbb{S}^1 \times \mathbb{R},\ \mathbb{S}^1 \times [0, \infty) \text{ or } \mathbb{M} \setminus \partial\mathbb{M}, \\ l_2 & \text{in all other cases.} \end{cases}$$

2.2. **Diffeomorphism groups of non-compact smooth manifolds.**

Suppose M is a separable C^∞ n-manifold without boundary. Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of M endowed with the Whitney C^∞-topology (cf. [15]). In this final subsection, we study the local topological type of the groups $\mathcal{D}(M)$ and $\mathcal{D}_c(M)$ for a non-compact manifold M.

For the group $G = \mathcal{D}(M)$ we use the following standard notations:

$$\mathcal{D}_0(M) = G_0, \quad \mathcal{D}_c(M) = G_c, \quad \mathcal{D}(M, K) = G_K, \quad \mathcal{D}_0(M, K) = (G_K)_0 \quad (K \subset M).$$

Since the inclusion $\mathcal{D}(M) \subset \mathcal{H}(M)$ is continuous, we have $\mathcal{D}_0(M) \subset \mathcal{D}_c(M)$. The quotient group $\mathcal{M}_c^\infty(M) = \mathcal{D}_c(M)/\mathcal{D}_0(M)$ (with the quotient topology) is called the mapping class group of M. Let $\mathcal{F} = \mathcal{F}^\infty(M)$ denote the collection of all smooth n-submanifolds of M which are closed in M. For $L \in \mathcal{F}$ and a subset $K \subset L$, let $\mathcal{E}_K^\infty(L, M)$ denote the space of

C^∞-embeddings $f : L \to M$ with $f|_K = i_K$ and let $\mathcal{E}_K^{\infty,*}(L, M) = \mathcal{E}_K^G(L, M)$. These spaces are endowed with the compact-open C^∞-topology. There is a natural restriction map

$$r : \mathcal{D}(M, K) \to \mathcal{E}_K^\infty(L, M), \quad r(h) = h|_L.$$

The following is the main result of this subsection.

Theorem 2.4. Suppose M is a non-compact separable C^∞ n-manifold without boundary.

(1) $(\mathcal{D}(M), \mathcal{D}_c(M)) \approx_\ell (\square^\omega l_2, \boxdot^\omega l_2)$. Hence, $\mathcal{D}_c(M)$ is an $(\mathbb{R}^\infty \times l_2)$-manifold.

(2) $\mathcal{D}_0(M)$ is a clopen subgroup of $\mathcal{D}_c(M)$ and $\mathcal{D}_c(M) \approx \mathcal{D}_0(M) \times \mathcal{M}_c^\infty(M)$.

(3) Suppose $(M_i)_{i\in\mathbb{N}}$ is an exhausting sequence of M consisting of compact C^∞ n-submanifolds of M. Let $K_i = M \setminus \mathrm{int}_M M_i$ $(i \in \mathbb{N})$.

 (i) If $\mathcal{D}_0(M, K_i)$ is contractible for each $i \in \mathbb{N}$, then $\mathcal{D}_0(M) \approx \mathbb{R}^\infty \times l_2$.

 (ii) If $\mathcal{E}_{K_{i+1}}^{\infty,*}(K_i, M)_0$ is contractible for each $i \in \mathbb{N}$, then

$$\mathcal{D}_0(M) \approx \mathcal{D}_0(M, K_1) \times (\mathbb{R}^\infty \times l_2).$$

Corollary 2.1. Let M be a non-compact connected C^∞ n-manifold without boundary.

(1) $\mathcal{D}_0(M) \approx \mathbb{R}^\infty \times l_2$ if $n = 1, 2$.

(2) $\mathcal{D}_0(M) \approx \mathbb{R}^\infty \times l_2$ if $n = 3$ and M is orientable and irreducible (i.e., any smooth 2-sphere in M bounds a 3-ball in M).

(3) $\mathcal{D}_0(M) \approx \mathcal{D}_0(X, \partial X) \times (\mathbb{R}^\infty \times l_2)$ if $M = \mathrm{Int}\, X$ for a compact C^∞ n-manifold X with boundary.

In particular, if $\mathcal{D}_0(X, \partial X)$ is contractible then $\mathcal{D}_0(M) \approx \mathbb{R}^\infty \times l_2$.

Theorem 2.4 follows from Theorem 1.2 and Theorem 2.5 (cf. [5], [11], [18], [23], [26]).

Theorem 2.5. Suppose $K, L \in \mathcal{F}$, $K \subset \mathrm{int}_M L$ and $\mathrm{cl}_M(L \setminus K)$ is compact $(\neq \emptyset)$.

(1) For any closed subset C of M with $C \cap L = \emptyset$, the restriction map $r : \mathcal{D}(M, K) \to \mathcal{E}_K^\infty(L, M)$ has a local section $s : \mathcal{U} \to \mathcal{D}(M, K \cup C) \subset \mathcal{D}(M, K)$ at i_L.

(2) The spaces $\mathcal{D}(M, K \cup (M \setminus L))$ and $\mathcal{E}_K^\infty(L, M)$ are infinite-dimensional separable Fréchet manifolds (thus topological l_2-manifolds) and $\mathcal{E}_K^{\infty,*}(L, M)$ is an open subset of $\mathcal{E}_K^\infty(L, M)$.

Corollary 2.1 (2) ($n = 3$ case) follows from Theorem 2.4 (3)(i) and the fact that $\mathcal{D}_0(N, \partial N)$ is contractible if N is a 3-ball or a compact orientable Haken 3-manifold with boundary (A. Hatcher [13, 14], N.V. Ivanov [16, 17]).

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