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INTRODUCTION TO CLUSTER TILTING IN 2-CALABI-YAU CATEGORIES

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ABSTRACT. Cluster tilting theory reveals combinatorial structure of 2-Calabi-Yau triangulated categories and is applied to categorify Fomin-Zelevinsky cluster algebras by many authors (Buan, Marsh, Reineke, Reiten Todorov, Caldero, Chapoton, Schiffler, Keller,...). In the first section, we will introduce cluster tilting theory in 2-Calabi-Yau triangulated category. In particular, a combinatorial description of change of endomorphism algebras of cluster tilting objects via mutation process is given in terms of Fomin-Zelevinsky quiver mutation rule. In the second section, a class of examples of 2-Calabi-Yau triangulated categories containing cluster tilting objects will be constructed from preprojective algebras and elements in the corresponding Coxeter groups.

In recent years, cluster tilting theory becomes a major subject in representation theory of associative algebras. It has the following three aspects:

(1) Categorification of combinatorics of Fomin-Zelevinsky cluster algebras [FZ2],
(2) Calabi-Yau analogue of classical tilting theory,
(3) Three dimensional Auslander-Reiten theory.

The aspect (2) with its application to (1) turns out to be so fruitful that there are a lot of applications outside of representation theory. Among others, Zamolodchikov's periodicity conjecture on Y-systems and T-systems associated to pairs of Dynkin diagrams is solved by Keller [Ke3] and Inoue-I.-Kuniba-Nakanishi-Suzuki [IIKNS].

In this paper, we will present results in cluster tilting theory from the viewpoint (2). The aim of representation theory is to understand the category of modules over finite dimensional algebras, and cluster tilting theory concerns special class of modules called cluster tilting objects. It turns out that the combinatorial behaviour of cluster tilting objects is very nice in 2-Calabi-Yau triangulated categories. In Section 1, we introduce domain of cluster tilting theory by giving a class of 2-Calabi-Yau triangulated categories associated with elements in Coxeter groups. In Section 2, we introduce the following three kinds of fundamental operations

(i) Cluster tilting mutation (Theorem 2.2),
(ii) Quiver mutation (Definition 2.5),
(iii) QP (=quivers with potentials) mutation (Definition 2.15)

in cluster tilting theory and give results on comparison of them. We are interested in the interrelation between categorical operation (i) and combinatorial operations (ii) and (iii).

We refer to surver articles [BM, GLS4, Ke3, Re, Ri] for more details in cluster tilting theory. We refer to [I1, I2] for the aspect (3) for experts in representation theory. We refer to [ARS, ASS] for general background in representation theory of associative algebras, and to [H, AHK] for classical tilting theory.

1. EXAMPLES OF 2-CY CATEGORIES WITH CLUSTER TILTING OBJECTS

Throughout this section, let K be an algebraically closed field, and let $\mathcal{C}$ be a K-linear triangulated category with the suspension functor $\Sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$. We assume the following conditions:

- $\mathcal{C}$ is Hom-*finite*, i.e. $\dim_K \mathrm{Hom}_{\mathcal{C}}(X,Y) < \infty$ for any $X, Y \in \mathcal{C}$.
- $\mathcal{C}$ is *Krull-Schmidt*, i.e. any object is isomorphic to a finite direct sum of objects whose endomorphism algebras are local.

There are the following important examples for any finite dimensional K-algebra Λ [H].

- The bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ of the category $\mathrm{mod}\,\Lambda$ of finite dimensional Λ-modules is a Hom-finite Krull-Schmidt triangulated category.

- If Λ is *selfinjective* i.e. Λ is an injective Λ-module, then the stable category $\underline{\mathrm{mod}}\Lambda$ [ARS, ASS, H] of $\mathrm{mod}\,\Lambda$ is a Hom-finite Krull-Schmidt triangulated category.

The following terminology was introduced by Kontsevich [Ko] (see [Ke2]).

Definition 1.1. We say that $\mathcal{C}$ is *2-Calabi-Yau (2-CY)* if there exists a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \simeq D\,\mathrm{Hom}_{\mathcal{C}}(Y,\Sigma^2 X)$$

for any $X, Y \in \mathcal{C}$, where $D = \mathrm{Hom}_K(-,K)$ is the K-dual.

We introduce the path algebras of quivers [ARS, ASS].

Definition 1.2. Let $Q = (Q_0, Q_1)$ be a quiver with the set Q_0 of vertices and the set Q_1 of arrows.

(1) We call a sequence
$$x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \cdots \xrightarrow{a_i} x_{i+1}$$
of arrows a *path* of length i. For example, vertices are paths of length zero, and arrows are paths of length one. We denote by Q_i the set of paths of length i. Let KQ_i be the K-vector space with the basis Q_i.

(2) The K-vector space
$$KQ := \bigoplus_{i\geq 0} KQ_i$$
forms a K-algebra where we define the multiplication by connecting paths. We call KQ the *path algebra* of Q.

The following class of 2-CY triangulated categories was introduced by Buan-Marsh-Reiten-Reineke-Todorov [BMRRT, Ke1].

Example 1.3. Let Q be a finite connected acyclic quiver and KQ the path algebra of Q. Let $\mathrm{mod}\,KQ$ be the category of finite dimensional KQ-modules and $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,KQ)$ the bounded derived category of $\mathrm{mod}\,KQ$. We call

$$\nu := D(KQ) \overset{\mathbf{L}}{\otimes}_{KQ} - : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$$

the *Nakayama functor*. This gives a *Serre functor* of $\mathcal{D}$ [H] in the sense of Bondal-Kapranov [BK], i.e. there exists a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(X,Y) \simeq D\,\mathrm{Hom}_{\mathcal{D}}(Y,\nu X)$$

for any $X, Y \in \mathcal{D}$. We put

$$F := \nu \circ [-2] : \mathcal{D} \xrightarrow{\sim} \mathcal{D}.$$

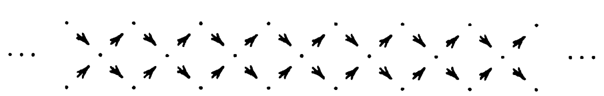
We define the *cluster category* $\mathcal{C} := \mathcal{D}/F$ of Q as follows:

- $\mathrm{Ob}\mathcal{C} = \mathrm{Ob}\mathcal{D}$,
- $\mathrm{Hom}_{\mathcal{C}}(X,Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, F^i Y)$ for any $X, Y \in \mathcal{C}$.

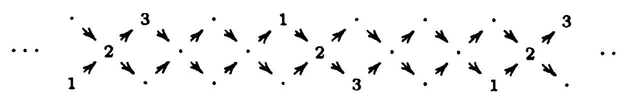
The composition of morphisms is defined naturally. Then $\mathcal{C}$ is a 2-CY triangulated category.

We can describe the derived category $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,KQ)$ and the cluster category $\mathcal{C}$ by drawing their *Auslander-Reiten quivers* [ARS, ASS, H], which display the structure of categories diagrammatically. Their vertices are isomorphism classes of indecomposable objects, and their arrows are certain morphisms called *irreducible*.

Example 1.4. Let Q be $1 \longrightarrow 2 \longrightarrow 3$. Then the Auslander-Reiten quiver of $\mathcal{D}$ is given by the following.



Identifying vertices in the same F-orbit, we obtain the following Auslander-Reiten quiver of $\mathcal{C}$.



In particular, there are 9 isomorphism classes of indecomposable objects in $\mathcal{C}$.

For a Dynkin quiver Q, there are $n+m$ isomorphism classes of indecomposable objects in the cluster category of Q, where n is the number of vertices in Q and m is the number of positive roots in the root system associated to Q [BMRRT].

We give another class of 2-CY triangulated categories [CB, GLS2].

Example 1.5. Let Q be a finite connected quiver. Define a new quiver $\overline{Q}$ by adding a new arrow $a^* : j \to i$ to Q for each arrow $a : i \to j$ in Q. We call

$$\Lambda := K\overline{Q}/\langle \sum_{a \in Q_1} (aa^* - a^*a) \rangle$$

the *preprojective algebra* of Q (see Example 1.12).

(1) If Q is Dynkin (i.e. ADE), then Λ is finite dimensional selfinjective and $\underline{\mathrm{mod}}\Lambda$ is a 2-CY triangulated category.
(2) If Q is non-Dynkin, then $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ is a 2-CY triangulated category.

The following is a key concept.

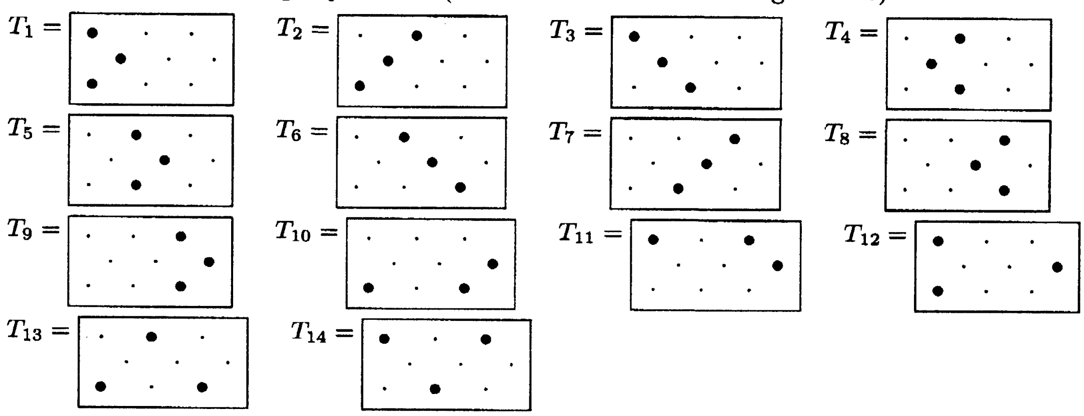
Definition 1.6. Let $\mathcal{C}$ be a 2-CY triangulated category. We say that an object $T \in \mathcal{C}$ is *cluster tilting* if

$$\mathrm{add}\,T = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(T, \Sigma X) = 0\}.$$

We give a few examples.

Example 1.7. (1) The cluster category of Q has a cluster tilting object KQ [BMRRT].
(2) The stable category $\underline{\mathrm{mod}}\Lambda$ of a preprojective algebra Λ of Dynkin type has a cluster tilting object [GLS1].
(3) $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ for a preprojective algebra Λ of non-Dynkin type does not have a cluster tilting object.

Example 1.8. Let Q be $1 \longrightarrow 2 \longrightarrow 3$ and $\mathcal{C}$ the cluster category of Q in Example 1.4. There are the following 14 basic cluster tilting objects in $\mathcal{C}$ (see Section 2 for the meaning of *basic*).



Notice that 14 is the Catalan number $\frac{1}{5}\binom{8}{4}$. In general, the number of basic cluster tilting objects in the cluster category is given by the generalized Catalan number [FZ1].

Aim 1.9. Construct a class of 2-CY triangulated categories with cluster tilting objects including Example 1.7(1) and (2).

In the rest of this section, we explain results by Buan-I.-Reiten-Scott in [BIRSc]. There is a related work by Geiss-Leclerc-Schröer [GLS3] by quite different methods.

Let Q be a finite connected quiver without loops which is non-Dynkin, and let $Q_0 = \{1, 2, \cdots, n\}$ be the set of vertices. We denote by Λ the preprojective algebra of Q. Then we have primitive orthogonal idempotents

$$1 = e_1 + \cdots + e_n$$

of Λ. Let

$$I_i := \Lambda(1 - e_i)\Lambda \subset \Lambda$$

be a two-sided ideal of Λ. We denote by

$$\langle I_1, \cdots, I_n \rangle$$

the ideal semigroup of Λ generated by $I_1, \cdots, I_n$.

The first observation is the following [IR, BIRSc].

Proposition 1.10. (1) *Any $I \in \langle I_1, \cdots, I_n\rangle$ is a tilting Λ-module.*
(2) $I_i^2 = I_i$.
(3) $I_iI_j = I_jI_i$ *if there is no arrow between i and j in Q,*
(4) $I_iI_jI_i = I_jI_iI_j$ *if there is precisely one arrow between i and j in Q.*

The above relations remind us braid relations. We denote by W the *Coxeter group* of Q (e.g. [BB]), i.e. W is presented by generators $s_1, \cdots, s_n$ with the following relations:

- $s_i^2 = 1$,
- $s_is_j = s_js_i$ if there is no arrow between i and j in Q,
- $s_is_js_i = s_js_is_j$ if there is precisely one arrow between i and j in Q.

We say that an expression $w = s_{i_1} \cdots s_{i_k}$ of $w \in W$ is *reduced* if k is the smallest possible number.

We have the following description of $\langle I_1, \cdots, I_n\rangle$ [IR, BIRSc].

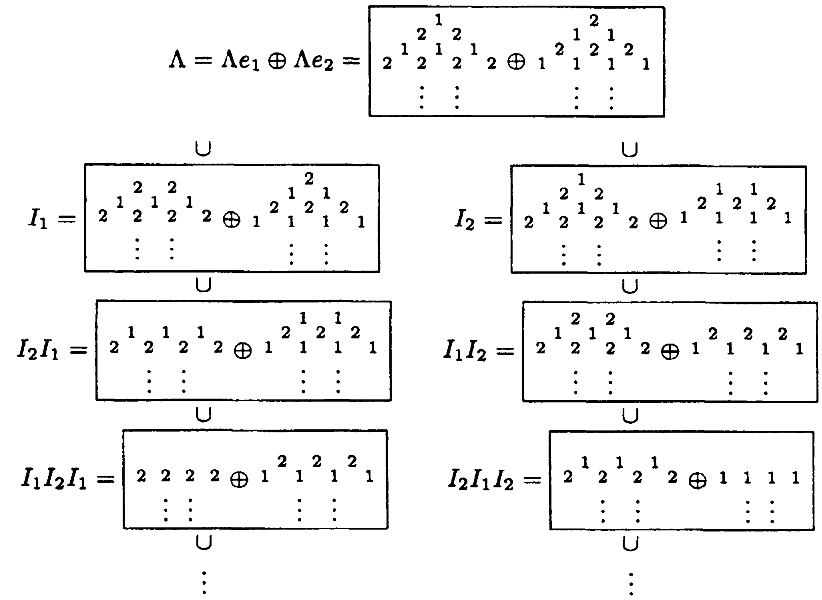
Proposition 1.11. *We have a well-defined bijection $W \xrightarrow{\sim} \langle I_1 \cdots, I_n\rangle$ given by*

$$w = s_{i_1} \cdots s_{i_k} \mapsto I_w := I_{i_1} \cdots I_{i_k}$$

for any reduced expression $w = s_{i_1} \cdots s_{i_k}$.

We give a simple example.

Example 1.12. Let Q be $1 \underset{b}{\overset{a}{\rightrightarrows}} 2$. Then $\overline{Q}$ is $1 \begin{smallmatrix} \xrightarrow{a} \\ \xrightarrow{b} \\ \xleftarrow{a^*} \\ \xleftarrow{b^*} \end{smallmatrix} 2$, and Λ is the factor algebra of $K\overline{Q}$ by two relations $aa^* + bb^* = 0$ and $a^*a + b^*b = 0$. Then $\langle I_1, I_2\rangle$ consists of the following ideals.



For $w \in W$, we put

$$\Lambda_w := \Lambda/I_w.$$

We have the following properties [BIRSc].

Proposition 1.13. (1) *Λ_w is a finite dimensional k-algebra.*
(2) *Λ_w is Iwanaga-Gorenstein of dimension at most one, i.e.*

$$\mathrm{inj.dim}_{\Lambda_w}(\Lambda_w) = \mathrm{inj.dim}(\Lambda_w)_{\Lambda_w} \leq 1.$$

We define a full subcategory of $\mathrm{mod}\,\Lambda_w$ by

$$\mathrm{Sub}\,\Lambda_w := \{X \in \mathrm{mod}\,\Lambda_w \mid X \text{ is a submodule of a projective } \Lambda_w\text{-module}\}.$$

This forms a Frobenius category in the sense of Happel [H]. In particular, the stable category

$$\mathcal{C}_w := \underline{\mathrm{Sub}}\Lambda_w$$

forms a triangulated category. Moreover, we have the following property [BIRSc].

Proposition 1.14. *$\mathcal{C}_w$ is a 2-CY triangulated category.*

As special cases of $\mathcal{C}_w$, we recover Examples 1.3 and 1.5.

Example 1.15. (1) Let $c = s_1 \cdots s_n \in W$ be a Coxeter element. Then the category $\mathcal{C}_{c^2}$ associated to $c^2 \in W$ is equivalent to the cluster category of Q given in Example 1.3.
(2) Let Q' be a full subquiver of Q which is Dynkin. Let w be the element of W corresponding to the longest element of Q'. Then we have

$$\Lambda_w \simeq \Lambda' \text{ and } \mathcal{C}_w \simeq \underline{\mathrm{mod}}\Lambda'$$

for the preprojective algebra Λ' of Q'.

Now we will construct cluster tilting objects in our 2-CY triangulated category $\mathcal{C}_w$.
Fix a reduced expression $w = s_{i_1} \cdots s_{i_k}$. Then we have a decreasing chain

$$\Lambda \supset I_{i_1} \supset I_{i_1}I_{i_2} \supset \cdots \supset I_{i_1}I_{i_2}\cdots I_{i_k} = I_w$$

of two-sided ideals of Λ. In particular, we have a chain

$$\Lambda/I_{i_1} \leftarrow \Lambda/I_{i_1}I_{i_2} \leftarrow \cdots \leftarrow \Lambda/I_{i_1}I_{i_2}\cdots I_{i_k} = \Lambda_w$$

of surjective K-algebra homomorphisms. In particular, we can regard each $\Lambda/I_{i_1}\cdots I_{i_\ell}$ as a Λ_w-module. We put

$$T(i_1, \cdots, i_k) := \bigoplus_{\ell=1}^{k} \Lambda/I_{i_1}\cdots I_{i_\ell} \in \mathrm{mod}\,\Lambda_w.$$

Now we can state the following main result in [BIRSc].

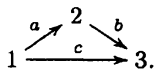
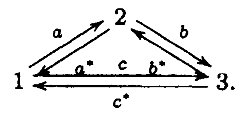
Theorem 1.16. (1) $T(i_1, \cdots, i_k) \in \mathrm{Sub}\,\Lambda_w$.
(2) *$T(i_1, \cdots, i_k)$ is a cluster tilting object in $\mathcal{C}_w$.*

Remark 1.17. (1) $T(i_1, \cdots, i_k)$ has precisely k indecomposable direct summands

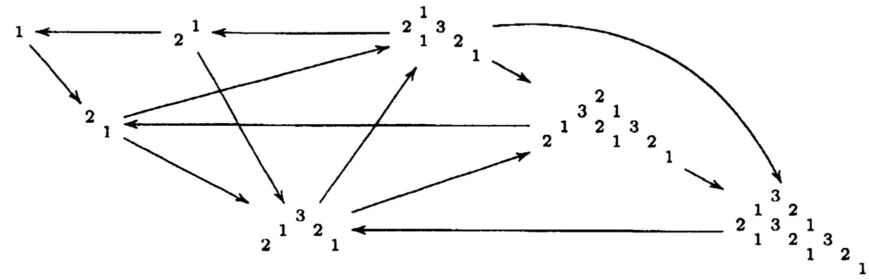
$$(\Lambda/I_{i_1})e_{i_1}, \quad (\Lambda/I_{i_1}I_{i_2})e_{i_2}, \quad \cdots, \quad (\Lambda/I_{i_1}I_{i_2}\cdots I_{i_k})e_{i_k}$$

up to isomorphisms.
(2) The quiver of the endomorphism algebra of $T(i_1, \cdots, i_k)$ is given in [BIRSc]. Moreover, it is shown in [BIRSm] that the endomorphism algebra is isomorphic to the Jacobian algebra of a quiver with a potential (see Definition 2.12).

We give an example.

Example 1.18. Let Q be  Then $\overline{Q}$ is 

Let $w = s_1s_2s_1s_3s_1s_2s_3$ be a reduced expression. Then $T(1,2,1,3,1,2,3)$ and the quiver of its endomorphism algebra is the following:



We end this section by giving other classes of 2-CY triangulated categories.

Example 1.19. (1) Let $(R, \mathfrak{m}, K)$ be a commutative complete local K-algebra and $\mathrm{CM}(R)$ the category of maximal Cohen-Macaulay R-modules [Y]. If R is a Gorenstein isolated singularity of dimension three, then the stable category $\underline{\mathrm{CM}}(R)$ is a 2-CY triangulated category by a classical result in Auslander-Reiten theory [Au]. See also [BIKR, I2, IR, IY].

(2) Based on a work of Keller [Ke4], Amiot introduced *generalized cluster categories* [Am1, Am2] associated to finite dimensional K-algebras of global dimension at most two and to quivers with potentials (see Definition 2.12). These categories play a key role in the solution of periodicity conjecture in [Ke3, IIKNS].

2. Cluster tilting mutation in 2-CY triangulated categories

Throughout this section, let K be an algebraically closed field, and let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category over K with the suspension functor Σ.

Let T be a cluster tilting object in $\mathcal{C}$. We always assume that T is *basic*, i.e.

$$T = T_1 \oplus \cdots \oplus T_n$$

with mutually non-isomorphic indecomposable objects $T_i \in \mathcal{C}$. We denote by

$$Q_T$$

the quiver of the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$ [ARS, ASS]. Then we have a presentation

$$\mathrm{End}_{\mathcal{C}}(T) \simeq KQ_T/I$$

of $\mathrm{End}_{\mathcal{C}}(T)$ for some ideal I of the path algebra KQ_T.

Aim 2.1. Study Q_T and I.

The following result was given by I.-Yoshino [IY] (see also [BMRRT]).

Theorem 2.2. (cluster tilting mutation) *Let $\mathcal{C}$ be a triangulated category and $T = T_1 \oplus \cdots \oplus T_n \in \mathcal{C}$ a basic cluster tilting object. Let $k \in \{1, \cdots, n\}$.*

(1) *There exists a unique indecomposable object $T_k^* \in \mathcal{C}$ such that $T_k^* \not\simeq T_k$ and $\mu_k(T) := (T/T_k) \oplus T_k^*$ is a basic cluster tilting object in $\mathcal{C}$.*

(2) *There exist triangles (called* exchange sequences*)*

$$T_k^* \xrightarrow{g} U_k \xrightarrow{f} T_k \to \Sigma T_k^* \quad and \quad T_k \xrightarrow{g'} U_k' \xrightarrow{f'} T_k^* \to \Sigma T_k$$

such that f and f' are right $\mathrm{add}(T/T_k)$*-approximations and g and g' are left* $\mathrm{add}(T/T_k)$*-approximations.*

Clearly we have $\mu_k \circ \mu_k(T) \simeq T$.

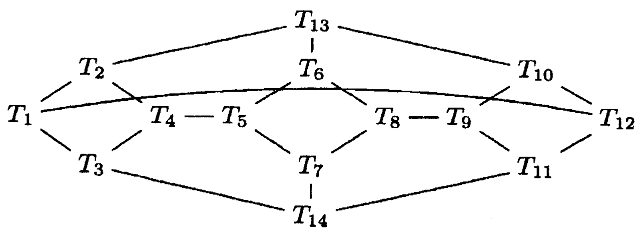
Example 2.3. Let Q be $1 \longrightarrow 2 \longrightarrow 3$ and $\mathcal{C}$ the cluster category of Q given in Example 1.4. Consider a basic cluster tilting object

$$T = \boxed{\begin{array}{ccccc} \cdot & & 3 & & \cdot \\ & 2 & & \cdot & & \cdot \\ 1 & & \cdot & & \cdot \end{array}}$$

given in Example 1.8. Then cluster tilting mutation of T is given by the following.

$$\mu_1(T) = \boxed{\begin{array}{ccccc} \cdot & & 3 & & \cdot \\ & 2 & & \cdot & & \cdot \\ \cdot & & 1^* & & \cdot \end{array}} \qquad \mu_2(T) = \boxed{\begin{array}{ccccc} \cdot & & 3 & & \cdot \\ & \cdot & & \cdot & & \cdot \\ 1 & & \cdot & & 2^* \end{array}} \qquad \mu_3(T) = \boxed{\begin{array}{ccccc} 3^* & & \cdot & & \cdot \\ & 2 & & \cdot & & \cdot \\ 1 & & \cdot & & \cdot \end{array}}$$

Moreover, the behaviour of cluster tilting mutation for 14 basic cluster tilting objects in $\mathcal{C}$ given in Example 1.8 is the following graph.



In general, the behaviour of cluster tilting mutation in the cluster category is described by the generalized Stasheff associahedron [FZ1].

Cluster tilting mutation plays a key role in the study of cluster tilting objects in 2-CY triangulated categories. For example, we have the following result for cluster categories [BMRRT].

Theorem 2.4. *Let $\mathcal{C}$ be the cluster category of a quiver Q. Then any cluster tilting object in $\mathcal{C}$ is reachable from the cluster tilting object $KQ \in \mathcal{C}$ by a successive cluster tilting mutation.*

We say that a path in a quiver is a *cycle* if the head coincides with the tail. A cycle of length one is called a *loop*, and a cycle of length two is called a *2-cycle*.

The following combinatorial operation was introduced by Fomin-Zelevinsky [FZ2].

Definition 2.5. (*quiver mutation*) Let Q be a quiver without loops. Let $k \in Q_0$ be a vertex which is not contained in 2-cycles. We define a quiver $\widetilde{\mu}_k(Q)$ by applying the following (i)-(iii) to Q.

(i) For each pair $i \xrightarrow{a} k \xrightarrow{b} j$ of arrows in Q, create a new arrow $i \xrightarrow{[ab]} j$.

(ii) Replace each arrow $i \xrightarrow{a} k$ by a new arrow $i \xleftarrow{a^*} k$.

(iii) Replace each arrow $k \xrightarrow{b} j$ by a new arrow $k \xleftarrow{b^*} j$.

Define a quiver $\mu_k(Q)$ by applying the following (iv) to $\widetilde{\mu}_k(Q)$.

(iv) Remove a maximal disjoint collection of 2-cycles.

Remark 2.6. (1) $\mu_k(Q)$ has no loops and k is not contained in 2-cycles in $\mu_k(Q)$.

(2) We have $\mu_k \circ \mu_k(Q) \simeq Q$.

(3) We can regard quiver mutation as a generalization of Bernstein-Gel'fand-Ponomarev reflection [BGP].

We give an example.

Example 2.7. For the following quiver Q of type A_3, we calculate $\mu_1(Q)$, $\mu_2(Q)$ and $\mu_2 \circ \mu_2(Q)$. (For simplicity we denote a^{**} and b^{**} by a and b respectively.)

$$Q = \left(1 \xrightarrow{a} 2 \xrightarrow{b} 3 \right) \xrightarrow{\mu_1} \left(1 \xleftarrow{a^*} 2 \xrightarrow{b} 3 \right)$$

$$\downarrow \mu_2$$

$$\left(\begin{tikzcd} 1 \arrow[rr, bend right, "{[ab]}"'] & 2 \arrow[l, "a^*"'] & 3 \arrow[l, "b^*"'] \end{tikzcd}\right) \xrightarrow{\widetilde{\mu}_2} \left(\begin{tikzcd} 1 \arrow[r, "a"] \arrow[rr, bend right, "{[ab]}"'] & 2 \arrow[r, "b"] & 3 \arrow[ll, bend right, "{[b^*a^*]}"'] \end{tikzcd}\right) \xrightarrow{\text{(iv)}} \left(1 \xrightarrow{a} 2 \xrightarrow{b} 3 \right)$$

In the rest of this section, we assume that C has a *cluster structure* [BIRSc], i.e. Q_T has no loops and 2-cycles for any cluster tilting object $T \in C$. In this case, we have the following.

Remark 2.8. Combining the exchange sequences in Theorem 2.2, we have a complex

$$T_k \xrightarrow{g'} U_k' \xrightarrow{f'g} U_k \xrightarrow{f} T_k$$

such that the sequences

$$\mathrm{Hom}_C(T, U_k') \xrightarrow{f'g} \mathrm{Hom}_C(T, U_k) \xrightarrow{f} J_C(T, T_k) \to 0,$$
$$\mathrm{Hom}_C(U_k, T) \xrightarrow{f'g} \mathrm{Hom}_C(U_k', T) \xrightarrow{g'} J_C(T_k, T) \to 0$$

are exact for the Jacobson radical J_C of C. Such a complex is called a *2-almost split sequence* in [I1] and an *AR 4-angle* in [IY]. Consequently the quiver and relations of $\mathrm{End}_C(T)$ can be controlled by exchange sequences.

Example 2.9. The 2-CY triangulated category C_w given in Proposition 1.14 has a cluster structure [BIRSc]. In particular, cluster categories in Example 1.3 and the stable category $\underline{\mathrm{mod}}\Lambda$ for preprojective algebras Λ of Dynkin type in Example 1.5(1) have a cluster structure.

Using Remark 2.8, we have the following result [BIRSc] which shows that cluster tilting mutation is compatible with quiver mutation.

Theorem 2.10. *Let C be a 2-CY triangulated category with a cluster structure and $T \in C$ a cluster tilting object. Then $Q_{\mu_k(T)} \simeq \mu_k(Q_T)$ holds for any $k \in (Q_T)_0$.*

For example, cluster tilting mutation given in Example 2.3 is compatible with quiver mutation in Example 2.7.

As an appication of Theorem 2.10, we have the following result [BIRSm].

Corollary 2.11. *Let C_i be a cluster category and $T_i \in C_i$ a cluster tilting object for $i = 1, 2$. If $Q_{T_1} \simeq Q_{T_2}$, then $\mathrm{End}_{C_1}(T_1) \simeq \mathrm{End}_{C_2}(T_2)$.*

The following was introduced by Derksen-Weyman-Zelevinsky [DWZ].

Definition 2.12. Let Q be a quiver without loops.

(1) We denote by Q_i the set of paths of length i, and by $Q_{i,\mathrm{cyc}}$ the set of cycles of length i. Let KQ_i be the K-vector space with the basis Q_i, and let $KQ_{i,\mathrm{cyc}}$ the subspace of KQ_i spanned by $Q_{i,\mathrm{cyc}}$. Similar to the path algebra KQ, the K-vector space

$$\widehat{KQ} := \prod_{i \geq 0} KQ_i$$

forms a K-algebra which we call the *complete path algebra* of Q. The Jacobson radical of $\widehat{KQ}$ is given by $J_{\widehat{KQ}} = \prod_{i \geq 1} KQ_i$. We regard $\widehat{KQ}$ as a topological algebra with respect to the $(J_{\widehat{KQ}})$-adic topology.

(2) A *quiver with a potential* (or *QP*) is a pair (Q, W) consisting of a quiver Q without loops and an element

$$W \in \prod_{i \geq 2} KQ_{i,\mathrm{cyc}}$$

called a *potential.* It is called *reduced* if $W \in \prod_{i \geq 3} KQ_{i,\mathrm{cyc}}$. Define $\partial_a W \in \widehat{KQ}$ by

$$\partial_a(a_1 \cdots a_\ell) := \sum_{a_i = a} a_{i+1} \cdots a_\ell a_1 \cdots a_{i-1}$$

and extend linearly and continuously. The *Jacobian algebra* is defined by

$$\mathcal{P}(Q, W) := \widehat{KQ}/\overline{\langle \partial_a W \mid a \in Q_1 \rangle}$$

where $\overline{I}$ is the closure of I.

Remark 2.13. (1) The behaviour of Jacobian algebras is very nice thanks to the completion.

(2) Two potentials W and W' are called *cyclically equivalent* if $W - W' \in \overline{[KQ, KQ]}$, where $[KQ, KQ]$ is the K-vector subspace of $\widehat{KQ}$ spanned by commutators. In this case, we clearly have $\mathcal{P}(Q, W) = \mathcal{P}(Q, W')$.

We give an example.

Example 2.14. Let (Q, W) be a (non-reduced) QP

$$\left(\begin{tikzcd} 1 \arrow[r, "a"] \arrow[rr, bend right, "c"'] & 2 \arrow[r, "b"] & 3 \arrow[ll, bend right, "d"'] \end{tikzcd}, cd + abd \right).$$

Then $\partial_a W = bd$, $\partial_b W = da$, $\partial_c W = d$ and $\partial_d W = c + ab$. Thus the Jacbian algebra $\mathcal{P}(Q, W)$ coincides with the Jacobian algebra of

$$(Q', W') = \left(1 \xrightarrow{a} 2 \xrightarrow{b} 3, 0 \right).$$

In general, for any QP (Q, W), a reduced QP (Q', W') satisfying $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q', W')$ was associated in [DWZ] and called the *reduced part* of (Q, W). We omit the detailed definition here. For example, the reduced part of the QP (Q, W) in Example 2.14 is given by (Q', W') there.

The following operation is introduced by Derksen-Weyman-Zelevinsky [DWZ].

Definition 2.15. (*QP mutation*) Let (Q, W) be a QP. Assume that $k \in Q_0$ is not contained in 2-cycles. Replacing W by a cyclically equivalent potential, we assume that no cycles in W start at k. Define a QP $\widetilde{\mu}_k(Q, W) := (\widetilde{\mu}_k(Q), [W] + \Delta)$ as follows:

- $\widetilde{\mu}_k(Q)$ is given in Definition 2.5.
- $[W]$ is obtained by replacing each factor $i \xrightarrow{a} k \xrightarrow{b} j$ in W by $i \xrightarrow{[ab]} j$.
- $\Delta := \sum_{(i \xrightarrow{a} k \xrightarrow{b} j) \text{ in } Q} a^*[ab]b^*$.

Define a QP $\mu_k(Q, W)$ as a reduced part of $\widetilde{\mu}_k(Q, W)$.

Remark 2.16. Clearly k is not contained in 2-cycles in $\mu_k(Q, W)$. Moreover, $\mu_k \circ \mu_k(Q, W)$ is right-equivalent to (Q, W) [DWZ] in the following sense:

Two QP's (Q, W) and (Q', W') are called *right-equivalent* if $Q_0 = Q'_0$ and there exists a K-algebra isomorphism $\phi : \widehat{KQ} \to \widehat{KQ'}$ such that $\phi|_{Q_0} = \mathrm{id}$ and $\phi(W)$ and W' are cyclically equivalent. In this case ϕ induces an isomorphism $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q', W')$.

We give an example.

Example 2.17. For a QP (Q,W) below, we calculate $\mu_2(Q,W)$ and $\mu_2\circ\mu_2(Q,W)$.

$$(Q,W)=\left(\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \end{tikzcd},0\right) \quad \xrightarrow{\mu_2} \quad \left(\begin{tikzcd} 1 \arrow[rr, bend right, "{[ab]}"'] & 2 \arrow[l, "a^*"'] & 3 \arrow[l, "b^*"'] \end{tikzcd},a^*[ab]b^*\right)$$

$$\xrightarrow{\widetilde{\mu}_2} \quad \left(\begin{tikzcd} 1 \arrow[r, "a"] \arrow[rr, bend right, "{[ab]}"'] & 2 \arrow[r, "b"] & 3 \arrow[ll, bend right, "{[b^*a^*]}"'] \end{tikzcd},[ab][b^*a^*]+b[b^*a^*]a\right) \quad \xrightarrow{\text{reduced}} \quad \left(\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \end{tikzcd},0\right)$$

The reduced part of $\widetilde{\mu}_2\circ\mu_2(Q,W)$ was calculated in Example 2.14.

The following result [BIRSm] shows that cluster tilting mutation is compatible with QP mutation.

Theorem 2.18. *Let $\mathcal{C}$ be a 2-CY triangulated category and $T\in\mathcal{C}$ a cluster tilting object. Let (Q,W) be a QP. If $\operatorname{End}_{\mathcal{C}}(T)\simeq\mathcal{P}(Q,W)$, then $\operatorname{End}_{\mathcal{C}}(\mu_k(T))\simeq\mathcal{P}(\mu_k(Q,W))$.*

Immediately we have the following.

Corollary 2.19. *Let $\mathcal{C}$ be a 2-CY triangulated category and $T\in\mathcal{C}$ a cluster tilting object. If $\operatorname{End}_{\mathcal{C}}(T)$ is a Jacobian algebra of a QP, then so is $\operatorname{End}_{\mathcal{C}}(T')$ for any cluster tilting object $T'\in\mathcal{C}$ reachable from T by a successive cluster tilting mutation.*

For example, we have the following.

Example 2.20. (1) *Cluster tilted algebras* (=endomorphism algebras of cluster tilting objects in cluster categories) are Jacobian algebras of QP's by Theorem 2.4 and Corollary 2.19 since $KQ=\mathcal{P}(Q,0)$.

(2) Let $\mathcal{C}_w$ be a 2-CY triangulated category in Proposition 1.14 and $T(i_1,\cdots,i_k)\in\mathcal{C}_w$ a cluster tilting object in Theorem 1.16. For any cluster tilting object $T\in\mathcal{C}_w$ reachable from $T(i_1,\cdots,i_k)$ by a successive cluster tilting mutation, $\operatorname{End}_{\mathcal{C}_w}(T)$ is a Jacobian algebra of a QP by Example 1.17(2) and Corollary 2.19.

We end this note by the following *nearly Morita equivalence* for Jacobian algebras [BIRSm] (see also [BMR]), where mod is the category of modules with finite length.

Theorem 2.21. *For a QP (Q,W), we have an equivalence*

$$\operatorname{mod}\mathcal{P}(Q,W)/[S_k]\simeq\operatorname{mod}\mathcal{P}(\mu_k(Q,W))/[S'_k],$$

where S_k and S'_k are simple modules associated with the vertex k.

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