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On k-vertex guarding simple polygons

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Abstract

The k-vertex guarding problem is to find a smallest set G of vertices in a simple polygon P such that every point in P is visible from at least k vertices of G. Recently, Salleh [8] proved an upper bound of $\lfloor 2n/3\rfloor$ for 2-vertex guarding a simple polygon and an upper bound of $\lfloor 3n/4\rfloor$ for 3-vertex guarding a convexly quadrilateralizable simple polygon.

In this paper we show that Fisk's coloring argument can be used to prove these bounds. The proofs lead to linear time guard placement algorithms. We also show that the problem of k-vertex guarding of a spiral polygon can be solved in linear time.

1 Introduction

Let P be a simple polygon. Two points p, q of P are *visible* if the line segment joining p to q does not intersect the exterior of P. Recently, Salleh [8] studied k-vertex guarding. A polygon P is called *k-vertex guardable* if there is a subset G of the vertices of P such that each point in P is visible from at least k vertices of G. Obviously, the 1-vertex guarding is the classical art gallery problem. Salleh proved an upper bound for the number of guards for 2-vertex guarding a simple polygon.

Proposition 1 (Salleh [8]) *For any n-gon P, $\lfloor 2n/3\rfloor$ vertex guards are sufficient and sometimes necessary to 2-vertex guard P. In particular, for every triangulation T for P, there exists a guard set G that 2-vertex guards P such that*

(i) *each side of P contains at least one guard, and*

(ii) *each ear e of T has guards at both of the vertices of e in $P \setminus e$.*

Salleh also proved an upper bound for the number of guards for 3-vertex guarding a convexly quadrilateralizable simple polygon.

Proposition 2 (Salleh [8]) *For any convexly quadrilateralizable n-gon P, $\lfloor 3n/4\rfloor$ vertex guards are sufficient and sometimes necessary to 3-vertex guard P. In particular, for any convex quadrilateralization Q for P, there exists a guard set that 3-vertex guards P such that*

(i) *each side of P contains at least one guard, and*

(ii) *each ear e of Q has guards at the vertices of e in $P \setminus e$ and one guard at any vertex of e not in $P \setminus e$.*

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Propositions 1 and 2 are proven using Chvátal's inductive argument. For 2-vertex guarding polygons, the proof uses Chvátal's Lemma [3] that any triangulation of a simple n-gon with at least 6 vertices contains a diagonal cutting off exactly 4,5, or 6 edges. For 3-vertex guarding polygons, Salleh [8] proved that, if a simple n-gon with at least 8 vertices admits a convex quadrilateralization Q, then Q contains a diagonal cutting off exactly 5 or 7 edges. Using these proofs Salleh provides algorithms for finding guards in $O(n^2)$ time.

In this paper we show that Fisk's coloring argument can be used to prove the bounds of Propositions 1 and 2. The proofs lead to linear time guard placement algorithms.

The *k-vertex guarding problem* is to find a smallest set G of vertices in a simple polygon P such that every point in P is visible from at least k vertices of G. Lee and Lin [6] proved that 1-vertex guarding problem is NP-hard. However, polynomial time algorithms exist for some classes of polygons. For example, Nilsson and Wood [7] designed a linear time algorithm for finding minimum number of guards in a spiral polygon. We design two algorithms solving k-vertex guarding problem for spiral polygons for $k = 1$ and $k = 2$. Both algorithms run in linear time.

A different kind of k-guardability has been previously studied by Belleville *et al.* [1]. A simple polygon is called *k-guardable* if there exists a set G of points that belong to the interior of edges of P such that no edge of P contains more than one element of G, and such that every point of P is visible from at least k elements of G. It has been shown [1] that (i) not all simple polygons are 3-guardable, and (ii) every simple polygon with n vertices is 2-guardable using at most $n - 1$ guards.

2 Proofs and Algorithms

Theorem 3 *For any simple polygon P with n vertices, a 2-vertex guard set of size at most $\lfloor 2n/3 \rfloor$ can be computed in $O(n)$ time.*

Necessity of $\lfloor 2n/3 \rfloor$ is shown in [8] using a hooked version of Chvátal's comb for $n = 3m + 2$. We show a slightly different comb for $n = 3m + 2$ in Fig. 1 (c). For completeness, we also show examples for $n = 3m$ and $n = 3m + 1$ in Figure 1.

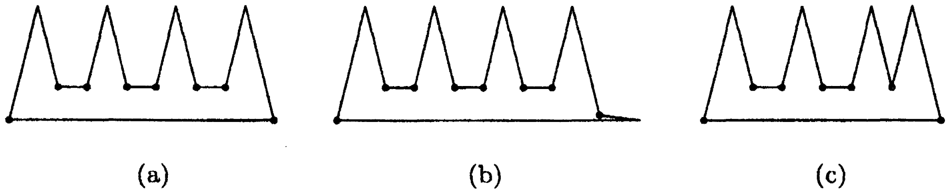

Figure 1: Lower bound for 2-vertex guarding. (a) $n = 3m, m = 4$. (b) $n = 3m + 1, m = 4$. (c) $n = 3m + 2, m = 3$.

Proof: As in Fisk's proof [4] take any triangulation and 3-color it. Let $V_i, i = 1, 2, 3$ be the set of vertices of ith color and let $n_i = |V_i|$. Without loss of generality $n_1 \leq n_2 \leq n_3$. We show that $G = V_1 \cup V_2$ is a 2-vertex guard set of size at most $\lfloor 2n/3 \rfloor$. Indeed, every point in P is visible from at least 2 vertices in G. The size of G can be bounded as follows. We have $n_3 \geq \lceil n/3 \rceil$. Then $n_1 + n_2 = n - n_3 \leq n - \lceil n/3 \rceil = \lfloor 2n/3 \rfloor$. The later equality can be verified by taking n modulo 3.

If $n = 3k$ then $n_3 = k$ and $n - k = \lfloor 2n/3 \rfloor$. If $n = 3k + 1$ then $n_3 = k + 1$ and $n - (k+1) = 2k = \lfloor 2n/3 \rfloor$. If $n = 3k + 2$ then $n_3 = k + 1$ and $n - (k + 1) = 2k + 1 = \lfloor 2n/3 \rfloor$.

Computation. A triangulation T of P can be computed in linear time [2]. The coloring of T can be done in linear time by constructing the dual graph and pruning its leaves. Therefore the total time is $O(n)$. ∎

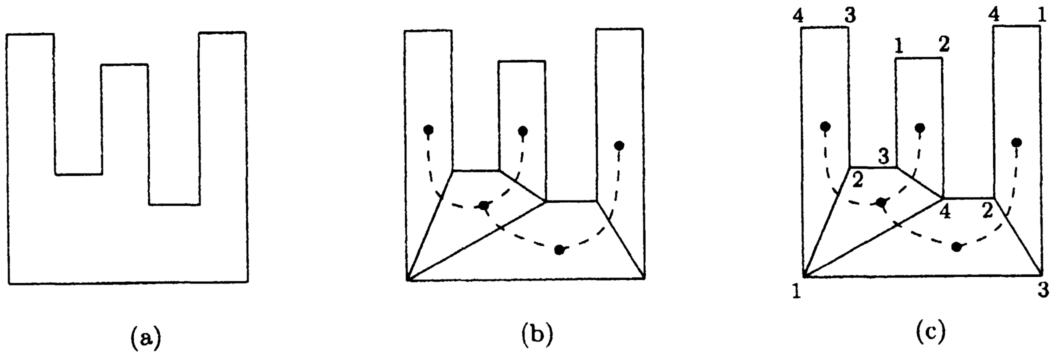

Figure 2: (a) An orthogonal polygon P. (b) A quadrangulation Q and its dual graph D. (c) 4-coloring.

For 3-vertex guarding polygons, we consider convexly quadrilateralizable simple polygons. Orthogonal polygons (simple polygons whose edges are horizontal and vertical) fall into this class. Kahn, Klawe, and Kleitman [5] proved that $\lfloor n/4 \rfloor$ guards suffice for 1-vertex guarding. They proved that any orthogonal polygon is convexly quadrilateralizable. Then a quadrangulation is colored into four colors such that the vertices of every quadrangle are distinct, see Fig. 2. The guards are placed at each vertex colored by the least frequently used color. We use the fact that any convexly quadrilateralizable simple polygon (not just an orthogonal polygon) can be 4-colored to prove the following theorem.

Theorem 4 *Let Q be a convex quadrangulation of a simple polygon P with n vertices. A 3-vertex guard set of size at most $\lfloor 3n/4 \rfloor$ can be computed in $O(n)$ time.*

Note that any convexly quadrilateralizable simple polygon has even number of vertices. Necessity of $\lfloor 3n/4 \rfloor$ is shown in [8] using the orthogonal version of Chvátal's comb for $n = 4m$. For completeness, we show an example for $n = 4m + 2$ in Figure 3.

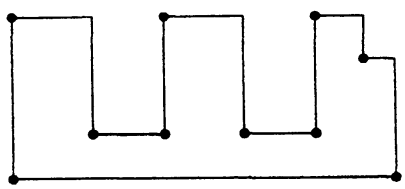

Figure 3: Lower bound for 3-vertex guarding a convexly quadrilateralizable polygon with $n = 4m+2$ vertices. This 14-gon requires 10 guards.

Proof: Color Q in four colors such that the vertices of every face are colored using all four colors as follows. Let D be the dual graph of Q. It has no cycles and is a tree (since it is connected). If $n = 4$ then the coloring is obvious. Suppose that $n > 4$. Remove the quadrangle q corresponding to a leaf. Then $P \setminus q$ is a simple polygon with $n - 2$ vertices and $Q \setminus q$ is its quadrangulation. q shares two vertices with $Q \setminus q$. Assuming that $Q \setminus q$ is 4-colored (the induction hypothesis), we color two remaining vertices of q.

Let $V_i, i = 1, 2, 3, 4$ be the set of vertices of ith color and let $n_i = |V_i|$. Without loss of generality $n_1 \leq n_2 \leq n_3 \leq n_4$. We show that $G = V_1 \cup V_2 \cup V_3$ is a 3-vertex guard set of size at most $\lfloor 3n/4 \rfloor$. Indeed, every point in P is visible from at least 3 vertices in G. The size of G can be bounded as follows. We have $n_4 \geq \lceil n/4 \rceil$. Then $n_1 + n_2 + n_3 = n - n_4 \leq n - \lceil n/4 \rceil = \lfloor 3n/4 \rfloor$. The later can be verified by taking n modulo 4.

If $n = 4k$ then $n_4 = k$ and $n - k = \lfloor 3n/4 \rfloor$. If $n = 4k + 1$ then $n_4 = k + 1$ and $n - (k + 1) = 3k = \lfloor 3n/4 \rfloor$. If $n = 4k + 2$ then $n_4 = k + 1$ and $n - (k + 1) = 3k + 1 = \lfloor 3n/4 \rfloor$. If $n = 4k + 3$ then $n_4 = k + 1$ and $n - (k + 1) = 3k + 2 = \lfloor 3n/4 \rfloor$.

The algorithm follows from the proof. A coloring of Q can be computed in linear time by constructing the dual graph and pruning its leaves. ∎

3 Spiral Polygons

Lee and Lin [6] proved that 1-vertex guarding problem is NP-hard. Nilsson and Wood [7] studied guarding of spiral polygons. A polygon is *spiral* if its convex vertices form a chain and its reflex vertices form a chain. Their approach is based on the following lemmas.

Lemma 5 (Nilsson and Wood [7]) *A collection of guards sees a spiral polygon if and only if they see all the edges of the reflex chain.*

Lemma 6 (Nilsson and Wood [7]) *Let m be the minimum number of guards guarding a spiral polygon. The polygon can be guarded by m guards placed on the convex chain of the polygon.*

Note that the guards in Lemma 6 can be placed anywhere in the polygon not just at vertices. We study the problem of guarding spiral polygons with vertex guards. Lemma 5 still holds for vertex guards. However, Lemma 6 cannot be used for vertex guards. Figure 4 shows an example where the minimum number of vertex guards is two but they must be selected from the reflex chain.

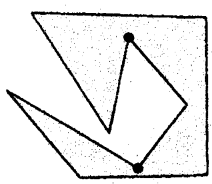

Figure 4: A spiral polygon such (i) the minimum number of guards is two and (ii) a unique set of two guards.

3.1 Vertex Guards in Spiral Polygons

Let P be a spiral polygon with reflex vertices $p_1, p_2, \ldots, p_k$ and convex vertices $q_1, q_2, \ldots, q_{n-k}$ such that q_1p_1 and p_kq_{n-k} are the edges of P. By Lemma 5 it suffices to guard only edges of the chain $q_1p_1p_2\ldots p_kq_{n-k}$. Consider the first segment q_1p_1 of the chain. Draw the ray q_1p_1 and let p_1' be the first crossing with the boundary of P, see Fig. 5. The polygon $q_1q_2q_3q_4p_1'$ can be guarded by one vertex guard. It can be placed at p_1 or q_4. The edge p_1p_2 is visible from both p_1 and q_4. Since p_2p_3 is visible from q_4 but not p_1 we place a guard at q_4. This can be turned into an algorithm.

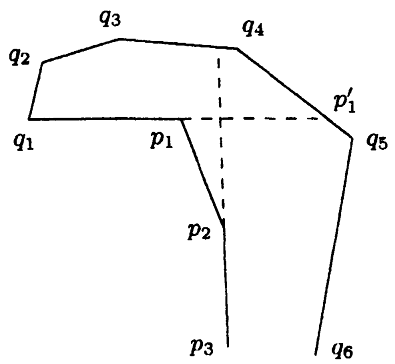

Figure 5: The best position for guarding segment q_1p_1 is q_4.

Theorem 7 *The minimum number of vertex guards in a spiral polygon with n vertices can be computed in $O(n)$ time.*

Proof: We show that the following algorihm VERTEXGUARDS finds the minimum number of guards. In the first **for** loop, we find all vertices of the convex chain that see $p_{i-1}p_i$. Then G is initialized as the empty set. The last guard added to G is denoted by g.

In the second **for** loop, we check weather $p_{i-1}p_i$ is guarded by g or not. If it is not guarded then a guard should be placed at one of the vertices $p_{i-1}, p_i, q_{a_i}, q_{a_i+1}, \ldots, q_{b_i}$. Since none of the vertices $p_{i+1}, p_{i+2}, \ldots, p_{n-k}$ is visible from p_{i-1} and p_{i+1} is visible from p_i, p_i has a preference over p_{i-1} for placing a guard. q_{b_i} has a preference over $p_{i-1}, p_i, q_{a_i}, q_{a_i+1}, \ldots, q_{b_i-1}$.

If $i \geq k$ then only one guard at p_i can be used to guard both $p_{i-1}p_i$ and p_ip_{i+1} (if $i = k$). Suppose that $i < k$. If $b_i < a_{i+2}$ then $p_{i+1}p_{i+2}$ is not visible from q_{b_i} and we place a guard at p_i. If $b_i \geq a_{i+2}$ then $p_{i+1}p_{i+2}$ is visible from q_{b_i} and we place a guard at q_{b_i}.

Running time. The first **for** loop can be implemented in $O(n)$ time since the sequences $a_1, a_2, \ldots, a_{k+1}$ and $b_1, b_2, \ldots, b_{k+1}$ are non-decreasing. Clearly, the second **for** loop takes linear time. ■

Figure 6 illustrates the guard placement by VERTEXGUARDS.

Algorithm 1: VERTEXGUARDS(P)

input : A spiral polygon P with reflex vertices $p_1, p_2, \ldots, p_k$ and convex vertices $q_1, q_2, \ldots, q_{n-k}$.
output: A set G of vertex guards.

```
p_0 ← q_1
p_{k+1} ← q_{n-k}
for i = 1 to k + 1 do
    Find a_i and b_i such that p_{i-1}p_i is visible from the convex vertices {q_{a_i}, q_{a_i+1}, ..., q_{b_i}}.
G ← ∅
g ← null
for i = 1 to k + 1 do
    if g ≠ null then
        if g = p_{i-1} then
            continue
        if g = q_j and j ≥ a_i then
            continue
    // p_{i-1}p_i is not guarded yet
    if i ≥ k or b_i < a_{i+2} then
        g = p_i
    else
        g = q_{b_i}
    G = G ∪ {g}
```

3.2 2-Vertex Guards in Spiral Polygons

Theorem 8 *The minimum number of 2-vertex guards in a spiral polygon with n vertices can be computed in $O(n)$ time.*

Proof: A set G of vertices in a spiral polygon P is 2-vertex guarding if every edge of the reflex chain is visible from at least two guards in G. The algorithm is similar to computing vertex guards by VERTEXGUARDS. First, compute a_i and b_i for $i = 1, 2, \ldots, k+1$. Then, for all $i = 1, 2, \ldots, k+1$, find two guards as follows.

If the segment $p_{i-1}p_i$ is already guarded by two vertices of G, then proceed with next i.

If the segment $p_{i-1}p_i$ is not guarded by any vertex of G, then find a guard as in VERTEXGUARDS and add it to G.

Suppose that the segment $p_{i-1}p_i$ is guarded by exactly one vertex g of G. If $g = p_i$, then add q_{b_i} to G. Suppose that $g = q_{b_i}$. Then $i < k$. If $b_i - 1 \geq a_{i+2}$ then add q_{b_i-1} to G; otherwise add p_i to G.

Similar to VERTEXGUARDS, this algorithm can be implemented in linear time. ∎

References

[1] P. Belleville, P. Bose, J. Czyzowicz, J. Urrutia, and J. Zaks. k-guarding polygons on the plane. In *Proc. 6th Canad. Conf. Comput. Geom.*, pp. 381–386, 1994.

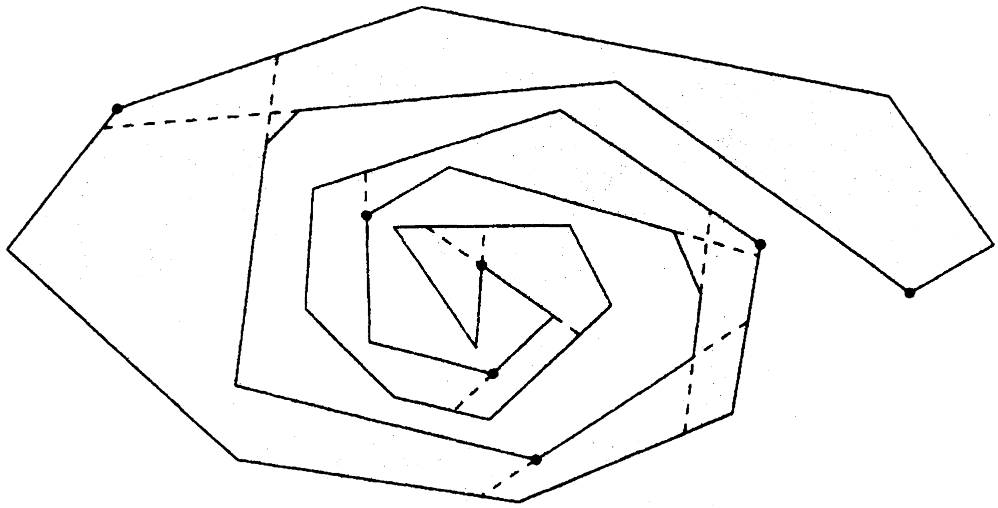

Figure 6: Vertex guards in a spiral polygon.

[2] B. Chazelle. Triangulating a simple polygon in linear time. *Discrete Comput. Geom.*, 6:485–524, 1991.

[3] V. Chvátal. A combinatorial theorem in plane geometry. *J. Combin. Theory Ser. B*, 18:39–41, 1975.

[4] S. Fisk. A short proof of Chvátal's watchman theorem. *J. Combin. Theory Ser. B*, 24:374, 1978.

[5] J. Kahn, M. M. Klawe, and D. Kleitman. Traditional galleries require fewer watchmen. *SIAM J. Algebraic Discrete Methods*, 4(2):194–206, 1983.

[6] D. Lee and A. Lin. Computational complexity of art gallery problems. *IEEE Trans. Inform. Theory*, 32(2):276–282, 1986.

[7] B. J. Nilsson and D. Wood. Optimum watchmen routes in spiral polygons. Technical Report LU-CS-TR:90-55, Dept. of Computer Science, 1990, `http://citeseer.ist.psu.edu/158091.html`. An extended abstract of a preliminary version in Proc. of 2nd Canad. Conf. Comput. Geom., 1990, 269–272.

[8] I. Salleh. k-vertex guarding simple polygons. *Comput. Geom. Theory Appl.*, 2008, `http://dx.doi.org/10.1016/j.comgeo.2008.07.004`. to appear.

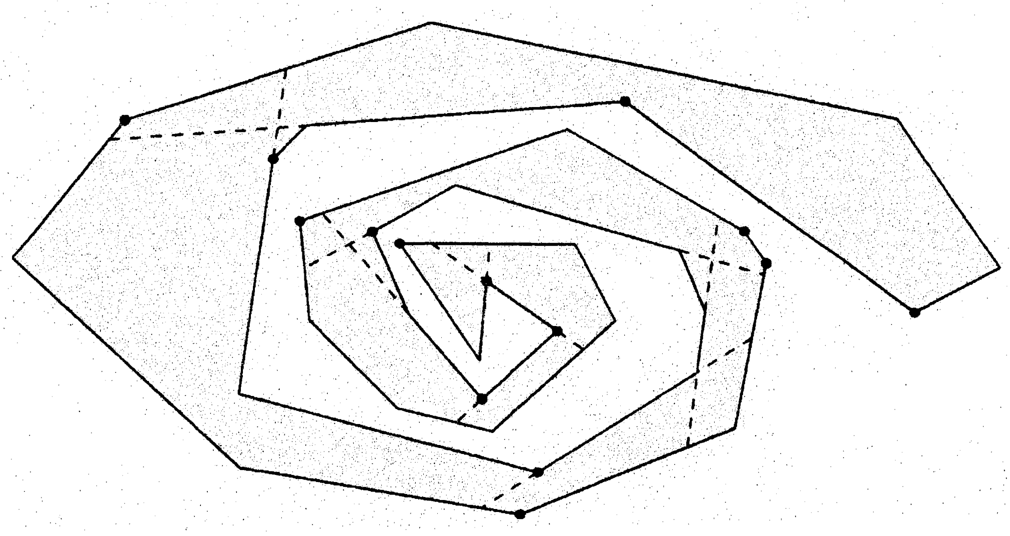

Figure 7: 2-Vertex guards in a spiral polygon.