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An Extension of the Boolean Global Convergence

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Abstract. In 999, there is a global convergent theorem for boolean network that have been proved. Next, the global convergent theorem for XOR boolean network have been proved in 2007, it is a counterpart of the global convergent theorem for boolean network. This result, we extended the global convergent behaviours to any map from the product of n finite Boolean algebra into itself, where n is a positive integer.

Key words: Global convergent theorem, Boolean network, XOR boolean network, Finite boolean algebra.

1. Introduction

Let $\{0,1\}$ be with operations $+$, $\oplus$, and $\cdot$ defined as follows,

$$\begin{aligned}
&0+0=0\oplus 0=1\oplus 1=1\cdot 0=0\cdot 1=0\cdot 0=0,\\
&1+1=1+0=0+1=1\oplus 0=0\oplus 1=1\cdot 1=1,\\
&\bar{0}=1, \text{ and } \bar{1}=0.
\end{aligned}$$

For $a,b\in\{0,1\}$, ab is the abbreviation of $a\cdot b$. For each positive integer n, let $\{0,1\}^n$ be the set of ordered n-tuples,

$$x=\begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix},$$

with components $x_i\in\{0,1\}\,(i=1,\ldots,n)$. We may think of x as a *bit string* of length n, thus we may write $x=x_1x_2\cdots x_n$. We also write $x=(x_1,x_2,\cdots,x_n)$. The zero element of $\{0,1\}^n$ is the n-tuple $\mathbf{0}=(0,0,\cdots,0)$. For $j\in\{1,\ldots,n\}$, the j-th unit vector e_j is the element of $\{0,1\}^n$, all of whose coordinates are 0 except for the j-th component is 1. The order "$\leq$" in $\{0,1\}$ is given by $0\leq 0\leq 1\leq 1$. For $x,y\in\{0,1\}^n$, $x\leq y$ is meant that.$x_i\leq y_i\,(i=1,\ldots,n)$. For $x,y\in\{0,1\}^n$, $\lambda,\gamma\in\{0,1\}$, define

$$x+y=\begin{pmatrix}\max\{x_1,y_1\}\\ \vdots\\ \max\{x_n,y_n\}\end{pmatrix},\ \lambda x=\begin{pmatrix}\max\{\lambda,x_1\}\\ \vdots\\ \max\{\lambda,x_n\}\end{pmatrix},\ \text{and}\ x\oplus y=\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_n\end{pmatrix},$$

where $\gamma_i = 0$ if $x_1 = y_1$; otherwise, $\gamma_i = 1\,(i = 1, \ldots, n)$. Hence

$$x+y=\begin{pmatrix}x_1+y_1\\ \vdots\\ x_n+y_n\end{pmatrix},\ cx=\begin{pmatrix}c+x_1\\ \vdots\\ c+x_n\end{pmatrix},\ \text{and}\ x\oplus y=\begin{pmatrix}x_1\oplus y_1\\ \vdots\\ x_n\oplus y_n\end{pmatrix}.$$

Boolean network of n elements is a mapping $F : \{0,1\}^n \to \{0,1\}^n$. XOR boolean network is a boolean network that replace the operation $+$ with $\oplus$. Throughout this paper, a *boolean matrix* is meant to be a matrix over $\{0,1\}$. The set of $n \times n$ boolean matrix is denoted by Ω_n. The symbol I stands for the identity matrix in Ω_n. Let α be a nonempty subset of $\{1, 2, \cdots, n\}$. For any $M \in \Omega_n$, $M(\alpha)$ stands for the principal submatrix of M that lies in rows and columns indexed by α. Boolean matrix multiplication is the same as in the case of complex matrices but the concerned products of entries are boolean. For a boolean network, boolean matrix addition is the operation $+$, it is the same as in the case of complex matrices but the concerned sums of entries are boolean. For an XOR boolean network, boolean matrix addition is the operation $\oplus$ instead of the operation $+$.

A non-zero element $u \in \{0,1\}^n$ is called a *(boolean) eigenvector* of $M \in \Omega_n$ if there exists an λ in $\{0,1\}$ such that $Mu = \lambda u$; λ is called the *(boolean) eigenvalue* associated with eigenvector. For $M \in \Omega_n$, the symbol $\sigma(M)$ denote the *(boolean) spectrum* of M, it is the set of all eigenvalues of M, so that $\sigma(M) \subset \{0,1\}$. The *(boolean) spectral radius* of M, which is denoted by $\rho(M)$, is defined to be the largest eigenvalue of M.

For an element x of $\{0,1\}^n$, the *von Neumann neighborhood* of x is the set $V_x = \{x, \tilde{x}^1, \cdots, \tilde{x}^n\}$. Here $\tilde{x}^j\,(i = 1, \ldots, n)$ is the *j-th neighbor* of x, which is defined to be the element $(x_1, \cdots, \bar{x}_j, \cdots, x_n)$. According to Robert(see[6, p.97]), the *boolean Jacobian matrix* of the map F from $\{0,1\}^n$ to itself evaluated at x is defined by $F'(x) = (f_{ij}(x))$, where

$$f_{ij}(x) = \begin{cases} 1 & \text{if } f_i(x) \neq f_i(\tilde{x}^j), \\ 0 & \text{otherwise.} \end{cases}$$

Robert usually called the boolean Jacobian matrix of a map as the *boolean* *derivative* of a map. The incidence matrix of F is the $n \times n$ Boolean matrix $B(F) = (b_{ij})$ where $b_{ij} = 0$ if f_i does not depend on x_j; otherwise $b_{ij} = 1$.

Let us remark that the boolean network $F : \{0,1\}^n \to \{0,1\}^n$ is global convergent to a fixed point ξ if ξ is a global attractor for the boolean network, that is, the trajectory $x^{t+1} = F(x^t)$ tends forward to ξ for any starting at x^0 of $\{0,1\}^n$; that is, there exists a positive integer $p(\leq 2^n)$ such that $F^p(x^0) = x^p = \xi$ for any starting $x^0 \in \{0,1\}^n$. The material of following notations can be found in the fundamental paper by Robert[1], [2] and [3], and also in the book by Robert[4], [5] and [6].

2. Boolean Global Convergent Theorems

In 1999, Shih and Ho proved a global convergent theorem for boolean network[7]:

Theorem 1. Suppose the map F from $\{0,1\}^n$ to itself defines a boolean network satisfies following conditions:

(1) $F(V_x) \subset V_{F(x)}$ for all $x \in \{0,1\}^n$;
(2) $\rho(F'(x)) = 0$ for all $x \in \{0,1\}^n$.

Then F has a unique fixed point and the boolean network is global convergent to this fixed point.

In 2007, Ho proved a global convergent theorem for boolean network[8] that is equipped with a XOR boolean structure:

Theorem 2. Suppose the map F from $\{0,1\}^n$ to itself defines a XOR boolean network satisfies following conditions:

(1) $F(V_x) \subset V_{F(x)}$ for all $x \in \{0,1\}^n$;
(2) $1 \notin \sigma(F'(x))$ for all $x \in \{0,1\}^n$.

Then F has a unique fixed point and the boolean network is global convergent to this fixed point.

In this paper, this result is extended to any map F from A^n into itself, where A is a finite Boolean algebra[9] and n is a positive integer.

Define $a \in A$ to be an atom of A if $0 < a$ but there is no x in A satisfying $0 < x < a$. We denoted by $At(A)$ the set of atoms of A. We say A is atomic if for each positive element x of A , there is some atom a such that $a \leq x$. We say A is complete if the least upper bound and the greatest lower bound of D belong to A for each $D \subseteq A$. Write the cardinality of $At(A)$ by $\#At(A)$.

Remark that for every Boolean algebra A, the map f from A into the power set algebra $P(At(A))$ defined by

$$f(x) = \{a \in At(A) : a \leq x\}$$

is a homomorphism. It is an embedding if A is atomic, and f is an epimorphism if A is complete.(see [9, Proposition 2.6])

Let the map F from A^n into itself, where A is a finite Boolean algebra and n is a positive integer. Let m be the cardinality of the set of atoms of A. We will construct a map $\hat{F}$ from $\{0,1\}^{mn}$ into itself and conclude that if this map $\hat{F}$ satisfies conditions(1)and(2) then F has a unique fixed point ξ and there exists a positive integer $p(\leq 2^n)$ such that $F^p(x) = \xi$ for any element x in A^n.

Lemma 1. Let A be a finite Boolean algebra and let

$$F : A^n \longrightarrow A^n \ (n \text{ is a positive integer})$$

be a map. Then there is a map

$$\hat{F} : \{0,1\}^{mn} \longrightarrow \{0,1\}^{mn} \ (m = \#At(A))$$

and two isomorphisms η and φ such that

$$F = (\eta\varphi)^{-1} \hat{F}\eta\varphi$$

Proof. Define the map from A into the power set algebra $P(At(A))$ by

$$\varphi^*(x) = \{a \in At(A) : a \leq x\}$$

Since A is a finite Boolean algebra, $At(A)$ is finite and A is both complete and atomic. Hence φ^* is an isomorphism.(see [9, Corollary 2.7])

Define the map from A^n into $[P(At(A))]^n$ by

$$\varphi(x) = \varphi(x_1, ..., x_n)$$

$$= (\varphi^* (x_1), ..., \varphi^* (x_n))$$

Then φ is also an isomorphism.

Since A is finite ,we may assume

$$At(A) = \{a_1, ..., a_m\}$$

and define $\eta^* : P(At(A)) \to \{0,1\}^m$ by

$$\eta^*(D) = \begin{cases} \mathbf{0} & \text{if } D = \phi, \\ e_j & \text{if } D = \{a_j\}, \\ \sum_{a_j \in D} e_j & \text{otherwise.} \end{cases}$$

Obviously, η^* is a bijection. For any D_1, $D_2 \in P(At(A))$

$$\eta^* (D_1 \cup D_2) = \sum_{a_j \in D_1 \cup D_2} e_j = \sum_{a_j \in D_1} e_j + \sum_{a_j \in D_2} e_j \qquad \text{(Boolean sum)}$$

$$= \eta^* (D_1) + \eta^* (D_2)$$

$$\eta^* (D_1 \cap D_2) = \sum_{a_j \in D_1 \cap D_2} e_j = \sum_{a_j \in D_1} e_j \cdot \sum_{a_j \in D_2} e_j$$

$$= \eta^* (D_1) \cdot \eta^* (D_2)$$

$$\eta^* (\phi) = (0, ..., 0)$$

$$\eta^* (At(A)) = (1, ..., 1)$$

$$\eta^*(D^c) = \sum_{a_j \in D^c} e_j = \overline{\sum_{a_j \in D} e_j} = \overline{\eta^* (D)}.$$

Hence η^* is a homorphism(see [9, p.8]), and so η^* is an isomorphism.

Define the map from $[P(At(A))]^n$ into $\{0,1\}^{mn}$ by

$$\begin{aligned} \eta(D) &= \eta (D_1, ..., D_n) \\ &= (\eta^* (D_1), ..., \eta^* (D_n)) \end{aligned}$$

Then η is also an isomorphism.

Define $\tilde{f}_j : \{0,1\}^{mn} \to \{0,1\}^m$ by

$$\tilde{f}_j = \eta^* \varphi^* f_j \, (\eta\varphi)^{-1},$$

where f_j are the components of F. i.e.,

$$\begin{array}{ccccc} A^n & \xrightarrow{\varphi} & [P(At(A))]^n & \xrightarrow{\eta} & \{0,1\}^{mn} \\ f_j \downarrow & & & & \downarrow \tilde{f}_j \\ A & \xrightarrow{\varphi^*} & P(At(A)) & \xrightarrow{\eta^*} & \{0,1\}^{m} \end{array}$$

For $x = (x_1, ..., x_m)$ in $\{0,1\}^m$., π_j is defined by

$$\pi_j \, (x) = x_j \; (j = 1, ..., m)$$

Now we define the components $\hat{f}_j$ of $\hat{F}$ by

$$\hat{f}_j = \begin{cases} \pi_\beta \tilde{f}_{\alpha+1} \text{ if } \; j = \alpha m + \beta, 0 < \beta < m, \\ \pi_m \tilde{f}_\alpha \quad \text{ if } \; j = \alpha m, \qquad j = 1, ...mn. \end{cases}$$

Then $\hat{F}$ is a map from $\{0,1\}^{mn}$ into $\{0,1\}^{mn}$.

For any x in $\{0,1\}^{mn}$,

$$\hat{F}\,(x) = \begin{pmatrix} \hat{f}_1 \\ \vdots \\ \hat{f}_{mn} \end{pmatrix} (x) = \begin{pmatrix} [\hat{f}_1(x), \cdots, \hat{f}_m(x)]^T \\ \vdots \\ [\hat{f}_{m(n-1)+1}(x), \cdots, \hat{f}_{mn}(x)]^T \end{pmatrix}$$

$$= \begin{pmatrix} [\pi_1 \tilde{f}_1\,(x)\,, \cdots, \pi_m \tilde{f}_1\,(x)]^T \\ \vdots \\ [\pi_1 \tilde{f}_n\,(x)\,, \cdots, \pi_m \tilde{f}_n\,(x)]^T \end{pmatrix} = \begin{pmatrix} \left[\eta^* \varphi^* f_1 \, (\eta\varphi)^{-1}\,(x)\right]^T \\ \vdots \\ \left[\eta^* \varphi^* f_n \, (\eta\varphi)^{-1}\,(x)\right]^T \end{pmatrix}$$

$$= \eta\varphi \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (\eta\varphi)^{-1}\,(x) = \eta\varphi F\,(\eta\varphi)^{-1}\,(x)\,.$$

Therefore, $F = (\eta\varphi)^{-1}\,\hat{F}\eta\varphi$. $\square$

We call $\hat{F}$ is the *$\eta\varphi$-mapping* of F. The aim of this paper is to prove the following theorem.

Theorem 3. Let A be a finite Boolean algebra and let

$$F : A^n \longrightarrow A^n \ (n \text{ is a positive integer})$$

be a map such that its $\eta\varphi$-mapping $\hat{F}$ satisfies following conditions:

(1) $\hat{F}(V_x) \subset V_{\hat{F}(x)}$ for all $x \in \{0,1\}^n$;
(2) $\rho(\hat{F}'(x)) = 0$ for all $x \in \{0,1\}^n$.

Then F has a unique fixed point ξ and there exists a positive integer $p(\leq 2^n)$ such that $F^p(x) = \xi$ for any element x in A^n.

3. Proof of Theorem 3

Let A be a finite Boolean algebra and let $F : A^n \longrightarrow A^n$. Lemma 1 and the hypothesis of Theorem 3 shows that its $\eta\varphi$-mapping $\hat{F} : \{0,1\}^{mn} \longrightarrow \{0,1\}^{mn}$ $(m = \#At(A))$ is actually a boolean network that satisfies conditions(1)and(2). By Theorem 1, there is a unique fixed point c of $\hat{F}$ such that this boolean network $\hat{F} : \{0,1\}^{mn} \longrightarrow \{0,1\}^{mn}$ is global convergent to c.

Let $\xi = (\eta\varphi)^{-1}(c)$. Then $\xi \in A^n$ and we have

$$\begin{aligned}
&\hat{F}(c) = c \\
&\Longrightarrow \eta\varphi F(\eta\varphi)^{-1}(c) = c \\
&\Longrightarrow F(\eta\varphi)^{-1}(c) = (\eta\varphi)^{-1}(c) \\
&\Leftrightarrow F(\xi) = \xi.
\end{aligned}$$

Since $(\eta\varphi)^{-1}$ is an isomorphism, ξ is a unique fixed point of F. Since this boolean network $\hat{F} : \{0,1\}^{mn} \longrightarrow \{0,1\}^{mn}$ is global convergent to c, there is a positive integer $p(\leq 2^n)$ such that $\hat{F}^p(x) = c$ for any $x \in \{0,1\}^{mn}$. For any y in A^n, there exist an element x in $\{0,1\}^{mn}$ such that $y = (\eta\varphi)^{-1}(x)$. Then

$$\begin{aligned}
&\hat{F}^p(x) = c \\
&\Rightarrow \eta\varphi F^p(\eta\varphi)^{-1}(x) = c \\
&\Rightarrow F^p(\eta\varphi)^{-1}(x) = (\eta\varphi)^{-1}(c) \\
&\Rightarrow F^p(y) = \xi
\end{aligned}$$

Hence p is also the positive integer such that $F^p(y) = \xi$ for any $y \in A^n$. This completes the proof of Theorem 3.

4. Remarks

The incidence matrix of F is the $n \times n$ Boolean matrix $B(F) = (b_{ij})$ where $b_{ij} = 0$ if f_i does not depend on x_j; otherwise $b_{ij} = 1$. f_i are the components of F. [6] Now we compare $B(F)$ with $B(\hat{F})$.

Remark 1. Let $A, F, \hat{F}$ be described above. Let $B(F) = (b_{ij})_{n\times n}$ and $B(\hat{F}) = (B_{ij})_{n\times n}$ where B_{ij} is an $m \times m$ Boolean matrix with $m = \#At(A)$. Then $b_{ij} = 0$ if and only if $B_{ij} = O_{n\times n} = O$,the $m \times m$ zero matrix.
Proof. $b_{ij} = 0$
$\Leftrightarrow f_i$ does not depend on x_j for $x = (x_1, ...x_n) \in A^n$
$\Leftrightarrow \eta^*\varphi^* f_i$ does not depend on x_j
$\Leftrightarrow \tilde{f}_i\eta\varphi$ does not depend on x_j
$\Leftrightarrow \tilde{f}_i$ does not depend on $a_{(j-1)m+1}, ...a_{jm}$ for $a = (a_1, ...a_{mn}) \in \{0,1\}^{mn}$.
$\Leftrightarrow \pi_1\tilde{f}_i, ..., \pi_m\tilde{f}_i$ does not depend on $a_{(j-1)m+1}, ..., a_{jm}$
$\Leftrightarrow \hat{f}_{(i-1)m+1,...,}\hat{f}_{im}$ does not depend on $a_{(j-1)m+1}, ..., a_{jm}$
$\Leftrightarrow b_{(i-1)m+s,(j-1)m+t} = 0$ ($s = 1, ..., m$; $t = 1, ..., m$; $B(\hat{F}) = (\hat{b}_{ij})_{mn\times mn}$)
$\Leftrightarrow B_{ij} = O$. $\square$

If $B_{ij} \neq O$ then $b_{ij} = 1$; but $b_{ij} = 1$ can not imply $B_{ij} = I$,the $m \times m$ identity matrix. We will show it with the following counterexample. So we can not apply this condition to Theorem 3.

Example 1. If $A = \{a, b, c, d\}$ with $a < b < d$ and $a < c < d$, then it is a finite Boolean algebra and $m = \#At(A) = \#\{b, c\} = 2$.
Consider $n = 2$. Let the map $F : A^2 \to A^2$ be defined by

y	$(a,*)$	$(b,*)$	$(c,*)$	$(d,*)$
$F(y)$	(a,a)	(a,c)	(a,b)	(a,d)

where $*$ is any element in A , we have

$$B(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

From

$$\{a, b, c, d\} \xrightarrow{\varphi^*} \{\phi, \{b\}, \{c\}, \{b, c\}\} \xrightarrow{\eta^*} (0,0), (0,1), (1,0), (1,1)$$

we have, for any $x = (x_1, x_2, x_3, x_4) \in \{0,1\}^{mn} = \{0,1\}^4$,

$$\hat{F}(x) = \eta\varphi F (\eta\varphi)^{-1} (x) = (0, 0, x_2, x_1)$$

Hence

$$B(\hat{F}) = \begin{pmatrix} 0\,0 & 0\,0 \\ 0\,0 & 0\,0 \\ 0\,1 & 0\,0 \\ 1\,0 & 0\,0 \end{pmatrix}$$

Though $B_{11} = B_{12} = B_{22} = O, B_{21} \neq I$. The claim follows. □

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