数理解析研究所講究録
第 1646 巻 2009 年 1-3

BOREL DEFINABLE SUBGROUPS AND ALMOST INTERNALITY IN ROSY DEPENDENT GROUPS

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ABSTRACT. Let G be a rosy dependent group with a countable language, weak canonical bases and elimination of hyperimaginaries. Let p be a global $\mathfrak{p}$-generic type and Σ be an $\emptyset$-invariant family of partial types. Under a certain assumption on independence relations of global types we show that if $p|\emptyset$ is not foreign to Σ, then there exists a Borel definable subgroup H such that G/H is almost Σ-internal.

1. PRELIMINARIES

We review some definitions and facts. We only consider theories having elimination of hyperimaginaries and we will work in the eq-structures.
ROSY THEORIES (See [A]): Let T be rosy and $\mathcal{M}$ be a big model. We work in $\mathcal{M}^{\mathrm{eq}}$. We say that T has weak canonical bases if for any type p there exists the smallest algebraically closed subset $\mathrm{wcb}_{\mathfrak{p}}(p)$ such that p does not $\mathfrak{p}$-forks over $\mathrm{wcb}(p)$. In [A] Adler showed that if T has weak canonical bases then $\mathrm{wcb}_{\mathfrak{p}}(\bar{a}/B) = \mathrm{aker}((a_i)_{i<\omega}) := \{d \in \mathrm{acl}^{\mathrm{eq}}((a_i)_{i<\omega}) : (a_i)_{i<\omega} \text{ is } d\text{-indiscernible } \}$, where $B = \mathrm{acl}^{\mathrm{eq}}(B)$ and $(a_i)_{i<\omega}$ is a Morley sequence of $\mathrm{tp}(\bar{a}/B)$.
DEPENDENT THEORIES (See [HP]): Let T be dependent. In [HP], it is shown that a global type $p \in S(\mathcal{M})$ does not fork over an algebraically closed set $A \subseteq \mathcal{M}$ if and only if p is $\mathrm{acl}^{\mathrm{eq}}(A)$-invariant if and only if p is strongly Borel definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. (We say that p is strongly Borel definable over A if for any formula $\varphi(x,y)$, there exists a finite Boolean combination of partial types over A, say $D_{p,\varphi}(y)$, such that $\varphi(x,b) \in p$ if and only if $\models D_{p,\varphi}(b)$ for any b. It is also mentioned that a global type p is $A(\subset \mathcal{M})$-invariant, then any Morley sequence of p over A has the same type over A.
ROSY GROUPS (See [EKP]): Let $(G(x),\cdot)$ be a rosy group over $\emptyset$ and let $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}$ be thorn-non-forking relation. There exists $\mathfrak{p}$-generic type $p \in \mathrm{S(A)}$ for G over A: $p(x) \vdash G(x)$ and if $a, b \in G$ with $a \models p$ and $a \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_A b$, then $b \cdot a \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_{\emptyset} A, b$ holds. And then $\mathrm{tp}(b \cdot a/A)$ is also $\mathfrak{p}$-generic.
FOREIGNNESS and ALMOST INTERNALITY in ROSY THEORIES: Let

Date: August 29, 2008.
1991 *Mathematics Subject Classification.* 03C45.
Key words and phrases. Rosy dependent groups, internality.

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Σ be A-invariant family of partial types. $\bar{b} \models \Sigma|B$ denotes that $\mathrm{tp}(\bar{b}/B) \supseteq \Sigma_0$ for some $\Sigma_0 \subseteq \Sigma$. We say that $p \in \mathrm{S(A)}$ is foreign to Σ if for any $B(\supseteq A), \bar{a}, \bar{b}$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_A B$ and $\bar{b} \models \Sigma|B$ we have $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_B \bar{b}$. We say that $p \in \mathrm{S(A)}$ is Σ-internal if for any $\bar{a} \in p$ there exist $B \supseteq A$ and $\bar{b} \models \Sigma|B$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_A B$ and $\bar{a} \in \mathrm{acl^{eq}}(B, \bar{b})$.

2. The proposition

Definition 2.1. Let G be a rosy group and let H be a Borel definable subgroup over B. We say that G/H is almost Σ-internal if for each $gH \in G/H$ there exist C and $D \models \Sigma|B$ such that $g \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_B C$ and gH is $\mathrm{acl^{eq}}(B, C, D)$-invariant.

Here, gH is an ultraimaginary element, we do not consider thorn-independence relation on ultraimaginaries, so we do not define $gH \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_B C$.

Proposition 2.2. *Let G be a rosy dependent group having weak canonical bases and a countable langage. Suppose that G is suficiently saturated and any global type (i.e. over G) does not $\mathfrak{p}$-fork over A if and only if it does not fork over A. Let p be a global $\mathfrak{p}$-generic type of G. If $p|\emptyset$ is not foreign to $\emptyset$-invariant family of partial types Σ, then there exists a Borel definable subgroup H such that $|G/H| \geq \omega$ and G/H is almost Σ-internal.*

Proof. Note that p is $\mathrm{acl^{eq}}(\emptyset)$-invariance: By the definition of $\mathfrak{p}$-generics, p does not $\mathfrak{p}$-fork over $\emptyset$. By our assumption p does not fork over $\emptyset$, so we see $\mathrm{acl^{eq}}(\emptyset)$-invariance of p by dependence of G.
As $p|\emptyset$ is not foreign to Σ, there exist $A \subset G, a \models p|A$ and $\bar{b} \models \Sigma$ such that $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_A \bar{b}$. We may assume $a, \bar{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_A G$. By our assumption we have $a, \bar{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_A G$, where $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ denotes non-forking relation. Let $B := \mathrm{wcb}_{\mathfrak{p}}(a, \bar{b}/G) (= \mathrm{wcb}_{\mathrm{f}}(a, \bar{b}/G))$.

Claim. *Put $H = \{g \in G : g \cdot a, \bar{b} \equiv_B a, \bar{b}\}$ is a Borel definable (over B) subgroup of G.*

$H \leq G$: Let $g \in K$. As $g^{-1} \in G$, it holds that $a, \bar{b} = g^{-1} \cdot g \cdot a, \bar{b} \equiv_G g^{-1} \cdot a, \bar{b}$, we have $g^{-1} \in H$. Let $g, g' \in H$. As $g' \in G$, we have $g' \cdot g \cdot a, \bar{b}, \equiv_G g' \cdot a, \bar{b} \equiv_G a, \bar{b}$, $g' \cdot g \in H$ follows.
The Borel definability of H over B: Let $\varphi(u \cdot x, \bar{y}, \bar{z}) \in L$ and put $q = \mathrm{tp}(a, \bar{b}/G)$. There exists a $D_{q,\varphi}(u, \bar{z})$ which defines a strongly Borel definable set over B such that $\varphi(d \cdot x, \bar{y}, \bar{c}) \in q$ if and if only $\models D_{q,\varphi}(d, \bar{c})$ for any $d \in G, \bar{c} \subset_\omega G$. As $H = \bigcap_{\varphi \in L}\{g \in G : D_{q,\varphi}(d, \bar{c}) \leftrightarrow D_{q,\varphi}(d \cdot g^{-1}, \bar{c})$ for any $d \in G, \bar{c} \subset_\omega G\}$, the Borel definability of H over B follows.

Claim. $|G/H| \geq \omega$

As p is $\mathrm{acl^{eq}}(\emptyset)$-invariant, take $(g_i)_{i<\omega} \subset G$ be the Morley sequence of $p|B$. As $a, \bar{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_B (g_i)_{i<\omega}$, $g := g_j^{-1} \cdot g_i$ is $\mathfrak{p}$-generic over $B, a, \bar{b}$. So we have $g \cdot a \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_B \bar{b}$.

As $a \not\downarrow^{\mathfrak{p}}_B \bar{b}$, we see that $g \notin H$ as desired.

Fix $g \in G$ and take a Morley sequence $(a_i)_{i<\omega} \subset G$ of $p|B, g$. Note that $g \downarrow^{\mathfrak{p}}_B (a_i)_{i<\omega}$.

Again take a Morley sequence $(g \cdot a'_i, \bar{b}_i)_{i<\omega}$ of $\mathrm{tp}(a, \bar{b}/G)|B, g, (a_i)_{i<\omega}$. ($\mathrm{tp}(a, \bar{b}/G)$ is B-invariant.) As $(a_i)_{i<\omega}$ and $(g \cdot a'_i)_{i<\omega}$ are Morley sequences of $p|B, g$, we have $(a_i)_{i<\omega} \equiv_{B,g} (g \cdot a'_i)_{i<\omega}$. Let $(b'_i)_{i<\omega} \subset G$ be such that $(a_i, b'_i)_{i<\omega}$ is a Morley sequence of $\mathrm{tp}(a, \bar{b}/G)|B, h$. So $\mathrm{wcb}_{\mathfrak{f}}(a, \bar{b}/G) = \mathrm{wcb}_{\mathfrak{p}}(a, \bar{b}/G) \subseteq \mathrm{acl}^{\mathrm{eq}}((a_i, b'_i)_{i<\omega})$. Therefore $\mathrm{tp}(a, \bar{b}/G)$ is $\mathrm{acl}^{\mathrm{eq}}((a_i, b'_i)_{i<\omega})$-invariant and $b'_i \models \Sigma|B$.

Claim. *Let $\sigma \in \mathrm{Aut}(G/\mathrm{acl}^{\mathrm{eq}}((a_i, b'_i)_{i<\omega}, B))$. Then $\sigma(gH) = gH$.*

As $\mathrm{tp}(a, \bar{b}/G)$ is $\mathrm{acl}^{\mathrm{eq}}((a_i, b'_i)_{i<\omega})$-invariant, we have $g \cdot a, \bar{b} \equiv_G \sigma(g) \cdot a', \bar{b}'$. As $\sigma(g) \in G$ and $\mathrm{tp}(a, \bar{b}/G)$ is B-invariant, we see $g \cdot a, \bar{b} \equiv_G \sigma(g) \cdot a', \bar{b}' \equiv_G \sigma(g) \cdot a, \bar{b}$, as desired.

We get $g \downarrow^{\mathfrak{p}}_B (a_i)_{i<\omega}$, $b'_i \models \Sigma|B$ and gH is $\mathrm{acl}^{\mathrm{eq}}((a_i, b'_i)_{i<\omega}, B)$-invariant. □

Question 2.3. (1) *Let $\mathcal{M}$ be a sufficiently saturated rosy model and any global type (i.e. over $\mathcal{M}$) does not $\mathfrak{p}$-fork over A if and only if it does not fork over A. Then is $\mathcal{M}$ simple?*

(2) *Can we find a Borel definable NORMAL subgroup H as in the Proposition?*

(3) *Can we find a Σ-connected component G^{Σ} in a Borel definable way? (G^{Σ} want to be foreign to Σ, connected and invariant under any definable automorphism as in* [W].*)*

(4) *Does any superrosy field has monomial $U^{\mathfrak{p}}$-rank? If so, Nubling's proof* [N] *that any supersimple field is n-ample for any $n < \omega$ works for any superrosy field.*

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