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Algebraic independence of certain series involving continued fractions and generated by linear recurrences

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In this paper we consider the necessary and sufficient conditions for the values of functions in question at algebraic points to be algebraically independent. The first such result in history is Lindemann–Weierstrass Theorem asserting that the values $e^{\alpha_1}, \dots, e^{\alpha_s}$ of exponential function at algebraic numbers $\alpha_1, \dots, \alpha_s$ are algebraically independent if and only if $\alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Q}$ (cf. Shidlovskii [10]). First we introduce here some previous results including the author's ones. In what follows, let $q_1, \dots, q_s$ be algebraic numbers with $0 < |q_i| < 1$ $(1 \le i \le s)$.

Let $\{R_k\}_{k\ge1}$ be a linear recurrence of positive integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \ge 1), \tag{1}$$

where $n \ge 2$ and $c_1, \dots, c_n$ are nonnegative integers with $c_n \ne 0$. We define a polynomial associated with (1) by

$$\Phi(X) = X^n - c_1 X^{n-1} - \cdots - c_n.$$

We assume that $\Phi(\pm1) \ne 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity and that $\{R_k\}_{k\ge1}$ is not a geometric progression. Let

$$f(z) = \sum_{k=1}^{\infty} z^{R_k}, \quad g(z) = \sum_{k=1}^{\infty} \frac{z^{R_k}}{1 - z^{R_k}}, \quad h(z) = \prod_{k=1}^{\infty} (1 - z^{R_k}).$$

Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers. The author [12, 15] proved that the following four properties are equivalent:

(i) $f(q_1), \dots, f(q_s), g(q_1), \dots, g(q_s), h(q_1), \dots, h(q_s)$ are algebraically dependent.

(ii) $f(q_1), \dots, f(q_s)$ are algebraically dependent.

(iii) $1, f(q_1), \dots, f(q_s)$ are linearly dependent over $\overline{\mathbb{Q}}$.

(iv) There exist a nonempty subset $\{q_{i_1}, \dots, q_{i_t}\}$ of $\{q_1, \dots, q_s\}$, roots of unity $\zeta_1, \dots, \zeta_t$, an algebraic number γ with $q_{i_l} = \zeta_l \gamma$ $(1 \le l \le t)$, and algebraic numbers $\xi_1, \dots, \xi_t$, not all zero, such that

$$\sum_{l=1}^{t} \xi_l \zeta_l^{R_k} = 0$$

for all sufficiently large k.

Although the necessary and sufficient conditions mentioned above are on any number of points, e.g. $\alpha_1, \dots, \alpha_s$ or $q_1, \dots, q_s$, there are some results on such conditions only on two points among $q_1, \dots, q_s$ as follows:

Let $F(z) = \sum_{k=1}^{\infty} z^{k!}$. Nishioka [5] settled Masser's conjecture asserting that the values $F(q_1), \dots, F(q_s)$ are algebraically dependent if and only if there exist distinct i, j $(1 \le i, j \le s)$ such that q_i/q_j is a root of unity.

Next we define

$$\Theta(z) = \sum_{k=1}^{\infty} \frac{z^{R_1+R_2+\cdots+R_k}}{(1-z^{R_1})(1-z^{R_2})\cdots(1-z^{R_k})}.$$

For any $k \ge 1$, let N_k be the greatest common divisor of n consecutive terms $R_k, R_{k+1}, \dots, R_{k+n-1}$. The author [14] proved that the values $\Theta(q_1), \dots, \Theta(q_s)$ are algebraically dependent if and only if there exist some $k \ge 1$ and distinct i, j $(1 \le i, j \le s)$ such that q_i/q_j is an N_k-th root of unity.

On the other hand, $\Theta(z)$ is expressed as the continued fraction

$$\Theta(z) = \cfrac{z^{R_1}}{1 - z^{R_1} + \cfrac{-z^{R_2}(1-z^{R_1})}{1 + \cfrac{-z^{R_3}(1-z^{R_2})}{1 + \ddots + \cfrac{-z^{R_n}(1-z^{R_{n-1}})}{1 + \ddots}}}},$$

which is obtained from the identity

$$\sum_{k=1}^{\infty} \frac{z_1 z_2 z_3 \cdots z_k}{(1-z_1)(1-z_2)(1-z_3)\cdots(1-z_k)} = \cfrac{z_1}{1 - z_1 + \cfrac{-z_2(1-z_1)}{1 + \cfrac{-z_3(1-z_2)}{1 + \ddots + \cfrac{-z_n(1-z_{n-1})}{1 + \ddots}}}}, \tag{2}$$

where $\{z_n\}_{n\geq 1}$ is a sequence of complex numbers with $|z_n| < 1$ such that $\lim_{n\to\infty} z_n = 0$. Letting $z_k = aq^{k-1}$ $(k \geq 1)$ in the left-hand side series of (2), we have

$$\sum_{k=1}^{\infty} \frac{a^k q^{k(k-1)/2}}{(1-a)(1-aq)\cdots(1-aq^{k-1})},$$

which is the series obtained by letting $x = -a$ in

$${}_1\phi_1(q; a; q, x) = \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k-1)/2} x^k}{(1-a)(1-aq)\cdots(1-aq^{k-1})}, \tag{3}$$

where ${}_1\phi_1(\alpha; \beta; q, x)$ is the case $r = s = 1$ of q-hypergeometric series defined by

$$\begin{aligned} &{}_r\phi_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; q, x) \\ &= \sum_{k=1}^{\infty} \frac{\prod_{l=0}^{k-1}(1-\alpha_1 q^l)\cdots\prod_{l=0}^{k-1}(1-\alpha_r q^l)}{\prod_{l=1}^{k}(1-q^l)\prod_{l=0}^{k-1}(1-\beta_1 q^l)\cdots\prod_{l=0}^{k-1}(1-\beta_s q^l)} \left[(-1)^k q^{\frac{k(k-1)}{2}}\right]^{1+s-r} x^k \end{aligned}$$

(cf. Gasper and Rahman [1]).

Now we state the main theorem of this paper. Replacing $k-1$ in the exponent of q in the right-hand side series of (3) by R_k, where $\{R_k\}_{k\geq 1}$ is a linear recurrence satisfying (1), and replacing x by $-x$ in (3), which does not lose the generality since x runs through all the algebraic numbers in what follows, we have

$$\begin{aligned} \Theta(x, a, q) &= \sum_{k=1}^{\infty} \frac{x^k q^{R_1 + R_2 + \cdots + R_k}}{(1-aq^{R_1})(1-aq^{R_2})\cdots(1-aq^{R_k})} \\ &= \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1-aq^{R_l}}. \end{aligned}$$

We note that $\Theta(1, 1, z)$ equals $\Theta(z)$ mentioned above. Throughout this paper, let

$$U = \{(x, a, q) \mid x, a, q \in \overline{\mathbb{Q}} \setminus \{0\},\ |a| \leq 1,\ |q| < 1\}.$$

Then $\Theta(x,a,q)$ converges at any point in U. Let $(x_1,a_1,q_1),(x_2,a_2,q_2)\in U$. We write $(x_1,a_1,q_1)\sim(x_2,a_2,q_2)$ if $x_1/a_1=x_2/a_2$ and if $a_1q_1^{R_k}=a_2q_2^{R_k}$ for all sufficiently large k. Then $\sim$ is an equivalence relation.

Theorem. *Let $\{R_k\}_{k\geq 1}$ be a linear recurrence satisfying* (1). *Suppose that $\{R_k\}_{k\geq 1}$ is not a geometric progression. Assume that $\Phi(\pm 1)\neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Then the values*

$$\Theta(x,a,q)\quad((x,a,q)\in U)$$

are algebraically dependent if and only if there exist distinct $(x_1,a_1,q_1),(x_2,a_2,q_2)\in U$ such that $(x_1,a_1,q_1)\sim(x_2,a_2,q_2)$.

Corollary 1. *Let $\{R_k\}_{k\geq 1}$ be as in Theorem. Suppose in addition that* g.c.d.$(R_{k+1}-R_k,\ R_{k+2}-R_{k+1},\dots,R_{k+n}-R_{k+n-1})=1$ *for any $k\geq 1$. Then the values $\Theta(x,a,q)$ $((x,a,q)\in U)$ are algebraically independent, namely the infinite set $\{\Theta(x,a,q)\mid(x,a,q)\in U\}$ is algebraically independent.*

Proof. By the theorem, if the values $\Theta(x,a,q)$ $((x,a,q)\in U)$ are algebraically dependent, then there exist distinct $(x_1,a_1,q_1),(x_2,a_2,q_2)\in U$ such that $x_1/a_1=x_2/a_2$ and $a_1q_1^{R_k}=a_2q_2^{R_k}$ for all sufficiently large k. Then there exists a positive integer k_0 such that $a_1q_1^{R_k}=a_2q_2^{R_k}$ $(k_0\leq k\leq k_0+n)$. Thus $(q_1/q_2)^{R_{k+1}-R_k}=1$ $(k_0\leq k\leq k_0+n-1)$ and so $q_1/q_2=1$ since g.c.d.$(R_{k_0+1}-R_{k_0},\ R_{k_0+2}-R_{k_0+1},\dots,R_{k_0+n}-R_{k_0+n-1})=1$. Hence $(x_1,a_1,q_1)=(x_2,a_2,q_2)$, which is a contradiction.

Corollary 2. *Let $\{R_k\}_{k\geq 1}$ be as in Theorem. Let $q_1,\dots,q_s$ be algebraic numbers with $0<|q_i|<1$ $(1\leq i\leq s)$ such that none of q_i/q_j $(1\leq i<j\leq s)$ is a root of unity. Then the infinite set*

$$\bigcup_{i=1}^{s}\left\{\sum_{k=1}^{\infty}\prod_{l=1}^{k}\frac{xq_i^{R_l}}{1-aq_i^{R_l}}\ \middle|\ x,a\in\overline{\mathbb{Q}}\setminus\{0\},\ |a|\leq 1\right\}$$

is algebraically independent.

Corollary 3. *Let $\{R_k\}_{k\geq 1}$ be as in Theorem. Suppose in addition $c_n=1$. Let $N^*=$* g.c.d.$(R_2-R_1,\ R_3-R_2,\dots,R_{n+1}-R_n)$. *Let ζ be a primitive N^*-th root of unity and $G=\langle(\zeta^{R_1},\zeta^{R_1},\zeta^{-1})\rangle$ a cyclic group generated by $(\zeta^{R_1},\zeta^{R_1},\zeta^{-1})$ with componentwise multiplication. Then the values $\Theta(x,a,q)$ $((x,a,q)\in U)$ are algebraically dependent if and only if there exist distinct $(x_1,a_1,q_1),(x_2,a_2,q_2)\in U$ such that $(x_1/x_2,a_1/a_2,q_1/q_2)\in G$.*

Proof. Let $R_k^*=R_{k+1}-R_k$ and $N_k^*=$ g.c.d.$(R_k^*,R_{k+1}^*,\dots,R_{k+n-1}^*)$ $(k\geq 1)$. Since $\{R_k^*\}_{k\geq 1}$ satisfies (1), noting that $c_n=1$, we see that $N_k^*=N^*$ for any $k\geq 1$. If distinct $(x_1,a_1,q_1),(x_2,a_2,q_2)\in U$ satisfy $(x_1/x_2,a_1/a_2,q_1/q_2)\in G$, then $x_1/x_2=a_1/a_2$ and $(q_1/q_2)^{R_k^*}=1$ $(k\geq 1)$ and hence $x_1/a_1=x_2/a_2$ and $(q_1/q_2)^{R_k}=(q_1/q_2)^{R_1}=a_2/a_1$

$(k \geq 1)$, which implies that $\Theta(x,a,q)$ $((x,a,q) \in U)$ are algebraically dependent by the theorem. Conversely, if $\Theta(x,a,q)$ $((x,a,q) \in U)$ are algebraically dependent, then by the theorem there exist distinct $(x_1,a_1,q_1),(x_2,a_2,q_2) \in U$ such that $x_1/a_1 = x_2/a_2$ and $a_1 q_1^{R_k} = a_2 q_2^{R_k}$ for all sufficiently large k. Then there exists a positive integer k_0 such that $a_1 q_1^{R_k} = a_2 q_2^{R_k}$ $(k_0 \leq k \leq k_0+n)$. Thus $(q_1/q_2)^{R_k^*} = 1$ $(k_0 \leq k \leq k_0+n-1)$ and hence $(q_1/q_2)^{N^*} = 1$. Since $(q_1/q_2)^{R_k^*} = 1$ $(k \geq 1)$, we see that $a_2/a_1 = (q_1/q_2)^{R_{k_0}} = (q_1/q_2)^{R_1}$ and so $(x_1/x_2, a_1/a_2, q_1/q_2) \in G$.

Example 1. Let $\{G_k\}_{k\geq 0}$ be the generalized Fibonacci numbers defined by

$$G_0 = 0, \quad G_1 = 1, \quad G_{k+2} = bG_{k+1} + G_k \quad (k \geq 0), \tag{4}$$

where b is a positive integer, and let

$$\Theta(x,a,q) = \sum_{k=1}^{\infty} \frac{x^k q^{G_1+G_2+\cdots+G_k}}{(1-aq^{G_1})(1-aq^{G_2})\cdots(1-aq^{G_k})}.$$

By Corollary 3 with $N^* = \text{g.c.d.}(G_2 - G_1,\ G_3 - G_2) = \text{g.c.d.}(b-1, b^2-b+1) = 1$ the values $\Theta(x,a,q)$ $((x,a,q) \in U)$ are algebraically independent. In particular, the infinite set

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{F_1+F_2+\cdots+F_k}}{(1-aq^{F_1})(1-aq^{F_2})\cdots(1-aq^{F_k})} \;\middle|\; x,a,q \in \overline{\mathbb{Q}} \setminus \{0\},\ |a| \leq 1,\ |q| < 1 \right\}$$

is algebraically independent, where $\{F_k\}_{k\geq 0}$ is the sequence of Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k \quad (k \geq 0). \tag{5}$$

The following result on an analogue of q-exponential function

$$E_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k q^{1+2+\cdots+k}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

gives a generalization of the author's previous result [14] stated above.

Corollary 4. *Let $\{R_k\}_{k\geq 1}$ be as in Theorem. Let a be a fixed algebraic number with $0 < |a| \leq 1$ and define*

$$\begin{aligned} E(x,q) &= \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-aq^{R_1})(1-aq^{R_2})\cdots(1-aq^{R_k})} \\ &= \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xq^{R_l}}{1-aq^{R_l}}. \end{aligned}$$

Then the values

$$E(x,q) \quad (x,q \in \overline{\mathbb{Q}} \setminus \{0\},\ |q|<1)$$

are algebraically dependent if and only if there exist some distinct pairs (x_1,q_1) *and* (x_2,q_2) *of nonzero algebraic numbers* x_1,x_2,q_1,q_2 *with* $|q_1|,|q_2|<1$ *such that* $x_1=x_2$ *and* $q_1^{N_k}=q_2^{N_k}$ *for some* $k\geq 1$, *where* $N_k=\text{g.c.d.}(R_k,R_{k+1},\dots,R_{k+n-1})$.

In particular, if $N_k=1$ *for any* $k\geq 1$, *then the values* $E(x,q)$ *are algebraically independent for any distinct pairs* (x,q) *of nonzero algebraic numbers* x,q *with* $|q|<1$.

Proof. By the theorem, the values $E(x,q)$ $(x,q\in\overline{\mathbb{Q}}\setminus\{0\},\ |q|<1)$ are algebraically dependent if and only if there exist a positive integer k_0 and some distinct pairs (x_1,q_1) and (x_2,q_2) of nonzero algebraic numbers with $|q_1|,|q_2|<1$ such that $x_1=x_2$ and $q_1^{R_k}=q_2^{R_k}$ for any $k\geq k_0$, which implies that $q_1^{N_{k_0}}=q_2^{N_{k_0}}$. Conversely, if $q_1^{N_{k_0}}=q_2^{N_{k_0}}$, then $q_1^{R_k}=q_2^{R_k}$ for any $k\geq k_0$, since N_{k_0} divides R_k for any $k\geq k_0$ by (1).

Remark 1. Some functions are known to have the property that their values at any given nonzero distinct algebraic numbers are algebraically independent. Example of the entire function $f(x)$ having such a property, namely the values $f(\alpha_1),\dots,f(\alpha_s)$ are algebraically independent for any nonzero distinct algebraic numbers $\alpha_1,\dots,\alpha_s$, is $\sum_{k=0}^{\infty}q^{k!}x^k$ or $\sum_{k=0}^{\infty}q^{d^k}x^k$, which were given by Nishioka [6], [8], respectively, where q is an algebraic number with $0<|q|<1$ and d is an integer greater than 1, or $\sum_{k=0}^{\infty}q^{F_k}x^k$, which were shown by the author [11], where $\{F_k\}_{k\geq 0}$ is the sequence of Fibonacci numbers defined by (5). Example of the function $g(z)\in\mathbb{Z}[[z]]$ analytic inside the unit circle having such a property, namely the values $g(q_1),\dots,g(q_s)$ are algebraically independent for any distinct algebraic numbers $q_1,\dots,q_s$ with $0<|q_i|<1$ $(1\leq i\leq s)$, is $\sum_{k=0}^{\infty}z^{k!+k}$, given by Nishioka [7], or $\sum_{k=1}^{\infty}[k\omega]z^k$, shown by Masser [4], where $\omega>0$ is a quadratic irrational number and $[\sigma]$ denotes the largest integer not exceeding the real number σ. In the case of $\omega=(-b+\sqrt{b^2+4})/2$ with b a positive integer, we have

$$\sum_{k=1}^{\infty}[k\omega]z^k=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}z^{G_k+G_{k+1}}}{(1-z^{G_k})(1-z^{G_{k+1}})},$$

where $\{G_k\}_{k\geq 0}$ is the generalized Fibonacci numbers defined by (4) in Example 1 above (cf. Nishioka [9]).

Corollary 5. *Let* $\{R_k\}_{k\geq 1}$ *be as in Theorem. Define*

$$\begin{aligned}\Theta(a,q) &= \sum_{k=1}^{\infty} \frac{a^k q^{R_1+R_2+\cdots+R_k}}{(1-aq^{R_1})(1-aq^{R_2})\cdots(1-aq^{R_k})} \\ &= \cfrac{aq^{R_1}}{1-aq^{R_1}+\cfrac{-aq^{R_2}(1-aq^{R_1})}{1+\cfrac{-aq^{R_3}(1-aq^{R_2})}{1+ \ddots + \cfrac{-aq^{R_n}(1-aq^{R_{n-1}})}{1+\ddots}}}}. \end{aligned} \tag{6}$$

Then the values

$$\Theta(a,q) \quad (a,q \in \overline{\mathbb{Q}} \setminus \{0\},\ |a| \leq 1,\ |q| < 1)$$

are algebraically dependent if and only if there exist some distinct pairs (a_1, q_1) *and* (a_2, q_2) *of nonzero algebraic numbers* a_1, a_2, q_1, q_2 *with* $|a_1|, |a_2| \leq 1$ *and* $|q_1|, |q_2| < 1$ *such that* $a_1 q_1^{R_k} = a_2 q_2^{R_k}$ *for all sufficiently large* k.

In particular, if g.c.d.$(R_{k+1}-R_k,\ R_{k+2}-R_{k+1}, \ldots, R_{k+n}-R_{k+n-1}) = 1$ *for any* $k \geq 1$, *then the values* $\Theta(a,q)$ *are algebraically independent for any distinct pairs* (a,q) *of nonzero algebraic numbers* a, q *with* $|a| \leq 1$ *and* $|q| < 1$, *namely the infinite set* $\{\Theta(a,q) \mid a, q \in \overline{\mathbb{Q}} \setminus \{0\},\ |a| \leq 1,\ |q| < 1\}$ *is algebraically independent.*

Remark 2. The continued fraction expansion (6) in Corollary 5 is obtained also from the identity (2).

Example 2. Let $\{G_k\}_{k\geq 0}$ be the generalized Fibonacci numbers defined by (4) in Example 1 above and let

$$\begin{aligned}\Theta^*(a,q) &= -\sum_{k=1}^{\infty} \frac{(-a)^k q^{G_1+G_2+\cdots+G_k}}{(1+aq^{G_1})(1+aq^{G_2})\cdots(1+aq^{G_k})} \\ &= \cfrac{aq^{G_1}}{1+aq^{G_1}+\cfrac{aq^{G_2}(1+aq^{G_1})}{1+\cfrac{aq^{G_3}(1+aq^{G_2})}{1+ \ddots + \cfrac{aq^{G_n}(1+aq^{G_{n-1}})}{1+\ddots}}}}. \end{aligned}$$

Since g.c.d.$(G_{k+1}-G_k,\ G_{k+2}-G_{k+1})=$ g.c.d.$(G_2-G_1,\ G_3-G_2)=1$ for any $k\geq 1$ (see Example 1), by Corollary 5 with $\Theta^*(a,q)=-\Theta(-a,q)$, the values $\Theta^*(a,q)$ are algebraically independent for any distinct pairs (a,q) of nonzero algebraic numbers a,q with $|a|\leq 1$ and $|q|<1$. In particular, the continued fractions

$$\cfrac{aq^{F_1}}{1+aq^{F_1}+\cfrac{aq^{F_2}(1+aq^{F_1})}{1+\cfrac{aq^{F_3}(1+aq^{F_2})}{1+\ddots+\cfrac{aq^{F_n}(1+aq^{F_{n-1}})}{1+\ddots}}}} \qquad (a,q\in\overline{\mathbb{Q}}\setminus\{0\},\ |a|\leq 1,\ |q|<1)$$

are algebraically independent, where $\{F_k\}_{k\geq 0}$ is the sequence of Fibonacci numbers defined by (5).

Corollary 6. *Let $(x_1,a_1,q_1),(x_1',a_1',q_1'),(x_2,a_2,q_2),(x_2',a_2',q_2')\in U$. If the values of Θ satisfy*

$$\begin{aligned}&\prod_{k=1}^{k_1-1}\frac{1-a_1q_1^{R_k}}{x_1q_1^{R_k}}\left(\Theta(x_1,a_1,q_1)-\sum_{k=1}^{k_1-1}\prod_{l=1}^{k}\frac{x_1q_1^{R_l}}{1-a_1q_1^{R_l}}\right)\\ =&\prod_{k=1}^{k_1-1}\frac{1-a_1'q_1'^{R_k}}{x_1'q_1'^{R_k}}\left(\Theta(x_1',a_1',q_1')-\sum_{k=1}^{k_1-1}\prod_{l=1}^{k}\frac{x_1'q_1'^{R_l}}{1-a_1'q_1'^{R_l}}\right)\end{aligned}$$

and

$$\begin{aligned}&\prod_{k=1}^{k_2-1}\frac{1-a_2q_2^{R_k}}{x_2q_2^{R_k}}\left(\Theta(x_2,a_2,q_2)-\sum_{k=1}^{k_2-1}\prod_{l=1}^{k}\frac{x_2q_2^{R_l}}{1-a_2q_2^{R_l}}\right)\\ =&\prod_{k=1}^{k_2-1}\frac{1-a_2'q_2'^{R_k}}{x_2'q_2'^{R_k}}\left(\Theta(x_2',a_2',q_2')-\sum_{k=1}^{k_2-1}\prod_{l=1}^{k}\frac{x_2'q_2'^{R_l}}{1-a_2'q_2'^{R_l}}\right),\end{aligned}$$

where k_1 and k_2 are positive integers, then there exists a positive integer k_3 such that

$$\begin{aligned}&\prod_{k=1}^{k_3-1}\frac{1-a_1a_2(q_1q_2)^{R_k}}{x_1x_2(q_1q_2)^{R_k}}\left(\Theta(x_1x_2,a_1a_2,q_1q_2)-\sum_{k=1}^{k_3-1}\prod_{l=1}^{k}\frac{x_1x_2(q_1q_2)^{R_l}}{1-a_1a_2(q_1q_2)^{R_l}}\right)\\ =&\prod_{k=1}^{k_3-1}\frac{1-a_1'a_2'(q_1'q_2')^{R_k}}{x_1'x_2'(q_1'q_2')^{R_k}}\left(\Theta(x_1'x_2',a_1'a_2',q_1'q_2')-\sum_{k=1}^{k_3-1}\prod_{l=1}^{k}\frac{x_1'x_2'(q_1'q_2')^{R_l}}{1-a_1'a_2'(q_1'q_2')^{R_l}}\right).\end{aligned}$$

Proof. Since $\Theta(x_1, a_1, q_1)$ and $\Theta(x_1', a_1', q_1')$ are algebraically dependent and so are $\Theta(x_2, a_2, q_2)$ and $\Theta(x_2', a_2', q_2')$, by the theorem $(x_1, a_1, q_1) \sim (x_1', a_1', q_1')$ and $(x_2, a_2, q_2) \sim (x_2', a_2', q_2')$, respectively. Then $x_1/a_1 = x_1'/a_1'$, $x_2/a_2 = x_2'/a_2'$, and there exist positive integers k_1', k_2' such that $a_1 q_1^{R_k} = a_1'(q_1')^{R_k}$ $(k \geq k_1')$ and $a_2 q_2^{R_k} = a_2'(q_2')^{R_k}$ $(k \geq k_2')$. Hence $(x_1 x_2)/(a_1 a_2) = (x_1' x_2')/(a_1' a_2')$ and $a_1 a_2 (q_1 q_2)^{R_k} = a_1' a_2' (q_1' q_2')^{R_k}$ for all $k \geq k_3 = \max\{k_1', k_2'\}$ and so the corollary is proved by using (7) below.

Sketch of the proof of Theorem. First we prove that, if there exist distinct $(x_1, a_1, q_1), (x_2, a_2, q_2) \in U$ such that $(x_1, a_1, q_1) \sim (x_2, a_2, q_2)$, then the values $\Theta(x_1, a_1, q_1)$ and $\Theta(x_2, a_2, q_2)$ are algebraically dependent. Since $x_1/a_1 = x_2/a_2$ and since there exists a positive integer k_0 such that $a_1 q_1^{R_k} = a_2 q_2^{R_k}$ for all $k \geq k_0$, we have

$$\begin{aligned}
&\prod_{k=1}^{k_0-1} \frac{1 - a_1 q_1^{R_k}}{x_1 q_1^{R_k}} \left(\Theta(x_1, a_1, q_1) - \sum_{k=1}^{k_0-1} \prod_{l=1}^{k} \frac{x_1 q_1^{R_l}}{1 - a_1 q_1^{R_l}} \right) \\
= &\sum_{k=k_0}^{\infty} \prod_{l=k_0}^{k} \frac{x_1}{a_1} \frac{a_1 q_1^{R_l}}{1 - a_1 q_1^{R_l}} \\
= &\sum_{k=k_0}^{\infty} \prod_{l=k_0}^{k} \frac{x_2}{a_2} \frac{a_2 q_2^{R_l}}{1 - a_2 q_2^{R_l}} \\
= &\prod_{k=1}^{k_0-1} \frac{1 - a_2 q_2^{R_k}}{x_2 q_2^{R_k}} \left(\Theta(x_2, a_2, q_2) - \sum_{k=1}^{k_0-1} \prod_{l=1}^{k} \frac{x_2 q_2^{R_l}}{1 - a_2 q_2^{R_l}} \right),
\end{aligned} \tag{7}$$

which implies that $\Theta(x_1, a_1, q_1)$ and $\Theta(x_2, a_2, q_2)$ are algebraically dependent.

Next assume that the values

$$\Theta(x, a, q) \quad ((x, a, q) \in U)$$

are algebraically dependent. Then there exist distinct $(x_1, a_1, q_1), \dots, (x_s, a_s, q_s) \in U$ such that the values $\Theta(x_1, a_1, q_1), \dots, \Theta(x_s, a_s, q_s)$ are algebraically dependent. In what follows, we prove that there exist some distinct i, i' $(1 \leq i, i' \leq s)$ such that $(x_i, a_i, q_i) \sim (x_{i'}, a_{i'}, q_{i'})$, which yields the theorem by renumbering $(x_i, a_i, q_i) = (x_1, a_1, q_1)$ and $(x_{i'}, a_{i'}, q_{i'}) = (x_2, a_2, q_2)$. There exist multiplicatively independent algebraic numbers $\beta_1, \dots, \beta_t$ with $0 < |\beta_j| < 1$ $(1 \leq j \leq t)$ and a primitive N-th root of unity ζ such that

$$q_i = \zeta^{m_i} \prod_{j=1}^{t} \beta_j^{e_{ij}} \quad (1 \leq i \leq s), \tag{8}$$

where $m_1, \dots, m_s$ are integers with $0 \leq m_i \leq N-1$ and e_{ij} $(1 \leq i \leq s,\ 1 \leq j \leq t)$ are nonnegative integers (cf. Loxton and van der Poorten [3], Nishioka [9]). We can choose a positive integer p and a sufficiently large integer u, which will be determined later, such that $R_{k+p} \equiv R_k \pmod{N}$ for any $k \geq u + 1$.

For an $n \times n$ matrix $\Omega = (\omega_{ij})$ with nonnegative integer entries and for $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ we define a transformation $\Omega : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega \boldsymbol{z} = \left(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \dots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right).$$

If

$$\Omega = \begin{pmatrix} c_1 & 1 & 0 & \dots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \dots & \dots & 0 \end{pmatrix}$$

and if

$$M(\boldsymbol{z}) = z_1^{R_n} \cdots z_n^{R_1},$$

where $\{R_k\}_{k \geq 1}$ is a linear recurrence satisfying (1), then by induction we have

$$M(\Omega^k \boldsymbol{z}) = z_1^{R_{k+n}} \cdots z_n^{R_{k+1}} \quad (k \geq 0).$$

Let $y_{j\lambda}$ $(1 \leq j \leq t,\ 1 \leq \lambda \leq n)$ be variables and let $\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn})$ $(1 \leq j \leq t)$, $\boldsymbol{y} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_t)$. Define

$$f_i(\boldsymbol{y}) = \sum_{k=u}^{\infty} \prod_{l=u}^{k} \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^{t} M(\Omega^l \boldsymbol{y}_j)^{e_{ij}}}{1 - a_i \zeta^{m_i R_{l+1}} \prod_{j=1}^{t} M(\Omega^l \boldsymbol{y}_j)^{e_{ij}}} \quad (1 \leq i \leq s).$$

Letting

$$\boldsymbol{\beta} = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots\dots, \underbrace{1, \dots, 1}_{n-1}, \beta_t),$$

we see that

$$f_i(\boldsymbol{\beta}) = \sum_{k=u}^{\infty} \prod_{l=u}^{k} \frac{x_i q_i^{R_{l+1}}}{1 - a_i q_i^{R_{l+1}}} = \sum_{k=u+1}^{\infty} \prod_{l=u+1}^{k} \frac{x_i q_i^{R_l}}{1 - a_i q_i^{R_l}}$$

and so

$$\Theta(x_i, a_i, q_i) = \left(\prod_{k=1}^{u} \frac{x_i q_i^{R_k}}{1 - a_i q_i^{R_k}} \right) f_i(\boldsymbol{\beta}) + \sum_{k=1}^{u} \prod_{l=1}^{k} \frac{x_i q_i^{R_l}}{1 - a_i q_i^{R_l}}.$$

Since $\Theta(x_1, a_1, q_1), \dots, \Theta(x_s, a_s, q_s)$ are algebraically dependent, so are $f_i(\boldsymbol{\beta})$ $(1 \leq i \leq s)$. Let

$$\Omega' = \mathrm{diag}(\underbrace{\Omega^p, \dots, \Omega^p}_{t}),$$

where p is replaced by its multiple such that all the entries of Ω^p are positive. In fact, we can choose such a p. (For the proof see [12].) Then each $f_i(\boldsymbol{y})$ satisfies Mahler type functional equation

$$\begin{aligned} f_i(\boldsymbol{y}) &= \left(\prod_{k=u}^{p+u-1} \frac{x_i\zeta^{m_iR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_{ij}}}{1-a_i\zeta^{m_iR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_{ij}}}\right) f_i(\Omega'\boldsymbol{y}) \\ &\quad + \sum_{k=u}^{p+u-1}\prod_{l=u}^{k} \frac{x_i\zeta^{m_iR_{l+1}}\prod_{j=1}^t M(\Omega^l\boldsymbol{y}_j)^{e_{ij}}}{1-a_i\zeta^{m_iR_{l+1}}\prod_{j=1}^t M(\Omega^l\boldsymbol{y}_j)^{e_{ij}}}, \end{aligned} \tag{9}$$

where $\Omega'\boldsymbol{y} = (\Omega^p\boldsymbol{y}_1, \ldots, \Omega^p\boldsymbol{y}_t)$. Since $f_i(\boldsymbol{\beta})$ $(1 \le i \le s)$ are algebraically dependent, by the theorem of Kubota [2] with Lemma 4 and Proof of Theorem 2 in [12], $f_i(\boldsymbol{y})$ $(1 \le i \le s)$ are algebraically dependent over $\overline{\mathbb{Q}}(\boldsymbol{y})$.

The rest of the proof is abbreviated in some places. (For the complete proof, see [16].) We apply Kubota's criterion [2] on the algebraic independence of Mahler functions over the rational function field, which is stated as a condition concerning the functional equation (9) and

$$H = \left\{ \frac{h(\Omega'\boldsymbol{y})}{h(\boldsymbol{y})} \;\middle|\; h(\boldsymbol{y}) \in \overline{\mathbb{Q}}(\boldsymbol{y}) \setminus \{0\} \right\}.$$

We assert that

$$Q_{ii'}(\boldsymbol{y}) = \prod_{k=u}^{p+u-1} \frac{x_i\zeta^{m_iR_{k+1}}(1-a_{i'}\zeta^{m_{i'}R_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_{i'j}})}{x_{i'}\zeta^{m_{i'}R_{k+1}}(1-a_i\zeta^{m_iR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_{ij}})} \in H$$

if and only if $a_i\zeta^{m_iR_{k+1}} = a_{i'}\zeta^{m_{i'}R_{k+1}}$ $(u \le k \le p+u-1)$, $(e_{i1}, \ldots, e_{it}) = (e_{i'1}, \ldots, e_{i't})$, and $a_i^p x_{i'}^p = a_{i'}^p x_i^p$. It is clear that, if $a_i\zeta^{m_iR_{k+1}} = a_{i'}\zeta^{m_{i'}R_{k+1}}$ $(u \le k \le p+u-1)$, $(e_{i1}, \ldots, e_{it}) = (e_{i'1}, \ldots, e_{i't})$, and $a_i^p x_{i'}^p = a_{i'}^p x_i^p$, then $Q_{ii'}(\boldsymbol{y}) = 1 \in H$. Conversely, suppose that $Q_{ii'}(\boldsymbol{y}) \in H$. Then there exists an $F(\boldsymbol{y}) \in \overline{\mathbb{Q}}(\boldsymbol{y}) \setminus \{0\}$ satisfying

$$F(\boldsymbol{y}) = \left(\prod_{k=u}^{p+u-1} \frac{x_{i'}\zeta^{m_{i'}R_{k+1}}(1-a_i\zeta^{m_iR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_{ij}})}{x_i\zeta^{m_iR_{k+1}}(1-a_{i'}\zeta^{m_{i'}R_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_{i'j}})}\right) F(\Omega'\boldsymbol{y}). \tag{10}$$

Let P be a positive integer and let

$$\boldsymbol{y}_j = (y_{j1}, \ldots, y_{jn}) = (z_1^{P^j}, \ldots, z_n^{P^j}) \quad (1 \le j \le t).$$

We choose a sufficiently large P such that the following two properties are both satisfied:

(a) If $(e_{i1}, \ldots, e_{it}) \ne (e_{i'1}, \ldots, e_{i't})$, then $\sum_{j=1}^t e_{ij}P^j \ne \sum_{j=1}^t e_{i'j}P^j$.

(b) $F^*(\boldsymbol{z}) = F(z_1^P, \ldots, z_n^P, \ldots, z_1^{P^t}, \ldots, z_n^{P^t}) \in \overline{\mathbb{Q}}(z_1, \ldots, z_n) \setminus \{0\}$.

Then by (10), $F^*(\boldsymbol{z})$ satisfies the functional equation

$$F^*(\boldsymbol{z}) = \left(\prod_{k=u}^{p+u-1} \frac{x_{i'}\zeta^{m_{i'}R_{k+1}}(1 - a_i\zeta^{m_iR_{k+1}}M(\Omega^k\boldsymbol{z})^{E_i})}{x_i\zeta^{m_iR_{k+1}}(1 - a_{i'}\zeta^{m_{i'}R_{k+1}}M(\Omega^k\boldsymbol{z})^{E_{i'}})} \right) F^*(\Omega^p\boldsymbol{z}), \tag{11}$$

where $E_i = \sum_{j=1}^t e_{ij}P^j$ $(1 \le i \le s)$. By Theorem 2 of [13] we see that

$$\frac{x_{i'}\zeta^{m_{i'}R_{k+1}}(1 - a_i\zeta^{m_iR_{k+1}}X^{E_i})}{x_i\zeta^{m_iR_{k+1}}(1 - a_{i'}\zeta^{m_{i'}R_{k+1}}X^{E_{i'}})} \in \overline{\mathbb{Q}}^\times$$

for any k $(u \le k \le p+u-1)$, where X is a variable, and $F^*(\boldsymbol{z}) \in \overline{\mathbb{Q}}^\times$. Hence $E_i = E_{i'}$ and $a_i\zeta^{m_iR_{k+1}} = a_{i'}\zeta^{m_{i'}R_{k+1}}$ $(u \le k \le p+u-1)$. Thus $(e_{i1}, \dots, e_{it}) = (e_{i'1}, \dots, e_{i't})$ by the property (a), and the functional equation (11) becomes

$$F^*(\boldsymbol{z}) = \frac{a_i^p x_{i'}^p}{a_{i'}^p x_i^p} F^*(\Omega^p\boldsymbol{z}).$$

Since $F^*(\boldsymbol{z}) \in \overline{\mathbb{Q}}^\times$, we have $a_i^p x_{i'}^p = a_{i'}^p x_i^p$, and the assertion is proved.

Now let S be a nonempty subset of $\{1, \dots, s\}$ such that for any $i, i' \in S$ we have $a_i\zeta^{m_iR_{k+1}} = a_{i'}\zeta^{m_{i'}R_{k+1}}$ $(u \le k \le p+u-1)$, $(e_{i1}, \dots, e_{it}) = (e_{i'1}, \dots, e_{i't})$, and $a_i^p x_{i'}^p = a_{i'}^p x_i^p$. Fix a $\lambda \in S$ and let $a = a_\lambda$, $m = m_\lambda$, $\xi = x_\lambda^p/a_\lambda^p$, and $e_j = e_{\lambda j}$ $(1 \le j \le t)$. Then for any $i \in S$ we have $x_i^p/a_i^p = \xi$, $a_i\zeta^{m_iR_{k+1}} = a\zeta^{mR_{k+1}}$ $(u \le k \le p+u-1)$, $(e_{i1}, \dots, e_{it}) = (e_1, \dots, e_t)$, and so

$$\prod_{k=u}^{p+u-1} \frac{x_i\zeta^{m_iR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_{ij}}}{1 - a_i\zeta^{m_iR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_{ij}}} = \xi \left(\prod_{k=u}^{p+u-1} \frac{a\zeta^{mR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_j}}{1 - a\zeta^{mR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_j}} \right).$$

Hence by (9) the linear combination $G(\boldsymbol{y}) = \sum_{i \in S} c_i f_i(\boldsymbol{y})$ with $c_i \in \overline{\mathbb{Q}}$ satisfies the functional equation

$$\begin{aligned} G(\boldsymbol{y}) &= \xi \left(\prod_{k=u}^{p+u-1} \frac{a\zeta^{mR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_j}}{1 - a\zeta^{mR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_j}} \right) G(\Omega'\boldsymbol{y}) \\ &\quad + \sum_{k=u}^{p+u-1} \sum_{i \in S} c_i \prod_{l=u}^{k} \frac{x_i\zeta^{m_iR_{l+1}}\prod_{j=1}^t M(\Omega^l\boldsymbol{y}_j)^{e_{ij}}}{1 - a_i\zeta^{m_iR_{l+1}}\prod_{j=1}^t M(\Omega^l\boldsymbol{y}_j)^{e_{ij}}} \end{aligned}$$

or

$$\begin{aligned} G(\boldsymbol{y}) &= \xi \left(\prod_{k=u}^{p+u-1} \frac{a\zeta^{mR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_j}}{1 - a\zeta^{mR_{k+1}}\prod_{j=1}^t M(\Omega^k\boldsymbol{y}_j)^{e_j}} \right) G(\Omega'\boldsymbol{y}) \\ &\quad + \sum_{k=u}^{p+u-1} \sum_{i \in S} c_i \left(\frac{x_i}{a_i}\right)^{k-u+1} \prod_{l=u}^{k} \frac{a\zeta^{mR_{l+1}}\prod_{j=1}^t M(\Omega^l\boldsymbol{y}_j)^{e_j}}{1 - a\zeta^{mR_{l+1}}\prod_{j=1}^t M(\Omega^l\boldsymbol{y}_j)^{e_j}}. \end{aligned} \tag{12}$$

Since $f_i(\boldsymbol{y})$ $(1 \le i \le s)$ are algebraically dependent over $\overline{\mathbb{Q}}(\boldsymbol{y})$, by Kubota's criterion [2] mentioned above, there exist algebraic numbers c_i $(i \in S)$, not all zero, such that (12) has a solution $G^*(\boldsymbol{y}) \in \overline{\mathbb{Q}}(\boldsymbol{y})$. Let P be a positive integer and let

$$\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn}) = (z_1^{P^j}, \dots, z_n^{P^j}) \quad (1 \le j \le t).$$

We choose a sufficiently large P such that

$$H(\boldsymbol{z}) = G^*(z_1^P, \dots, z_n^P, \dots, z_1^{P^t}, \dots, z_n^{P^t}) \in \overline{\mathbb{Q}}(z_1, \dots, z_n).$$

Then by (12), $H(\boldsymbol{z})$ satisfies the functional equation

$$\begin{aligned} H(\boldsymbol{z}) &= \xi \left(\prod_{k=u}^{p+u-1} \frac{a\zeta^{mR_{k+1}} M(\Omega^k \boldsymbol{z})^E}{1 - a\zeta^{mR_{k+1}} M(\Omega^k \boldsymbol{z})^E} \right) H(\Omega^p \boldsymbol{z}) \\ &\quad + \sum_{k=u}^{p+u-1} \sum_{i \in S} c_i \left(\frac{x_i}{a_i} \right)^{k-u+1} \prod_{l=u}^{k} \frac{a\zeta^{mR_{l+1}} M(\Omega^l \boldsymbol{z})^E}{1 - a\zeta^{mR_{l+1}} M(\Omega^l \boldsymbol{z})^E}, \end{aligned} \tag{13}$$

where $E = \sum_{j=1}^t e_j P^j$. Letting $H(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z})$, where $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime polynomials in $\overline{\mathbb{Q}}[z_1, \dots, z_n]$ with $B \not\equiv 0$, by (13) we have

$$\begin{aligned} & A(\boldsymbol{z}) B(\Omega^p \boldsymbol{z}) \prod_{k=u}^{p+u-1} (1 - a\zeta^{mR_{k+1}} M(\Omega^k \boldsymbol{z})^E) \\ = \; & \xi A(\Omega^p \boldsymbol{z}) B(\boldsymbol{z}) \prod_{k=u}^{p+u-1} a\zeta^{mR_{k+1}} M(\Omega^k \boldsymbol{z})^E \\ & + \sum_{k=u}^{p+u-1} \sum_{i \in S} c_i \left(\frac{x_i}{a_i} \right)^{k-u+1} B(\boldsymbol{z}) B(\Omega^p \boldsymbol{z}) \prod_{l=u}^{k} a\zeta^{mR_{l+1}} M(\Omega^l \boldsymbol{z})^E \\ & \quad \times \prod_{l'=k+1}^{p+u-1} (1 - a\zeta^{mR_{l'+1}} M(\Omega^{l'} \boldsymbol{z})^E). \end{aligned} \tag{14}$$

We can put by Lemma 3.2.3 in Nishioka [9] the greatest common divisor of $A(\Omega^p \boldsymbol{z})$ and $B(\Omega^p \boldsymbol{z})$ as $\boldsymbol{z}^I$, where I is an n-dimensional vector with nonnegative integer components. Then $B(\Omega^p \boldsymbol{z})$ divides $B(\boldsymbol{z}) \boldsymbol{z}^I \prod_{k=u}^{p+u-1} M(\Omega^k \boldsymbol{z})^E$ by (14). Therefore $B(\boldsymbol{z})$ is a monomial in $z_1, \dots, z_n$ by Lemma 12 of [13] with Lemma 4 and Proof of Theorem 2 in [12]. If u is sufficiently large, since p and u are independent, the right-hand side of (14) is divided by $z_1 \cdots z_n B(\Omega^p \boldsymbol{z})$ and thus $A(\boldsymbol{z})$ is divided by $z_1 \cdots z_n$. Since $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime, $B(\boldsymbol{z}) \in \overline{\mathbb{Q}}^\times$. If $A(\boldsymbol{z}) \notin \overline{\mathbb{Q}}$, then by Lemma 6 of [14], $\deg_{\boldsymbol{z}} A(\Omega^p \boldsymbol{z}) > \deg_{\boldsymbol{z}} A(\boldsymbol{z})$, which is a contradiction by comparing the total degrees of both sides of (14). Hence $A(\boldsymbol{z}) = 0$. Then by (14), we see that $\sum_{i \in S} c_i (x_i/a_i)^{k-u+1} = 0$ $(u \le k \le p+u-1)$ and so $\sum_{i \in S} c_i (x_i/a_i)^k = 0$ $(1 \le k \le p)$. Hence $x_i/a_i = x_{i'}/a_{i'}$ for some distinct $i, i' \in S$ since c_i $(i \in S)$ are not all zero. Since $i, i' \in S$ and $R_{k+p} \equiv R_k \pmod{N}$ for any $k \ge u+1$, we have $a_i \zeta^{m_i R_{k+1}} = a_{i'} \zeta^{m_{i'} R_{k+1}}$ $(k \ge u)$. By (8) with $(e_{i1}, \dots, e_{it}) = (e_{i'1}, \dots, e_{i't})$ we get $a_i q_i^{R_{k+1}} = a_{i'} q_{i'}^{R_{k+1}}$ $(k \ge u)$. Therefore $(x_i, a_i, q_i) \sim (x_{i'}, a_{i'}, q_{i'})$, and the proof of the theorem is completed.

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