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A solution based on marginal contributions for multi-alternative games with restricted coalitions

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1 Introduction

The cooperative game theory provides useful tools to analyze cost allocation, voting power, and so on. The problems to be analyzed by the cooperative game theory include n entities called players and are usually expressed by characteristic functions called games which map each subset of players to a real number. The solutions to the problems are given by value functions which assign a real number to each player. The real number called a value can show the cost borne by the player, power of influence, and so on depending on the problem setting. Several value functions have been proposed. As representative examples of value functions, the Shapley value [5] and the Banzhaf value [1][3] are well-known. Each of them is uniquely specified by reasonable axiom systems.

In the conventional cooperative games, each player can take one from two options: cooperate and non-cooperate. However, in the real world problems, we may face a decision problem to choose one from several options. From this point of view, it is worthwhile to treat cooperative games in which each player has r options. Then multi-alternative games also called games with r alternatives have been proposed by Bolger [2]. A multi-alternative game is expressed by a generalized characteristic function which maps an arrangement showing all players' choices to an r-dimensional real vector. Bolger [2] proposed a generalized value function which maps a multi-alternative game to an n-dimensional real vector whose i-th component shows the value of player i. This function is a generalization of the Shapley function. On the other hand, Ono [4] proposed a multi-alternative Banzhaf value (an MBZ value) as a generalization of the Banzhaf value.

The value functions/generalized value functions described above are considered under the assumption that all coalitions/arrangements are formed with equal possibilities. In the real world, there are many cases when this assumption does not hold. For example, when a certain license is necessary to choose an option in a multi-alternative game, players without the licenses cannot choose it and then some arrangements cannot be realized.

In this paper, we introduce a kind of restriction on arrangements into multi-alternative games. In our model, the choice of an alternative is restricted. Such a restriction can be found in the real world. For example, when a license/skill is necessary for taking some alternatives, those alternatives cannot be chosen by unlicensed/unskillful players. Under the restrictions on choices, we propose a value based on marginal contributions for a given game. The value indicates an evaluation of an alternative by a player under the given game. Further, the proposed value is axiomatized.

In Section 2, we briefly introduce an extended multi-alternative games and related concepts given by Tsurumi et al. [6] and Bolger [2] and the Bolger value and the MBZ value are presented. In Section 3, we propose a restricted situation which is called *a restricted choice situation* and a value for multi-alternative games with the restricted situation. Further, related concepts and properties are presented. In Section 4, the proposed value is axiomatized. In Section 5, we give a numerical example which is called "Job Selection Game" to exemplify the usefulness of the restricted multi-alternative games and the proposed value.

2 Extended multi-alternative games and previous values

2.1 Extended multi-alternative games

In this section, we introduce the extended multi-alternative games proposed by Tsurumi et al. [6] which are extensions of multi-alternative games (games with r alternatives) by Bolger [2]. Extended multi-alternative games assume that each player chooses one from $r (r \geq 2)$ alternatives or none of them while original multi-alternative games assume that each player always chooses one alternative. The extended multi-alternative games are mathematically characterized as follows:

Let $N = \{1, \ldots, n\}$ be the set of players and $R = \{1, \ldots, r\}$ the set of alternatives. Let Γ_j be the set of players who have chosen the alternative $j \in R$. A finite sequence of subsets of players, $\Gamma = (\Gamma_1, \ldots, \Gamma_r)$, is called an arrangement. Each arrangement Γ satisfies $\Gamma_1 \cup \cdots \cup \Gamma_r \subseteq N$ and $\Gamma_k \cap \Gamma_l = \emptyset$ $(\forall k \neq l)$. Let Γ_0 be a subset of players who have chosen none of alternatives. Then we have $\Gamma_0 = N - \bigcup_{k \in R} \Gamma_k$. For the sake of convenience, we define $R_0 = \{0, 1, \ldots, r\}$. We denote $\exists k \in R, S = \Gamma_k$ by $S \in \Gamma$. For any $S \in \Gamma$, we call (S, Γ) an embedded coalition (ECL). Let $E(N, R)$ be the set of ECLs and $A(N, R)$ the set of arrangements on N and R. Then a function $v : A(N, R) \to \mathbb{R}^r$ such that $v_k(\Gamma) = 0$ if $\Gamma_k = \emptyset$ is called an extended multi-alternative game on N with r alternatives, where $v(\Gamma) = (v_1(\Gamma), v_2(\Gamma), \ldots, v_r(\Gamma))$ and $\mathbb{R}$ is the set of real numbers. Let $MG(N, R)$ be the set of extended multi-alternative games on N and R.

In order to exemplify an extended multi-alternative game, we present the following example.

Example 1 (Job Selection Game) *Three students A, B and C are considering to work part-time. There are two jobs 1 and 2 but students cannot take both. Then each student can take one job or nothing. They can take the same job. If only two students would take different jobs, the remaining student would not get any payoff but the students taking jobs would get some payoffs independently. The payoff does not depend on the job taken but on the student taking a job. The payoffs of students A, B and C would be 8, 6 and 4 units, respectively. If student A would work alone while students B and C would make the same choice, independent of the job taken by A, student A would get 5 units as a payoff. If students B and C would work together while student A would not work with them, independent of the job taken by them, students B and C would get 18 units as the total payoff. If student B would work alone while students A and C would make the same choice, independent of the job taken by B, student B would get 3 units as a payoff. If students A and C would work together while student B would not work with them, independent of the job taken by them, students A and C would get 25 units as the total payoff. If student C would work alone while students A and B would make the same choice, independent of the job* *taken by C, student C would get 1 unit as a payoff. If students A and B would work together while student C would not work with them, independent of the job taken by them, students A and B would get 30 units as the total payoff. If all students A, B and C would work together, independent of the job taken, they would get 50 units as the total payoff.*

This game can be represented by the following extended multi-alternative game v for $k = 1, 2$ with $N = \{A, B, C\}$ and $R = \{1, 2\}$:

$$
\begin{aligned}
&v_k(\Gamma) = 8, \ \textit{for } \Gamma_k = \{A\} \textit{ and } |\Gamma_j| = 1, j = 1, 2,\\
&v_k(\Gamma) = 6, \ \textit{for } \Gamma_k = \{B\} \textit{ and } |\Gamma_j| = 1, j = 1, 2,\\
&v_k(\Gamma) = 4, \ \textit{for } \Gamma_k = \{C\} \textit{ and } |\Gamma_j| = 1, j = 1, 2,\\
&v_k(\Gamma) = 5, \ \textit{for } \Gamma_k = \{A\} \textit{ and } (\{B, C\} \in \Gamma \textit{ or } \emptyset \in \Gamma),\\
&v_k(\Gamma) = 3, \ \textit{for } \Gamma_k = \{B\} \textit{ and } (\{A, C\} \in \Gamma \textit{ or } \emptyset \in \Gamma),\\
&v_k(\Gamma) = 1, \ \textit{for } \Gamma_k = \{C\} \textit{ and } (\{A, B\} \in \Gamma \textit{ or } \emptyset \in \Gamma),\\
&v_k(\Gamma) = 30, \ \textit{for } \Gamma_k = \{A, B\},\\
&v_k(\Gamma) = 25, \ \textit{for } \Gamma_k = \{A, C\},\\
&v_k(\Gamma) = 18, \ \textit{for } \Gamma_k = \{B, C\},\\
&v_k(\Gamma) = 50, \ \textit{for } \Gamma_k = N,\\
&v_k(\Gamma) = 0, \ \textit{for other cases,}
\end{aligned}
$$

where $|\Gamma_j|$ is the cardinality of Γ_j.

An important class of extended multi-alternative games is the set of voting games with r alternatives which are called extended multi-alternative voting games. We assume that only one alternative is elected.

Let $\Gamma = (\Gamma_1, \ldots, \Gamma_r)$ be an arbitrary arrangement. If alternative $j \in R$ is elected, we call (Γ_j, Γ) a pair of a winning coalition. If alternative j is not elected, we call (Γ_j, Γ) a pair of a losing coalition. Let WE be the set of pairs of winning coalitions. Let LE be the set of pairs of losing coalitions. Then the triple (N, R, WE) is called a voting game with r alternatives (or a multi-alternative voting game).

A multi-alternative voting game (N, R, WE) can be represented by a multi-alternative game v as follows:

$$
v_k(\Gamma) = \begin{cases} 1 & \text{if } (\Gamma_k, \Gamma) \in WE, \\ 0 & \text{otherwise,} \end{cases} \tag{1}
$$

where $k \in R$.

2.2 Previous axioms and values

We describe axioms proposed previously and previous values for extended multi-alternative games. In this paper, we regard an extended multi-alternative game as a multi-alternative game with $(r + 1)$ alternatives $R_0 = \{0, 1, \ldots, r\}$ where the set of players with no choice takes zero value for any arrangement.

Let π^j , $j = 1, \dots, r$, be a vector function which maps a multi-alternative game to an n-dimensional real vector whose i-th component shows the value of player i. The i-th component of π^j is denoted by π_i^j.

Axiom 1 (j-efficiency) *Value π^j satisfies*

$$\sum_{i \in N} \pi_i^j(v) = v_j(\Gamma_{(N:j)}),$$

where $\Gamma_{(N:j)} = (\emptyset, \dots, \emptyset, N, \emptyset, \dots, \emptyset)$ (N is the $(j+1)$-th component).

Axiom 2 (j-null player) *Value π^j satisfies $\pi_i^j(v) = 0$ for any j-null player $i \in N$, where player i is a j-null player in v if and only if for all arrangements Γ satisfying $\Gamma_j \ni i$ and for all $k \neq j$*

$$v_j(\Gamma) = v_j(\Gamma^{i \to k}),$$

where $\Gamma^{i \to k}$ is the arrangement obtained by changing player i's selection to the k-th alternative in Γ ($i \notin \Gamma_k$).

Axiom 3 (linearity) *Value π^j satisfies $\pi^j(v + w) = \pi^j(v) + \pi^j(w)$ and $\pi^j(cv) = c \cdot \pi^j(v)$ for a sum of extended multi-alternative games $v + w$ and a scalar multiplication of an extended multi-alternative game cv, where, for extended multi-alternative games v and w, we define $v + w$ and cv by $(v + w)_j(\Gamma) = v_j(\Gamma) + w_j(\Gamma)$ and $(cv)_j = c \cdot v_j(\Gamma)$; $j = 1, \dots, r$.*

Axiom 4 (symmetry) *Value π^j satisfies $\pi_i^j(v) = \pi_s^j(v)$ if players i and s are symmetric, where players $i \in N$ and $s \in N$ are said to be symmetric if and only if $v_j(\Gamma) = v_j(\Gamma')$ with arrangement Γ' obtained by interchange between players i and s in arrangement Γ.*

Axiom 5 (pivot move) *Value π^j satisfies $\pi_i^j(v) = \pi_i^j(w)$ for extended multi-alternative games v and w such that*

$$\sum_{k \neq j} v_j(\Gamma) - v_j(\Gamma^{i \to j}) = \sum_{k \neq j} w_j(\Gamma) - w_j(\Gamma^{i \to j}), \text{for all } \Gamma \text{ such that } i \in \Gamma_j.$$

Axiom 6 (mean of total contribution) *Value π^j satisfies*

$$\sum_{i \in N} \pi_i^j(v) = \frac{1}{(r+1)^{n-1} r} \sum_{i \in N} \sum_{\Gamma : i \in \Gamma_j, k \neq j} (v_j(\Gamma) - v_j(\Gamma^{i \to k})).$$

Theorem 1 (Bolger [2]) *The value function $\theta^j(v)$, $j = 1, \dots, r$ defined as follows is the unique function satisfying Axioms 1 through Axiom 5:*

$$\theta_i^j(v) = \sum_{\Gamma : \Gamma_j \ni i} \sum_{k \neq j} \frac{(|\Gamma_j| - 1)!(n - |\Gamma_j|)!}{n! r^{n - |\Gamma_j| + 1}} [v_j(\Gamma) - v_j(\Gamma^{i \to k})],$$

$$\forall i \in N, j \in R. \tag{2}$$

Theorem 2 (Ono [4]) *The value function $\beta^j(v)$, $j = 1, \dots, r$ defined as follows is the unique function satisfying Axioms 2, 3, 4, 5 and 6:*

$$\beta_i^j(v) = \sum_{\Gamma : \Gamma_j \ni i} \sum_{k \neq j} \frac{1}{(r+1)^{n-1} r} [v_j(\Gamma) - v_j(\Gamma^{i \to k})], \quad \forall i \in N, j \in R. \tag{3}$$

3 The proposed value for restricted multi-alternative games

In the conventional extended multi-alternative games, each player can choose any alternative from a given set of alternatives. However, in the real world, there exists a situation where some alternatives cannot be chosen by all players. For example, in Job Selection Game described in the previous section, some students cannot take some jobs due to their inabilities or conflicts with regular lessons. In order to treat such situations, we formulate restricted games with r alternatives (restricted multi-alternative games).

In this paper, we consider the restriction on the selection of alternatives for each player. Let R_i be the set of alternatives which player $i \in N$ can choose. Obviously, we have $R_i \subseteq R_0$ and $0 \in R_i$, $\forall i \in N$. Especially, $R_i = R_0$ holds if player i can choose any alternatives and $R_i = \{0\}$ holds if player i can choose none of alternatives. Then the set of feasible arrangements, W, is defined by

$$W = \{\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_r) \mid \forall j \in R\ \forall i \in \Gamma_j;\ j \in R_i\}. \tag{4}$$

We call the set of feasible arrangement W *a restricted choice situation.* Let $AR(N,R)$ be the set of restricted choice situations. We characterize a multi-alternative game with a restricted choice situation as a pair (v, W) where $v \in MG(N,R)$ and $W \in AR(N,R)$.

Now, we propose a value for multi-alternative games with restricted choice situations.

Definition 1 *Given $W \in AR(N,R)$, we define $W_{i,j}$ by*

$$W_{i,j} = \{\Gamma \in W \mid i \in \Gamma_j\}.$$

$W_{i,j}$ is a subset of W where player i chooses the j-th alternative.

We define a function $f^j : MG(N,R) \to (\mathbb{R}^n)^{AR(N,R)}$ $(j = 1, \ldots, r)$ by its i-th component,

$$f_i^j(v)(W) = \begin{cases} \displaystyle\sum_{\substack{\Gamma \in W \\ \Gamma_j \ni i}} \sum_{\substack{k \in R_0 - \{j\} \\ \Gamma^{i \to k} \in W}} \frac{1}{|W|}[v_j(\Gamma) - v_j(\Gamma^{i \to k})], & \text{if } W_{i,j} \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

Let us interpret the function defined by (5). The term $v_j(\Gamma) - v_j(\Gamma^{i \to k})$ can be interpreted as the marginal contribution of player i to Γ_j. $|W|$ shows the number of feasible arrangements. Therefore, the weight $\frac{1}{|W|}$ means that each feasible arrangement is formed with equal probability. Then $f_i^j(v)(W)$ is the expected value of the marginal contributions of player i to alternative j in restricted game (v, W).

Theorem 3 *When $W = A(N,R)$, the proposed value $f_i^j(v)(W)$ is proportional to the MBZ value [4]. More specifically, $f_i^j(v)(A(N,R)) = \frac{r}{r+1}\beta_i^j(v)$. Namely, the normalized $f_i^j(v)(W)$ equals to the normalized MBZ value of player i to alternative j.*

In the rest of this section, we give some concepts associated to the axiom system of value f^j.

Definition 2 (*j-null player* for restricted multi-alternative games) *Let $v \in MG(N,R)$, $W \in AR(N,R)$, $i \in N$ and $j \in R$. Player i is called a j-null player on (v, W) if and only if the following holds:*

$$\textit{if } W_{i,j} \neq \emptyset \textit{ then } v_j(\Gamma) - v_j(\Gamma^{i \to k}) = 0,\ \ \forall \Gamma \in W_{i,j}, k \in R_0 - \{j\}, \Gamma^{i \to k} \in W.$$

Note that player i is a j-null player if $W_{i,j} = \emptyset$.

This concept is a generalization of *j-null players* of Bolger's multi-alternative games.

Definition 3 *Let $v \in MG(N,R)$, $W \in AR(N,R)$ and $i \in N$. Then player i is called an unrelated player if $R_i = \{0\}$.*

Unrelated players are the players who cannot choose any alternatives and j-null players for all $j \in R$ because $W_{i,j} = \emptyset$ for all $j \in R$.

4 Axiomatic approach

In this section, we give axioms which are reasonable for a value function to restricted multi-alternative games. We consider four axioms concerning null players, linearity, the independence from unrelated players, and the proportionality to total deducted welcome difference in voting games. The first two axioms are generalizations of those of the Bolger value and the MBZ value. That is, Axiom 7 and 8 are generalizations of Axiom 2 and 3 for multi-alternative games with restricted choice situations.

Let π^j be a vector function from $MG(N,R)$ into $(\mathbb{R}^n)^{AR(N,R)}$. The i-th component of π^j is denoted by π_i^j. Note that for any $v, w \in MG(N,R)$, $v + w \in MG(N,R)$ holds.

Axiom 7 (j-null player) *Given $v \in MG(N,R)$, $i \in N$, $j \in R$ and $W \in AR(N,R)$, the following holds:*

$$\pi_i^j(v)(W) = 0 \Leftrightarrow i \text{ is a } j\text{-null player on } W$$

Axiom 8 (Linearity) *Given v^1, $v^2 \in MG(N,R)$ c_1, $c_2 \in \mathbb{R}$ and $W \in AR(N,R)$, the following holds:*

$$\pi^j(c_1v^1 + c_2v^2)(W) = c_1\pi^j(v^1)(W) + c_2\pi^j(v^2)(W), \; j = 1, \ldots, r$$

Axiom 9 (Independence from unrelated players) *Let $v \in MG(N,R)$ and $W \in AR(N,R)$. Let us add an unrelated player $n+1$ to the set of players N, and we denote v' the $(n+1)$-person game. Then, the following holds:*

$$\pi_i^j(v', W) = \pi_i^j(v, W), \;\; \forall i \in N, \;\; \forall j \in R.$$

Axiom 9 means that the value is not changed by the addition of unrelated players to a game. Note that because $R_{n+1} = \{0\}$, the set of feasible arrangements W does not change between the $(n+1)$-person game v' and the original n-person game v in Axiom 9.

Axiom 10 described in what follows is a property with respect to multi-alternative voting games. If Γ_j changed from a winning coalition to a losing coalition by player i's moving from Γ_j to Γ_k $(k \in R_0 - \{j\})$, the movement is called a negatively influential movement under arrangement Γ. On the contrary, if Γ_j changed from a losing coalition to a winning coalition by player i's moving from Γ_j to Γ_k $(k \in R_0 - \{j\})$, the movement is called a positively influential movement under arrangement Γ. Under an arrangement Γ, let $M_{i,j}^-(\Gamma|v)$ be the number of negatively influential movements of player i from Γ_j and $M^{+}_{i,j}(\Gamma|v)$ the number of positively influential movements of player i from Γ_j. Using $M^{-}_{i,j}(\Gamma|v)$ and $M^{+}_{i,j}(\Gamma|v)$, we define

$$M_{i,j}(v) = \begin{cases} \sum_{\substack{\Gamma\in W \\ i\in\Gamma_j}} (M^{-}_{i,j}(\Gamma|v) - M^{+}_{i,j}(\Gamma|v)) & \text{if there exists } \Gamma \text{ such that } \Gamma_j \ni i, \\ 0 & \text{otherwise.} \end{cases}$$

Then Axiom 10 is given as follows.

Axiom 10 (Proportionality to welcome degree difference) *Let $W \in AR(N,R)$, $\Gamma \in W$, $i,s \in N$ and $j \in R$, and let v be a voting game with r alternatives. If $M_{i,j}(v) - M_{s,j}(v) = l$, we have*

$$\pi^j_i(v)(W) = \pi^j_s(v)(W) + \frac{l}{|W|}.$$

Let us interpret Axiom 10. First, $M^{-}_{i,j}(\Gamma|v)$ and $M^{+}_{i,j}(\Gamma|v)$ can be interpreted as welcome and unwelcome degrees of player i to alternative j under arrangement Γ, respectively. Then the difference $M^{-}_{i,j}(\Gamma|v) - M^{+}_{i,j}(\Gamma|v)$ can show the deducted welcome degree of player i to alternative j under arrangement Γ. Accordingly, $M_{i,j}(v)$ stands for a total deducted welcome degree of i to j, which may indicate the power to victory of player i by choosing alternative j. Axiom 10 shows that the difference of values between players i and s should be proportional to the difference between their total deducted welcome degrees, more specifically, it should be the ratio of the difference between their total deducted welcome degrees to the number of feasible arrangements.

Theorem 4 *Function f^j, $j = 1,\ldots,r$ defined by (5) is the unique function which satisfies Axioms 7 through 10.*

5 Numerical example

We calculate the proposed value f^j_i in Job Selection Game described in Example 1 and demonstrate the effect by a restriction. We compare the values in two different situations: a situation where all students can choose all jobs and a situation when student A cannot choose job 2.

The proposed values are shown in Tables 1 and 2. Table 1 shows the proposed values when all students can choose all jobs while Table 2 shows the proposed values when student A cannot choose job 2. Because v is symmetric with respect to jobs, the proposed values are same independent of the jobs students choose when all students can choose all jobs. As shown in Table 2, the value of student A in Job 2 is zero, This would be natural from the restriction that A cannot take Job 2. The value of student A decreases not only in Job 2 but also in Job 1 by the restriction. This would be a reflection of Axiom 10 by the decrement of possible movements of student A. By the comparison between Tables 1 and 2, we can observe that the restriction can strongly change the strength (value) of players (students).

Table. 1: The proposed values when all students can choose all jobs

	Job 1		Job 2	
Player	Value	Normalized	Value	Normalized
A	11.407	0.413	11.407	0.413
B	8.963	0.327	8.963	0.327
C	6.963	0.254	6.963	0.254
Total	27.333	1	27.333	1

Table. 2: The proposed values when student A cannot choose job 2

	Job 1		Job 2	
Player	Value	Normalized	Value	Normalized
A	8.555	0.309	0	0
B	10.722	0.387	5.444	0.569
C	8.388	0.303	4.111	0.430
Total	27.666	1	9.555	1

6 Conclusion

We have investigated extended multi-alternative games with *restricted choice situations*. We have proposed a value based on marginal contributions for restricted multi-alternative games and shown that the proposed value is proportional to the MBZ value. Further, we axiomatized the proposed value. In numerical example, we observe that the restriction can strongly change the strength of players.

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