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Lattice-Valued Fuzzy Convex Geometry

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Abstract

There are five classical theorems on convex sets in Euclidean spaces, i.e., Carathéodory's theorem, Radon's theorem, finite and infinite Helly's theorems, and Kakutani's fixed point theorem. In this paper, by considering lattice-valued fuzzy convex sets in Euclidean spaces and some lattice-valued fuzzy topologies of Euclidean spaces, we extend these theorems to the lattice-valued fuzzy case in a uniform way using a-cut operators. This paper provides an example of lattice-valued fuzzyfication of classical mathematics.

1 Introduction

In this paper, we investigate the notion of convexity in Euclidean spaces from the viewpoint of lattice-valued fuzzy mathematics and reveal some combinatorial properties of lattice-valued fuzzy convex sets in Euclidean spaces.

Convex geometry has been studied extensively from different perspectives (for example, see [10, 6]) and has been applied to diverse research areas, including computer science and economics. In this paper, we consider Carathéodory's theorem, Radon's theorem and (finite and infinite) Helly's theorems (see [8, 10]), which are fundamental results in combinatorial convex geometry and have many applications to computer science (for example, see [1, 21]). As a very significant fact, these theorems characterize the dimension of a Euclidean space. We also consider Kakutani's fixed point theorem (see [14, 4]), which is a generalization of Brouwer's fixed point theorem and has widespread applications to economics and game theory (see [4, 2]). It is well known that J. F. Nash used Kakutani's fixed point theorem to show the existence of so called Nash equilibrium (see [20, 4]).

Let us briefly review historical developments of fuzzy convex geometry. In 1965, Zadeh [26] introduced the concept of $[0,1]$-valued fuzzy sets ($[0,1]$-fuzzy sets, for short) by considering $[0,1]$-valued membership functions. Zadeh [26] also introduced the concept of $[0,1]$-fuzzy convex sets, based on which several authors have studied the various properties of $[0,1]$-fuzzy convex sets (for exapmle, see [9, 15, 22, 24, 25, 27]). Among them, in 1980, Lowen [17] showed separation theorems for $[0,1]$-fuzzy convex sets. Meanwhile, in 1967, Goguen [12] generalized the concept of $[0,1]$-fuzzy sets to L-valued fuzzy sets (L-fuzzy sets, for short) for an arbitrary completely distributive lattice L. However, Goguen [12] did not consider the convexity of L-fuzzy sets. Thereafter, in 2008, Huang and Shi [11] introduced the notion of L-fuzzy convex sets, some properties of which are investigated in this paper.

In this paper, we show some L-fuzzy versions of the above classical theorems on convex fuzzy sets, i.e, Carathéodory's theorem, Radon's theorem, finite and infinite Helly's theorem, and Kakutani's fixed point theorem. As a related work, in 1991, Feiyue [9] extended Carathéodory's theorem to the $[0,1]$-fuzzy case. In this paper, we extend Carathéodory's theorem to the L-fuzzy case for an arbitrary completely distributive lattice L. By using some properties of a-cut operators for $a \in L$ (for the definition, see Definition 2.4), we can prove L-fuzzy versions of the five theorems in a uniform way. It is expected that we can extend other theorems on convex sets to the L-fuzzy case in similar ways. We remark that a-cut operators play important roles also in the context of mathematical fuzzy logics, especially topological duality theory for them (see [18, 19]).

This paper is organized as follows. In Section 2, we first review basic notions on L-fuzzy sets. Then we introduce a-cut operators for $a \in L$ and show some properties of them. In Section 3, we review basic notions on L-fuzzy convex sets and show some properties of them, among which it is shown that a-cut operators transform an L-fuzzy convex set into a convex set in the ordinary (i.e., two-valued) sense. In Section 4, we first review basic notions on L-fuzzy topology and then equip n-dimensional Euclidean space with a natural L-fuzzy topology. Then we show that, under some assumption on $a \in L$, a-cut operators transform an L-fuzzy compact (resp. closed) set into a compact (resp. closed) set in the ordinary sense. Section 4 is needed only for L-fuzzy versions of infinite Helly's theorem and Kakutani fixed point theorem. In Section 5, using a-cut operators, we show the main theorems in this paper, i.e., some L-fuzzy versions of the above five theorems in convex geometry. Finally, in Section 6, we consider some future work, including a mechanical procedure to get fuzzy versions of theorems in classical mathematics. Our final aim is to obtain such a "fuzzyfication procedure", and this paper is a first step for it.

In Sections 2, 3 and 4, we omit the proofs of all the propositions and, in Section 5, we sketch the proofs of the main theorems. The details of the proofs will be presented in a subsequent paper.

2 Basic Concepts and Propositions

ω denotes the set of all non-negative integers.

A lattice L is called a completely distributive lattice iff L is a complete lattice and arbitrary joins in L distribute over arbitrary meets in L (see [7, 16]). Throughout this paper, L denotes an arbitrary completely distributive lattice and 0 (resp. 1) denotes the least (resp. greatest) element of L. L_2 denotes the two-element Boolean algebra $\{0, 1\}$.

$\mathbb{R}^n$ denotes n-dimensional Euclidean space. An L-fuzzy set on $\mathbb{R}^n$ is defined as a function from $\mathbb{R}^n$ to L, where "on $\mathbb{R}^n$" is often dropped. Since L is complete, the class of all L-fuzzy sets is closed under arbitrary meets (i.e., infimum). We call a set in the ordinary sense a crisp set.

For two L-fuzzy sets μ and λ, we define $\mu \leq \lambda$ by $\mu(x) \leq \lambda(x)$ for any $x \in \mathbb{R}^n$. We do not distinguish between an element a of L and the L-fuzzy set λ defined by $\lambda(x) = a$ for any $x \in \mathbb{R}^n$. For instance, for an L-fuzzy set μ, when we write $\mu \leq a$, we mean that $\mu(x) \leq a$ for any $x \in \mathbb{R}^n$.

Definition 2.1. *Let μ be an L-fuzzy set on $\mathbb{R}^n$. The a-cut set of μ, which is denoted by $\mathrm{C}_a(\mu)$, is defined by*

$$\mathrm{C}_a(\mu) = \{x \in \mathbb{R}^n \,;\, \mu(x) \geq a\}.$$

Definition 2.2. *Let $a \in L$ and μ an L-fuzzy set on $\mathbb{R}^n$. Then, μ is said to be non-empty relative to a iff there exists $x \in \mathbb{R}^n$ such that $\mu(x) \geq a$.*

It is clear that μ is non-empty relative to 1 iff $\mu^{-1}(\{1\})$ is not empty.

Definition 2.3. *For $a \in L$, a is said to be upward inaccessible in L iff the following holds:*

$$\bigvee\{x \in L \,;\, x < a\} < a.$$

If L is a finite totally ordered lattice, then every element in L is upward inaccessible in L.

Definition 2.4. *Let $a \in L$. We define a function τ_a from L to L_2 as follows.*

$$\tau_a(x) = \begin{cases} 1 & (\text{if } x \geq a) \\ 0 & (\text{otherwise}). \end{cases}$$

We call τ_a an a-cut operator.

Throughout this paper, a denotes an element of L.
Note that $\mathrm{C}_a(\mu) = (\tau_a \circ \mu)^{-1}(\{1\})$ for an L-fuzzy set μ.

Proposition 2.5. *The following hold:*

- *τ_a is idempotent, i.e., $\tau_a \circ \tau_a = \tau_a$;*
- *τ_a is monotone, i.e., if $x \leq y$ then $\tau_a(x) \leq \tau_a(y)$.*

τ_a commutes with arbitrary meets as follows.

Proposition 2.6. *Let μ_i be an L-fuzzy sets for each index $i \in I$. Then,*

$$\tau_a \circ \bigwedge_{i \in I} \mu_i = \bigwedge_{i \in I} (\tau_a \circ \mu_i).$$

While τ_a does not necessarily commutes with joins, we have the following:

Proposition 2.7. *Let $a \in L$ be upward inaccessible in L. Then,*

$$\tau_a \circ \bigvee_{i \in I} \mu_i = \bigvee_{i \in I} (\tau_a \circ \mu_i).$$

We give an example which shows that τ_a does not necessarily commutes with joins. Let L be the four-element Boolean algebra $\{0, a, b, 1\}$, where a and b are incomparable. Define an L-fuzzy set μ by $\mu(x) = a$ for $x \in \mathbb{R}^n$ and an L-fuzzy set λ by $\lambda(x) = b$ for $x \in \mathbb{R}^n$. Then we have

$$\tau_1 \circ (\mu \vee \lambda) \neq (\tau_1 \circ \mu) \vee (\tau_1 \circ \lambda).$$

Note that 1 is not upward inaccessible in L.

3 Basic Results on L-Fuzzy Convex Sets

We call a convex subset of $\mathbb{R}^n$ in the ordinary sense a crisp convex set on $\mathbb{R}^n$.

Definition 3.1. *An L-fuzzy set μ on $\mathbb{R}^n$ is an L-fuzzy convex set on $\mathbb{R}^n$ iff*

$$\mu(rx + (1-r)y) \geq \mu(x) \wedge \mu(y)$$

for any $x, y \in \mathbb{R}^n$ and any $r \in [0,1]$.

$\mathcal{C}_L$ denotes the set of all L-fuzzy convex sets on $\mathbb{R}^n$. For a set X, let L^X denote the set of all functions from X to L.

Definition 3.2. *For a set X and a subset $\mathcal{C}$ of L^X, $(X, \mathcal{C})$ is called an L-fuzzy convexity space iff $(X, \mathcal{C})$ satisfies the following properties:*

- *$0, 1 \in \mathcal{C}$;*
- *$\mathcal{C}$ is closed under arbitrary meets;*
- *if $\{\mu_i \in \mathcal{C}\ ;\ i \in I\}$ is totally ordered, then $\bigvee\{\mu_i\ ;\ i \in I\} \in \mathcal{C}$,*

where 0 (resp. 1) is the constant function whose value is always 0 (resp. 1).

Note that L_2-fuzzy convexity spaces coincide with ordinary convexity spaces (for the definition of ordinary convexity spaces, see [6]).

Proposition 3.3. *$(\mathbb{R}^n, \mathcal{C}_L)$ is an L-fuzzy convexity space.*

Since the class of all L-fuzzy convex sets is closed under arbitrary meets, we can define the L-fuzzy convex hull of an L-fuzzy set as follows.

Definition 3.4. *For an L-fuzzy set μ on $\mathbb{R}^n$,*

$$\mathrm{H}_L(\mu) = \bigwedge\{\lambda\ ;\ \mu \leq \lambda \text{ and } \lambda \text{ is an } L\text{-fuzzy convex set}\}.$$

Then, $\mathrm{H}_L(\mu)$ is called the L-fuzzy convex hull of μ.

Let X be a subset of $\mathbb{R}^n$. Then, the L_2-fuzzy convex hull (of the indicator function) of X coincides with the ordinary convex hull of X, where the indicator function μ_X of X is defined by $\mu_X(x) = 1$ for $x \in X$ and $\mu_X(x) = 0$ for $x \notin X$. Throughout this paper, we do not distinguish between a set and the indicator function of the set.

τ_a transforms an L-fuzzy convex set on $\mathbb{R}^n$ into a crisp convex subset of $\mathbb{R}^n$ as follows.

Proposition 3.5. *Let $\mu \in \mathcal{C}_L$. Then, $\tau_a \circ \mu$ is a crisp convex set.*

The converse of the above proposition also holds:

Proposition 3.6. *Let μ be an L-fuzzy set on $\mathbb{R}^n$. If $\tau_a \circ \mu$ is a crisp convex subset of $\mathbb{R}^n$ for any $a \in L$, then μ is an L-fuzzy convex set on $\mathbb{R}^n$.*

The following proposition states that, for a subset X of $\mathbb{R}^n$, the L-fuzzy convex hull (of the indicatior function) of X coincides with the L_2-fuzzy convex hull (of the indicatior function) of X.

Proposition 3.7. *Let μ be an L-fuzzy set on $\mathbb{R}^n$. If the range of μ is contained in $\{0,1\}$, then we have*

$$\mathrm{H}_L(\mu) = \mathrm{H}_{L_2}(\mu).$$

τ_a commutes with H_L as follows.

Proposition 3.8. *Let μ be an L-fuzzy set on $\mathbb{R}^n$. Then,*

$$\mathrm{H}_L(\tau_a \circ \mu) = \tau_a \circ \mathrm{H}_L(\mu).$$

4 Basic Results on L-Fuzzy Topology

This section is needed only for Theorem 5.9 and Theorem 5.11.

Chang [5] introduced the notion of [0, 1]-valued fuzzy topology and thereafter Goguen [13] extended that to the L-valued fuzzy case. Now there are a number of articles on fuzzy topology (for example, see **??**).

Definition 4.1. *Let S be a set and $\mathcal{O}$ a subset of L^S. $(S, \mathcal{O})$ is an L-fuzzy topological space iff the following hold:*

- $0, 1 \in \mathcal{O}$;
- *if $\mu, \lambda \in \mathcal{O}$, then $\mu \wedge \lambda \in \mathcal{O}$;*
- *if $\{\mu_i\,;\, i \in I\} \subset \mathcal{O}$, then $\bigvee\{\mu_i\,;\, i \in I\} \in \mathcal{O}$,*

where 0 (resp. 1) is the constant function whose value is always 0 (resp. 1). We call $\mathcal{O}$ an L-fuzzy topology of S and an element of $\mathcal{O}$ an L-fuzzy open set on S.

In order to define the notion of L-fuzzy closed sets, we equip L with an operation which is suppoed to have similar properties as the usual set theoretic complement:

Definition 4.2. *We assume that L is equipped with an operation*

$$(\text{-})^c : L \longrightarrow L$$

which satisfies the following properties:

- $0^c = 1$ *and* $1^c = 0$*;*
- *if* $x < y$ *then* $y^c < x^c$*;*
- $(x^c)^c = x$*;*
- $(\bigwedge_{i \in I} x_i)^c = \bigvee_{i \in I}(x_i)^c$ *and* $(\bigvee_{i \in I} x_i)^c = \bigwedge_{i \in I}(x_i)^c$*;*

where x, y, x_i are arbitrary elements in L.

We give a typical example of $(\text{-})^c$. Let $L = [0,1]$. Define $(\text{-})^c : L \longrightarrow L$ by $a^c = 1 - a$. Then, $(\text{-})^c$ satisfies the above four properties. Note that τ_a does not commute with $(\text{-})^c$ in this case.

Definition 4.3. *Let $(S, \mathcal{O})$ be an L-fuzzy topological space. For an L-fuzzy open set μ on S, μ^c is called an L-fuzzy closed set on S, where μ^c is defined by $\mu^c(x) = (\mu(x))^c$ for $x \in S$.*

An open (resp. closed) subset of $\mathbb{R}^n$ equipped with the Euclidean topology is called a crisp open (resp. closed) subset of $\mathbb{R}^n$.

We equip $\mathbb{R}^n$ with an L-fuzzy topology as follows.

Definition 4.4. *For $a, b \in L$ with $a < b$, let $\mathcal{O}_{a,b}$ be the set of all L-fuzzy sets μ on $\mathbb{R}^n$ such that the range of μ is contained in $\{a, b\}$ and $\mu^{-1}(b)$ is a crisp open subset of $\mathbb{R}^n$.*

Then, we can equip $\mathbb{R}^n$ with the L-fuzzy topology generated by

$$\bigcup\{\mathcal{O}_{a,b}\ ;\ a, b \in L \text{ and } a < b\}.$$

Note that every constant function from $\mathbb{R}^n$ to L is an L-fuzzy open set on $\mathbb{R}^n$ and that the L_2-fuzzy open (resp. closed) sets on $\mathbb{R}^n$ coincide with the crisp open (resp. closed) subsets on $\mathbb{R}^n$.

If a is upward inaccessible in L, then τ_a transforms an L-fuzzy closed set on $\mathbb{R}^n$ into a crisp closed subset of $\mathbb{R}^n$:

Proposition 4.5. *Let $a \in L$ be upward inaccessible in L and μ an L-fuzzy closed set on $\mathbb{R}^n$. Then, $\tau_a \circ \mu$ is a crisp closed subset of $\mathbb{R}^n$.*

The ordinary definition (using open covers) of compactness is naturally extended to the L-fuzzy case as follows.

Definition 4.6. *An L-fuzzy set μ on $\mathbb{R}^n$ is an L-fuzzy compact set iff, for any L-fuzzy open sets λ_i $(i \in I)$ on $\mathbb{R}^n$, if $\mu \leq \bigvee\{\lambda_i; i \in I\}$ then $\mu \leq \bigvee\{\lambda_j; j \in J\}$ for some finite subset J of I.*

A compact subset of $\mathbb{R}^n$ equipped with the Euclidean topology is called a crisp compact subset of $\mathbb{R}^n$. The L_2-fuzzy compact sets on $\mathbb{R}^n$ coincide with the crisp compact subsets of $\mathbb{R}^n$.

Remark 4.7. *For any $[0,1]$-fuzzy set μ on $\mathbb{R}^n$, if $\mu \neq 0$, then μ is not a $[0,1]$-fuzzy compact set, which is shown as follows: Let $x_0 \in \mathbb{R}^n$ be such that $\mu(x_0) \neq 0$. Take a strictly increasing sequence $(a_n)_{n\in\omega}$ of real numbers such that $a_n > 0$ for any $n \in \omega$ and $(a_n)_{n\in\omega}$ converges to $\mu(x_0)$. For $n \in \omega$, define a $[0,1]$-fuzzy open set μ_n on $\mathbb{R}^n$ by*

$$\mu_n(x) = \begin{cases} a_n & \text{if } x \in (x_0 - 1, x_0 + 1) \\ 0 & \text{otherwise.} \end{cases}$$

Define a $[0,1]$-fuzzy open set λ on $\mathbb{R}^n$ by

$$\lambda(x) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{otherwise.} \end{cases}$$

Now we have

$$\mu \leq \lambda \vee \left(\bigvee_{n\in\omega} \mu_n \right).$$

By the choice of $(a_n)_{n\in\omega}$, we have, for any $n \in \omega$,

$$\mu \not\leq \lambda \vee \mu_0 \vee \ldots \vee \mu_n.$$

Therefore, μ is not a $[0,1]$-fuzzy compact set on $\mathbb{R}^n$.

Thus, 0 is the only $[0,1]$-fuzzy compact set on $\mathbb{R}^n$. If we replace the definition of L-fuzzy compactness with another one, we can see many non-trivial $[0,1]$-fuzzy compact sets on $\mathbb{R}^n$, though we omit the details here.

If a is upward inaccessible in L, then τ_a transforms an L-fuzzy compact set on $\mathbb{R}^n$ into a crisp compact subset of $\mathbb{R}^n$:

Proposition 4.8. *Let $a \in L$ be upward inaccessible in L and μ an L-fuzzy compact set on $\mathbb{R}^n$. Then, $\tau_a \circ \mu$ is a crisp compact subset of $\mathbb{R}^n$.*

If we drop the assumption that $a \in L$ is upward inaccessible in L, then the above proposition dose not necessarily hold: Consider the four-element Boolean algebra $L = \{0, a, b, 1\}$. Note that a and b are incomparable. Now, define $\mu : \mathbb{R} \to L$ by

$$\mu(x) = \begin{cases} a & \text{if } x \in [0,1) \\ b & \text{if } x \in [1,2] \\ 0 & \text{otherwise.} \end{cases}$$

Then, μ is L-fuzzy compact and $\tau_a \circ \mu$ is not crisp compact.

5 Main Results

In this section, we state main theorems and sketch the proofs of them. Roughly speaking, by using the above properties of τ_a, we can show that the main theorems follow from corresponding classical versions. Note that if L is isomorphic to L_2 then the main theorems coincide with the classical versions of them.

We first review the classical versions.

The following is the classical version of Carathéodory's theorem. For a set X, $|X|$ denotes the cardinality of X. For $X \subset \mathbb{R}^n$, $\mathrm{H}(X)$ denotes the convex hull of X in the ordinary sense.

Proposition 5.1. *For $X \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the following are equivalent:*

- *$x \in \mathrm{H}(X)$;*
- *there is $Y \subset \mathbb{R}^n$ such that $Y \subset X$, $x \in \mathrm{H}(Y)$, and $|Y| \leq n+1$.*

The following is the classical version of Radon's theorem.

Proposition 5.2. *Let $X \subset \mathbb{R}^n$ with $|X| \geq n+2$. Then, there are $Y, Z \subset \mathbb{R}^n$ such that $X = Y \cup Z$, $Y \cap Z = \emptyset$ and $\mathrm{H}(Y) \cap \mathrm{H}(Z) \neq \emptyset$.*

The following is the classical version of finite Helly's theorem.

Proposition 5.3. *Let $\mathcal{F}$ be a finite family of crisp convex subsets of $\mathbb{R}^n$ with $n+1 \leq |\mathcal{F}|$. If $\bigcap X \neq \emptyset$ for any $X \subset \mathcal{F}$ with $|X| = n+1$, then $\bigcap \mathcal{F} \neq \emptyset$.*

The following is the classical version of infinite Helly's theorem.

Proposition 5.4. *Let $\mathcal{F}$ be a family of crisp compact convex subsets of $\mathbb{R}^n$. If $\bigcap X \neq \emptyset$ for any $X \subset \mathcal{F}$ with $|X| = n+1$, then $\bigcap \mathcal{F} \neq \emptyset$.*

The following is the classical version of Kakutani's fixed point theorem. $\mathcal{P}(S)$ denotes the powerset of S for a set S.

Proposition 5.5. *Let $S \neq \emptyset$ be a compact convex subset of $\mathbb{R}^n$. Consider $\Phi : S \longrightarrow \mathcal{P}(S)$ with a closed graph such that, for every $x \in S$, $\Phi(x)$ is non-empty and convex. Then, there is $x \in S$ with $x \in \Phi(x)$.*

The following are the main theorems in this paper.

Theorem 5.6 (L-valued Carathéodory's theorem). *Let μ be an L-fuzzy set on $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $a \in L$. Then, the following are equivalent:*

1. $\mathrm{H}_L(\mu)(x) = a$;
2. *there exists an L-fuzzy set λ on $\mathbb{R}^n$ such that the following hold:*
   - $\lambda \leq \mu$;
   - $\mathrm{H}_L(\lambda)(x) = \mathrm{H}_L(\mu)(x) = a$;
   - $|\{y \,;\, \lambda(y) > 0\}| \leq n+1$.

Proof. We show that (1) implies (2). Assume $\mathrm{H}_L(\mu)(x) = a$. Consider $\tau_a \circ \mu$, which is a crisp convex subset of $\mathbb{R}^n$ by Proposition 3.5. By Proposition 3.8 and Proposition 3.7, we have

$$\mathrm{H}_{L_2}(\tau_a \circ \mu)(x) = \mathrm{H}_L(\tau_a \circ \mu)(x) = \tau_a(\mathrm{H}_L(\mu)(x)) = \tau_a(a) = 1.$$

Thus, by the classical version of Carathéodory's theorem, there exists a crisp set λ_0 such that $\lambda_0 \leq \tau_a \circ \mu$, $\mathrm{H}_{L_2}(\lambda_0)(x) = 1$ and $|\{y \,;\, \lambda_0(y) > 0\}| \leq n+1$. Define an L-fuzzy set λ by

$$\lambda(x) = \begin{cases} a & (\text{if } \lambda_0(x) = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, λ satisfies the required three properties.

The converse is easily verified. □

Theorem 5.7 (L-valued Radon's theorem). *Let μ be an L-fuzzy set on $\mathbb{R}^n$ and $a \in L$. Assume $|\mathrm{C}_a(\mu)| \geq n+2$. Then, there exist two L-fuzzy sets λ_1, λ_2 on $\mathbb{R}^n$ such that the following hold:*

- $\lambda_1 \vee \lambda_2 = \mu$;
- $\lambda_1 \wedge \lambda_2 = 0$;
- $\mathrm{H}_L(\lambda_1) \wedge \mathrm{H}_L(\lambda_2)$ *is non-empty relative to a.*

Proof. Consider a crisp set $\tau_a \circ \mu$. By the assumption, we have

$$|\{x\,;\,\tau_a \circ \mu(x) = 1\}| \geq n+2.$$

Thus, by the classical version of Radon's theorem, there exist two crisp sets ν_1, ν_2 such that $\nu_1 \vee \nu_2 = \tau_a \circ \mu$, $\nu_1 \wedge \nu_2 = 0$ and $(\mathrm{H}_{L_2}(\nu_1) \wedge \mathrm{H}_{L_2}(\nu_2))(x) = 1$ for some $x \in \mathbb{R}^n$.

Define an L-fuzzy set κ_1 by

$$\kappa_1(x) = \begin{cases} a & (\text{if } \nu_1(x) = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

SImilarly, define an L-fuzzy set κ_2 by

$$\kappa_2(x) = \begin{cases} a & (\text{if } \nu_2(x) = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, we have $\kappa_1 \wedge \kappa_2 = 0$ and $(\mathrm{H}_L(\kappa_1) \wedge \mathrm{H}_L(\kappa_2))(x) = a$ for some $x \in \mathbb{R}^n$.

Now, define an L-fuzzy set λ_1 by

$$\lambda_1(x) = \begin{cases} \mu(x) & (\text{if } a \leq \mu(x) \text{ and } \kappa_1(x) = a) \\ \mu(x) & (\text{if } a \not\leq \mu(x) \text{ and } \kappa_1(x) = 0) \\ 0 & (\text{otherwise}). \end{cases}$$

Define an L-fuzzy set λ_2 by

$$\lambda_2(x) = \begin{cases} \mu(x) & (\text{if } a \leq \mu(x) \text{ and } \kappa_2(x) = a) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, we can verify that $\lambda_1 \vee \lambda_2 = \mu$ and $\lambda_1 \wedge \lambda_2 = 0$. Recall that $(\mathrm{H}_L(\kappa_1) \wedge \mathrm{H}_L(\kappa_2))(x) = a$ for some $x \in \mathbb{R}^n$. Since $\kappa_1 \leq \lambda_1$ and $\kappa_2 \leq \lambda_2$, we have $(\mathrm{H}_L(\lambda_1) \wedge \mathrm{H}_L(\lambda_2))(x) \geq a$ for some $x \in \mathbb{R}^n$. $\square$

Theorem 5.8 (L-valued finite Helly's theorem). *Assume the following:*

- $\{\mu_1, ..., \mu_k\}$ *is a family of L-fuzzy convex sets on* $\mathbb{R}^n$*;*
- $|\{\mu_1, ..., \mu_k\}| \geq n+1$*;*
- *for any* $X \subset \{\mu_1, ..., \mu_k\}$ *with* $|X| = n+1$, $\bigwedge X$ *is non-empty relative to* a.

Then, $\bigwedge\{\mu_1, ..., \mu_k\}$ *is non-empty relative to* a.

Proof. Consider the set

$$\mathcal{F}_a = \{\tau_a \circ \mu_1, ..., \tau_a \circ \mu_k\}.$$

Note that $\tau_a \circ \mu_i$ is is a-crisp covex subset of $\mathbb{R}^n$ by Proposition 3.5.

In order to apply the classical version of Helly's theorerm, we need to show that for any $Y \subset \mathcal{F}_a$ with $|Y| = n+1$, there exists $y \in \mathbb{R}^n$ such that $(\bigwedge Y)(y) = 1$, which is proved by a similar argument as in the second paragraph of the proof of Theorem 5.9 below.

Therefore, by the classical version of Helly's theorerm, there is $z \in \mathbb{R}^n$ such that $(\bigwedge \mathcal{F}_a)(z) = 1$. By Proposition 2.6, we have $(\bigwedge\{\mu_1, ..., \mu_k\})(z) \geq a$. □

Theorem 5.9 (L-valued infinite Helly's theorem). *Assume the following:*

- $a \in L$ *is upward inaccessible in* L*;*
- $\mathcal{F}$ *is a family of L-fuzzy compact convex sets on* $\mathbb{R}^n$*;*
- *for any* $X \subset \mathcal{F}$ *with* $|X| = n+1$, $\bigwedge X$ *is non-empty relative to* a.

Then, $\bigwedge \mathcal{F}$ *s non-empty relative to* a.

Proof. Consider the set

$$\mathcal{F}_a = \{\tau_a \circ \mu;\ \mu \in \mathcal{F}\}.$$

For $\mu \in \mathcal{F}$, $\tau_a \circ \mu$ is a crisp compact covex set by Proposition 3.5 and Proposition 4.8.

In order to apply the classical version of infinite Helly's theorerm, we show that for any $Y \subset \mathcal{F}_a$ with $|Y| = n+1$, there exists $y \in \mathbb{R}^n$ such that $(\bigwedge Y)(y) = 1$. Let $Y \subset \mathcal{F}_a$ with $|Y| = n+1$. Then, there exists

$X \subset \{\mu \in \mathcal{F}\,;\ \tau_a \circ \mu \in Y\}$ such that $Y = \{\tau_a \circ \mu\,;\ \mu \in X\}$ and, if $\mu_1, \mu_2 \in X$ and $\tau_a \circ \mu_1 = \tau_a \circ \mu_2$ then we have $\mu_1 = \mu_2$. Clearly, $|X| = n + 1$. Thus, by the assumption of the theorem, there exists $y \in \mathbb{R}^n$ such that $(\bigwedge X)(y) \geq a$. Since $Y = \{\tau_a \circ \mu;\ \mu \in X\}$, it follows from Proposition 2.6 that $(\bigwedge Y)(y) = 1$.

Therefore, by the classical version of infinite Helly's theorerm, there exists $z \in \mathbb{R}^n$ such that $(\bigwedge \mathcal{F}_a)(z) = 1$. Hence, by Proposition 2.6, we have $(\bigwedge \mathcal{F})(z) \geq a$. □

Definition 5.10. *Let $a \in L$ and μ an L-fuzzy set on $\mathbb{R}^n$. For a function $\Phi : \mathrm{C}_a(\mu) \to L^{\mathrm{C}_a(\mu)}$, an L-fuzzy set $\mathrm{G}(\Phi)$ on $\mathbb{R}^{2n}$ is defined by*

$$\mathrm{G}(\Phi)((x, y)) = \begin{cases} \Phi(x)(y) & (\text{if } (x, y) \in \mathrm{C}_a(\mu) \times \mathrm{C}_a(\mu)) \\ 0 & (\text{otherwise}). \end{cases}$$

Intuitively, we can consider $\mathrm{G}(\Phi)$ as the L-fuzzy graph of Φ.

Theorem 5.11 (L-valued Kakutani's fixed point theorem). *Let μ be an L-fuzzy set on $\mathbb{R}^n$. Consider a function $\Phi : \mathrm{C}_a(\mu) \to L^{\mathrm{C}_a(\mu)}$. Assume:*

- *$a \in L$ is upward inaccessible in L;*
- *μ is non-empty relative to a and L-fuzzy compact convex;*
- *$\Phi(x)$ is non-empty relative to a and L-fuzzy convex for any $x \in \mathrm{C}_a(\mu)$;*
- *$\mathrm{G}(\Phi)$ is an L-fuzzy closed set on $\mathbb{R}^{2n}$.*

Then, there exists $x \in \mathrm{C}_a(\mu)$ with $\Phi(x)(x) \geq a$.

Proof. Consider $\tau_a \circ \mu$, which is a crisp compact convex subset of $\mathbb{R}^n$ by Proposition 3.5 and Proposition 4.8. Note that $(\tau_a \circ \mu)^{-1}(\{1\}) = \mathrm{C}_a(\mu)$. By the assumption, we have $\tau_a \circ \mu(x) = 1$ for some $x \in \mathbb{R}^n$.

Define a function Ψ from $\mathrm{C}_a(\mu)$ to ${L_2}^{\mathrm{C}_a(\mu)}$ by

$$\Psi(x) = \tau_a \circ \Phi(x).$$

Then, we have $\tau_a \circ \mathrm{G}(\Phi) = \mathrm{G}(\Psi)$. By Proposition 4.5, $\tau_a \circ \mathrm{G}(\Phi)$ is a crisp closed subset of $\mathbb{R}^n$. By the assumption, for every $x \in \mathrm{C}_a(\mu)$ there exists $y \in \mathrm{C}_a(\mu)$ with $\Psi(x)(y) = 1$.

Thus, by the classical version of Kakutani's fixed point theorem, there exists $x \in \mathrm{C}_a(\mu)$ with $\Psi(x)(x) = 1$. Therefore, there exists $x \in \mathrm{C}_a(\mu)$ with $\Phi(x)(x) \geq a$. □

Intuitively, we can consider $x \in \mathrm{C}_a(\mu)$ with $\Phi(x)(x) \geq a$ as an L-fuzzy fixed point of Φ relative to a.

6 Conclusions and Future Work

In this paper, we obtained the L-fuzzy versions of the five classical theorems in convex geometry, by using a-cut operators τ_a. By letting L be the two-element Boolean algebra, we can recover the classical version from the L-fuzzy version of each theorem.

It is expected that we can extend other theorems in convex geometry to the L-fuzzy case in similar ways. In particular, we conjecture that, as a generalization of Theorem 5.7, we can obtain an L-fuzzy version of Tverberg's theorem, which is a generalization of Radon's theorem, and that we can obtain a slightly stronger version of Theorem 5.6 by exploiting the idea of [9, Theorem 3.3].

The notion of L-fuzzy compactness in $\mathbb{R}^n$ employed in this paper (see Definition 4.6) become trivial in the $[0,1]$-fuzzy case as is explained in Remark 4.7 and so it may be better to employ another definition of L-fuzzy compactness in $\mathbb{R}^n$ and develop L-fuzzy versions of infinite Helly's theorem and Kakutani's fixed point theorem based on the new definition.

Our ultimate aim is to develop a mechanical procedure to obtain L-fuzzy versions of theorems in classical mathematics. This may be accomplished as follows: Fix an appropriate formal language of classical mathematics and that of L-fuzzy mathematics, and then construct a translation from the former to the latter so that the translation preserves the validity of any formula. This paper seems to provide some insights on how to construct such a translation.

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