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Applications of subspace theorem to the fractional parts of geometric series

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1 Introduction

Weyl's criterion states that a sequence x_n $(n=0,1,\ldots)$ is uniformly distributed modulo 1 if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp(2\pi i h x_n)=0 \tag{1.1}$$

for every nonzero integer h. As a corollary, an arithmetic progression $\xi n+\eta$ $(n=0,1,\ldots)$ is uniformly distributed modulo 1 if and only if its common difference is a irrational number. On the other hand, it is generally difficult to check the criterion (1.1) in the case where the sequence x_n $(n=0,1,\ldots)$ is a geometric progression $\xi\alpha^n$ $(n=0,1,\ldots)$.

In this paper we study the fractional parts of geometric sequences whose common ratio $\alpha>1$ is an algebraic number. We now review the fractional parts of powers of Pisot and Salem numbers. Pisot numbers are algebraic integers greater than 1 whose conjugates different from themselves have absolute values strictly less than 1. Salem numbers are algebraic integers greater than 1 which have at least one conjugate with modulus 1 and exactly one conjugate outside the unit circle. Let $||x||$ denote the distance from the real number x to the nearest integer. Moreover, we write $\{x\}$ and $[x]$ the fractional part of x and the integral part of x, respectively. Take a Pisot number α. Since the trace of α^n is a rational integer,

$$\lim_{n\to\infty}||\alpha^n||=0.$$

Next, let α be a Salem number. Then for any positive ε there exists a nonzero $\xi\in\mathbf{Q}(\alpha)$ satisfying

$$\limsup_{n\to\infty}||\xi\alpha^n||<\varepsilon$$

(see [4]). However, little is known about the fractional parts of the sequence $\xi\alpha^n$ $(n = 0, 1, \ldots)$ in the case of $\xi \notin \mathbf{Q}(\alpha)$. For example, suppose that $\alpha > 1$ is a natural number and that ξ is a positive number. Then $\xi\alpha^n$ $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1 if and only if ξ is normal in base α. However, we even do not know whether the numbers $\sqrt{2}$, $\sqrt[3]{5}$, and π are normal in base 10 or not. In section 2 we survey the normality of an algebraic irrational number ξ. In particular, we give a lower bound of the number $\lambda_N(\alpha, \xi)$ of nonzero digits among the first N digits of the α-ary expansion of ξ. In other words, we count the number of $n \in \mathbf{N}$ such that

$$\{\xi\alpha^n\} \geq \frac{1}{\alpha}.$$

In section 3 and 4, we estimate the number of $n \in \mathbf{N}$ satisfying

$$\{\xi\alpha^n\} \geq c(\alpha)$$

for an algebraic number α and a positive constant $c(\alpha)$ depending only on α. In this paper, we introduce results without proofs in this paper.

2 Borel conjecture

Borel [5] showed that almost all positive numbers are normal in every integral base $\alpha \geq 2$. He [6] also conjectured that all irrational numbers ξ are normal. However, there is no such an irrational ξ whose normality was proved. In the case of $\alpha \geq 3$, we even do not know whether all digits $0, 1, \ldots, \alpha - 1$ occur infinitely many times in the α-ary expansion of an irrational number. In this section we introduce some partial results.

Let $\alpha \geq 2$ be a natural number and $\xi > 0$ an irrational number. In what follows, we denote the α-ary expansion of ξ by

$$\xi = \sum_{i=-\infty}^{M} s_i(\xi)\alpha^i = s_M(\xi) \cdots s_0(\xi).s_{-1}(\xi)s_{-2}(\xi) \cdots .$$

Define the infinite word $\mathbf{s}$ by

$$\mathbf{s} = s_{-1}(\xi)s_{-2}(\xi) \cdots .$$

First, we measure the complexity of the α-ary expansion of ξ by the number $p(N)$ of distinct blocks of length N appearing in the words $\mathbf{s}$. If ξ is normal in base α, then $p(N) = \alpha^N$ for any positive N. Ferenczi and Mauduit [9] showed that

$$\lim_{N \to \infty} (p(N) - N) = \infty.$$

Adamczewski and Bugeaud [1] improved their results as follows:

$$\lim_{N\to\infty} \frac{p(N)}{N} = \infty.$$

Moreover, Bugeaud and Evertse [8] showed for any positive ξ with $\eta < 1/11$ that

$$\limsup_{N\to\infty} \frac{p(N)}{N(\log N)^{\eta}} = \infty.$$

Next, we give an lower bound of $\lambda_N(\alpha, \xi)$ in the case of $\alpha = 2$, which we define in the previous section. Put

$$\xi' = \frac{\xi}{2^{[\log_2 \xi]}}.$$

Note that $1 < \xi' < 2$. Let $D(\geq 2)$ be the degree of ξ' and A_D the leading coefficient of the minimum integer polynomial of ξ'. Bailey, Borwein, Crandall, and Pomerance [3] showed for any positive ϵ that there exists a positive $c(\varepsilon)$ satisfying

$$\lambda_N(2, \xi) > (1 - \varepsilon)(2A_D)^{-1/D} N^{1/D} \tag{2.1}$$

for $N \geq c(\varepsilon)$. Rivoal [15] improved the coefficient $(1-\varepsilon)(2A_D)^{-1/D}$ of (2.1) for certain classes of algebraic irrational numbers ξ. Namely, suppose that there exist two polynomials P, Q with positive integral coefficients and two positive integers a, b fulfilling $P(\xi) = a + bQ(\xi)^{-1}$. Let ε be an arbitrary positive number. Then we have for sufficiently large N (with threshold depending on ξ and ε)

$$\lambda_N(2, \xi) \geq (1 - \varepsilon)(B(p)B(q))^{-1/\delta} N^{1/\delta}, \tag{2.2}$$

where $\delta = \deg(PQ)$ and p, q are the dominant coefficients of P and Q, respectively.

For instance, let $\xi_0 = 0.558\ldots$ be the unique real zero of the polynomial $8X^3 - 2X^2 + 4X - 3$. (2.1) implies

$$\lambda_N(2, \xi_0) \geq (1 - \varepsilon)16^{-1/3} N^{1/3}.$$

On the other hand, since $4\xi_0 = 1 + 2(2\xi_0^2 + 1)^{-1}$, we can apply (2.2) to ξ_0. Thus,

$$\lambda_N(2, \xi_0) \geq (1 - \varepsilon) N^{1/3}.$$

3 Limit points of the fractional parts of powers of geometric series

Koksma [14] proved that, if any common ratio $\alpha > 1$ is given, then for almost all initial values ξ the geometric sequences $\xi\alpha^n$ $(n = 0, 1, \ldots)$ are uniformly distributed modulo 1. Similarly, let ξ be any nonzero initial value. Then α $\xi\alpha^n$ $(n = 0, 1, \ldots)$ are uniformly distributed modulo 1 for almost all common ratios.

Now we introduce the exceptional set of Koksma's theorem. In particular, we consider the maximal limit points $\limsup_{n\to\infty}\{\xi\alpha^n\}$. It is known for a fixed $\alpha > 1$ that there is a nonzero ξ satisfying

$$\limsup_{n\to\infty}\{\xi\alpha^n\} < 1.$$

Hence, the sequence $\xi\alpha^n$ $(n = 0, 1, \ldots)$ isn't uniformly distributed modulo 1. More precisely, let $\alpha > 2$. Then Tijdeman [16] constructed a nonzero $\xi = \xi(\alpha)$ such that

$$\limsup_{n\to\infty}\{\xi\alpha^n\} \le \frac{1}{\alpha - 1}. \tag{3.1}$$

Let $\alpha_0 = 2.025\ldots$ be the unique solution of $34X^3 - 102X^2 + 75X - 16 = 0$. Dubickas [11] showed for $1 < \alpha < \alpha_0$ that there exists a nonzero $\xi = \xi(\alpha)$ such that

$$\limsup_{n\to\infty}\{\xi\alpha^n\} \le 1 - \frac{2(\alpha-1)^2}{9(2\alpha-1)^2} \tag{3.2}$$

Note that if $2 < \alpha < \alpha_0$, then (3.2) is stronger than (3.1). In fact, it is easy to check

$$1 - \frac{2(\alpha-1)^2}{9(2\alpha-1)^2} < \frac{1}{\alpha-1}$$

for such an α. It is a interesting problem to estimate the value

$$\inf_{\xi\in\mathbb{R},\xi\neq 0}\ \limsup_{n\to\infty}\{\xi\alpha^n\} \tag{3.3}$$

for a given α. Let $\alpha > 1$ be an algebraic number with minimal polynomial $a_dX^d + a_{d-1}X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X]$ $(a_d > 0)$. Take a positive ξ. If α is a Pisot of Salem number, then suppose $\xi \notin \mathbb{Q}(\alpha)$. Then Dubickas [10] proved

$$\limsup_{n\to\infty}\{\xi\alpha^n\} \ge c(\alpha) := \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\},$$

where

$$L_+(\alpha) = \sum_{a_i>0} a_i,\ L_-(\alpha) = \sum_{a_i\leq 0} a_i.$$

Moreover, let

$$\lambda_N(\alpha,\xi) = \mathrm{Card}\,\{n \in \mathbb{Z}\,|\,0 \leq n < N, \{\xi\alpha^n\} \geq c(\alpha)\},$$

where Card denotes the cardinality. Note that if $\alpha > 1$ is a natural number, then $\lambda(\alpha,\xi)$ means the number of nonzero digits of α-ary expansion of ξ. For simplicity, suppose that α is an algebraic integer and that α has at least one conjugate different from itself which is outside the unit circle. Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_p$ be the conjugates of α whose absolute values are greater than 1. In the same way as that of Theorem 3 of [10], we can show that

$$\liminf_{N\to\infty} \frac{\lambda_N(\alpha,\xi)}{\log N} \geq \left(\log\left(1 + \frac{\log\alpha}{\log|\alpha_2| + \cdots + \log|\alpha_p|}\right)\right)^{-1}. \tag{3.4}$$

In the section 4, we improve this inequality in the case where ξ is an algebraic number with $\xi \notin \mathbb{Q}(\alpha)$.

In the last of this section, we consider geometric sequences $\xi\alpha^n$ ($n = 0, 1, \ldots$) for a fixed initial value. The author [12] gave an algorithm to construct common ratios α such that $||\xi\alpha^n|||$ is arbitrarily small for all n. Let ξ be a nonzero real number. Then for any positive numbers ε and M, there exists a common ratio α with $\alpha > M$ such that

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \leq \frac{1+\varepsilon}{2\alpha}.$$

Moreover, the set of α satisfying

$$\limsup_{n\to\infty} ||\xi\alpha^n|| \leq \frac{1+\varepsilon}{\alpha}. \tag{3.5}$$

is uncountable. In particular, there is an α transcendental over the field $\mathbb{Q}(\xi)$ satisfying (3.5).

4 Main results

In what follows, we assume that $\alpha > 1$ is an algebraic number with minimal polynomial $a_dX^d + a_{d-1}X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X]$ ($a_d > 0$). Write the conjugates of α by $\alpha_1 = \alpha, \ldots, \alpha_d$. Take an algebraic irrational positive number ξ with $\xi \notin \mathbb{Q}(\alpha)$. Then we have the following:

THEOREM 4.1. (1) *If α is a Pisot or Salem number, then*

$$\lim_{N\to\infty} \frac{\lambda_N(\alpha,\xi)}{\log N} = \infty.$$

(2) *Otherwise,*

$$\liminf_{N\to\infty} \frac{\lambda_N(\alpha,\xi)}{\log N} \geq \left(\log\left(\frac{\log M(\alpha)}{\log \alpha}\right)\right)^{-1},$$

where $M(\alpha)$ is the Mahler measure of α defined by

$$M(\alpha) = a_d \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.$$

Theorem 4.1 gives a good estimation if $\log M(\alpha)/\log\alpha$ is small. Now we give a numerical example in the case of $\alpha = 4+\sqrt{2}$. Let ξ be a positive number. By (3.4), we get

$$\liminf_{N\to\infty} \frac{\lambda_N(4+\sqrt{2},\xi)}{\log N} \geq \log\left(\frac{\log(14)}{\log(4-\sqrt{2})}\right)^{-1} = 0.978\ldots.$$

Moreover, if ξ is an algebraic number with $\xi \notin \mathbb{Q}(\sqrt{2})$, then Theorem 4.1 implies

$$\liminf_{N\to\infty} \frac{\lambda_N(4+\sqrt{2},\xi)}{\log N} \geq \log\left(\frac{\log(14)}{\log(4+\sqrt{2})}\right)^{-1} = 2.24\ldots.$$

If $\alpha = 2$, then there is a big gap between the estimation (2.1) and the first statement of Theorem 4.1. So we give a stronger lower bound for $\lambda_N(\alpha,\xi)$ than that of Theorem 4.1 in the case where α is a Pisot or Salem number.

THEOREM 4.2. *Let $\alpha > 1$ be a Pisot or Salem number. Let ξ be a positive algebraic number with $\xi \notin \mathbb{Q}(\alpha)$. Put*

$$D = [\mathbb{Q}(\alpha,\xi) : \mathbb{Q}(\alpha)].$$

Then there exists an effectively computable absolute constant $c > 0$ such that

$$\lambda_N(\alpha,\xi) \geq c\frac{(\log N)^{3/2}}{(\log(4D))^{1/2}(\log\log N)^{1/2}}$$

for every sufficiently large N.

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