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Pyramidal traveling fronts in the Allen-Cahn equations

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Abstract

Pyramidal traveling fronts in the Allen-Cahn equations have been studied in the three-dimensional whole space. For a given admissible pyramid a pyramidal traveling front is uniquely determined and it is asymptotically stable under the condition that given perturbations decay at infinity. A pyramidal traveling front is a combination of planar fronts on the lateral surfaces. Also it is a combination of two-dimensional V-form waves associated with the edges of a pyramid.

AMS Mathematical Classifications: 35K57, 35B35
Key words: pyramidal traveling wave, Allen-Cahn equation, stability

1 Introduction

For one-dimensional traveling waves in the Allen-Cahn equation or the Nagumo equation so many works have been studied. See [1, 4, 9, 10, 2]

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and so on. In the two-dimensional plane or higher dimensional spaces the simplest traveling waves are planar ones. Recently non-planar traveling waves in the whole space have been studied by [17, 18, 7, 8, 12, 3, 21, 22] and so on. For non-planar traveling waves researchers are interested in the shapes of the contour lines or surfaces. Constructing traveling waves with new shapes is an attracting motivation of the mathematical research. The mathematical study on these multi-dimensional traveling waves will give information for chemists or biochemists to study multi-dimensional chemical waves or nerve transmission phenomena in future.

The stability of planar traveling waves have been studied by [14, 13, 23, 15] and so on. The existence and stability of two-dimensional V-form waves are studied by [17, 18, 7, 8, 12]. The existence and the uniqueness and asymptotic stability of pyramidal traveling waves are studied in [21, 22].

In this paper we consider the following equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \qquad \text{in } \mathbb{R}^3,\ t > 0,$$
$$u|_{t=0} = u_0 \qquad \text{in } \mathbb{R}^3.$$

A given function u_0 belongs to $BU(\mathbb{R}^3)$. Here $BU(\mathbb{R}^3)$ is the space of bounded uniformly continuous functions from $\mathbb{R}^3$ to $\mathbb{R}$ with the supremum norm. The Laplacian Δ stands for $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. We study nonlinear terms of bistable type including cubic ones. This equation is called the Allen-Cahn equation or the Nagumo equation.

In the one-dimensional space, let $\Phi(x - kt)$ be a traveling wave that connects two stable equilibrium states ± 1 with speed k. By putting $\mu = x - kt$, Φ satisfies

$$\begin{aligned} -\Phi''(\mu) - k\Phi'(\mu) - f(\Phi(\mu)) = 0 \quad & -\infty < \mu < \infty, \\ \Phi(-\infty) = 1, \quad \Phi(\infty) = -1. & \end{aligned} \tag{1}$$

To fix the phase we set $\Phi(0) = 0$. See Figure 1.

The following is the assumptions on f in this paper.

(A1) f is of class $C^1[-1, 1]$ with $f(\pm 1) = 0$ and $f'(\pm 1) < 0$.

(A2) $\int_{-1}^{1} f > 0$ holds true.

(A3) $f(s) < 0$ holds true for $s > 1$. $f(s) > 0$ holds true for $s < -1$.

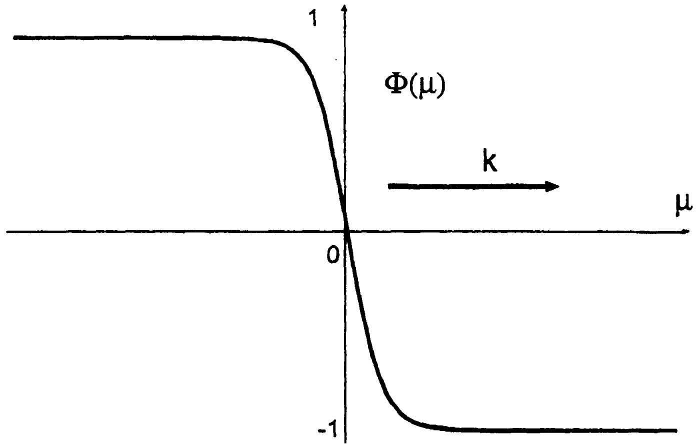

Figure 1: One-dimensional traveling wave Φ

(A4) There exists $\Phi(\mu)$ that satisfies (1) for some $k \in \mathbb{R}$.

We note that $k > 0$ follows from (A2) and (A4).

For $f(u) = -(u+1)(u+a)(u-1)$ with a given constant $a \in (0,1)$, $\Phi(\mu) = -\tanh(\mu/\sqrt{2})$ satisfies (A1)-(A4) for $k = \sqrt{2}a$. Another simple example is as follows. Let $G(u) \in C^2(\mathbb{R})$ satisfy

$$G(\pm 1) = 0, \quad G'(\pm 1) = 0, \quad G''(\pm 1) > 0$$
$$G(s) > 0 \quad \text{if } s^2 \neq 1,$$
$$\max\left\{0, \sup_{s<-1} \frac{G'(s)}{\sqrt{2G(s)}}\right\} < \inf_{s>1} \frac{G'(s)}{\sqrt{2G(s)}},$$

and let $f(u)$ be given by

$$f(u) = -G'(u) + k\sqrt{2G(u)}$$

for any constant k with

$$\max\left\{0, \sup_{s<-1} \frac{G'(s)}{\sqrt{2G(s)}}\right\} < k < \inf_{s>1} \frac{G'(s)}{\sqrt{2G(s)}},$$

Then $\Phi(\mu)$ given by

$$\mu = -\int_0^{\Phi} \frac{dv}{\sqrt{2G(v)}}, \quad \mu = x - kt$$

satisfies (A1)-(A4).

For more examples of one-dimensional traveling waves see [4, 1, 2, 3, 21].

We adopt the moving coordinate of speed c toward the z-axis without loss of generality. We put $s = z - ct$ and $u(x, y, z, t) = w(x, y, s, t)$. We denote $w(x, y, s, t)$ by $w(x, y, z, t)$ for simplicity. Then we obtain

$$\begin{aligned} w_t - w_{xx} - w_{yy} - w_{zz} - cw_z - f(w) = 0 \qquad & \text{in } \mathbb{R}^3,\ t > 0, \\ w|_{t=0} = u_0 \qquad & \text{in } \mathbb{R}^3. \end{aligned} \tag{2}$$

Here w_t stands for $\partial w/\partial t$ and so on. We write the solution as $w(x, y, z, t; u_0)$. If v is a traveling wave with speed c, it satisfies

$$\mathcal{L}[v] \stackrel{\text{def}}{=} -v_{xx} - v_{yy} - v_{zz} - cv_z - f(v) = 0 \quad \text{in } \mathbb{R}^3. \tag{3}$$

We assume

$$c > k$$

throughout this paper. Since the curvature often accelerates the speed, one has many traveling waves if $c > k$. As far as the author knows, it is an open problem to prove the existence or non-existence of traveling waves if $c < k$.

Let $n \geq 3$ be a given integer. We put

$$\tau \stackrel{\text{def}}{=} \frac{\sqrt{c^2 - k^2}}{k} > 0. \tag{4}$$

Assume $(A_j, B_j) \in \mathbb{R}^2$ satisfies

$$A_j^2 + B_j^2 = 1 \qquad \text{for all } j = 1, \ldots, n \tag{5}$$

and

$$\begin{aligned} A_j B_{j+1} - A_{j+1} B_j > 0 \qquad 1 \leq j \leq n-1, \\ A_n B_1 - A_1 B_n > 0. \end{aligned} \tag{6}$$

Now

$$\boldsymbol{\nu}_j \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+\tau^2}} \begin{pmatrix} -\tau A_j \\ -\tau B_j \\ 1 \end{pmatrix}$$

is the unit normal vector of a surface $\{z = \tau(A_j x + B_j y)\}$. We put

$$\begin{aligned} h_j(x, y) &\stackrel{\text{def}}{=} \tau\left(A_j x + B_j y\right), \\ h(x, y) &\stackrel{\text{def}}{=} \max_{1 \leq j \leq n} h_j(x, y) = \tau \max_{1 \leq j \leq n} \left(A_j x + B_j y\right). \end{aligned} \tag{7}$$

Then $z = h(x, y)$ gives a reverse pyramid in $\mathbb{R}^3$. We call it simply a pyramid hereafter. We set

$$\Omega_j = \{(x, y) \mid h(x, y) = h_j(x, y)\},$$

and obtain

$$\mathbb{R}^2 = \cup_{j=1}^{n} \Omega_j.$$

We locate $\Omega_1, \Omega_2, \ldots, \Omega_n$ counterclockwise. To ensure this location we assumed (6). Now the lateral surfaces of a pyramid are given by

$$S_j = \{(x, y, h_j(x, y)) \in \mathbb{R}^3 \mid (x, y) \in \Omega_j\}$$

for $j = 1, \ldots, n$. We put

$$\Gamma_j \stackrel{\text{def}}{=} \begin{cases} S_j \cap S_{j+1} & \text{if } 1 \leq j \leq n-1, \\ S_n \cap S_1 & \text{if } j = n. \end{cases}$$

Then Γ_j represents an edge of a pyramid. Also

$$\Gamma \stackrel{\text{def}}{=} \cup_{j=1}^{n} \Gamma_j$$

represents the set of all edges. See Figure 2.

By using (A_j, B_j) with $A_j^2 + B_j^2 = 1$, Equation (3) has a solution $\Phi\left((k/c)(z - h_j(x, y))\right)$. It is called a planar traveling front associated with the lateral surface S_j. Now we put

$$\underline{v}(x, y, z) \stackrel{\text{def}}{=} \Phi\left(\frac{k}{c}(z - h(x, y))\right) = \max_{1 \leq j \leq n} \Phi\left(\frac{k}{c}(z - h_j(x, y))\right).$$

We define

$$D(\gamma) \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 \mid \operatorname{dist}((x, y, z), \Gamma) \geq \gamma\} \tag{8}$$

for $\gamma \geq 0$.

The existence of pyramidal traveling fronts is proved in [21]. See Figure 3.

Theorem 1 (**[21]**) *Let $c > k$ and let $h(x, y)$ be given by (7). Under the assumptions (A1), (A2), (A3) and (A4) there exists a solution $V(x, y, z)$ to (3) with*

$$\lim_{\gamma \to +\infty} \sup_{(x,y,z) \in D(\gamma)} \left| V(x, y, z) - \Phi\left(\frac{k}{c}(z - h(x, y))\right) \right| = 0. \tag{9}$$

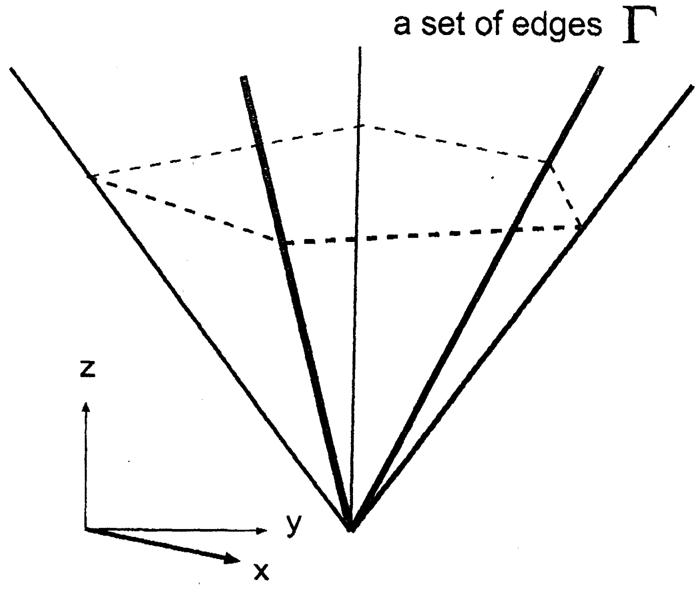

Figure 2: The edge lines Γ

Moreover one has

$$V_z(x,y,z) < 0, \quad \Phi\left(\frac{k}{c}(z - h(x,y))\right) < V(x,y,z) < 1 \qquad \text{for all } (x,y,z) \in \mathbb{R}^3.$$

The following theorem is the main assertion on the uniqueness and the stability of pyramidal traveling fronts.

Theorem 2 ([22]) *In addition to the assumptions as in Theorem 1 suppose*

$$\lim_{\gamma \to +\infty} \sup_{(x,y,z) \in D(\gamma)} |u_0(x,y,z) - V(x,y,z)| = 0. \tag{10}$$

Then

$$\lim_{t \to +\infty} \sup_{(x,y,z) \in \mathbb{R}^3} |u(x,y,z-ct,t) - V(x,y,z)| = 0$$

holds true. Especially $V(x,y,z)$ as in Theorem 1 is uniquely determined by (3) and (9).

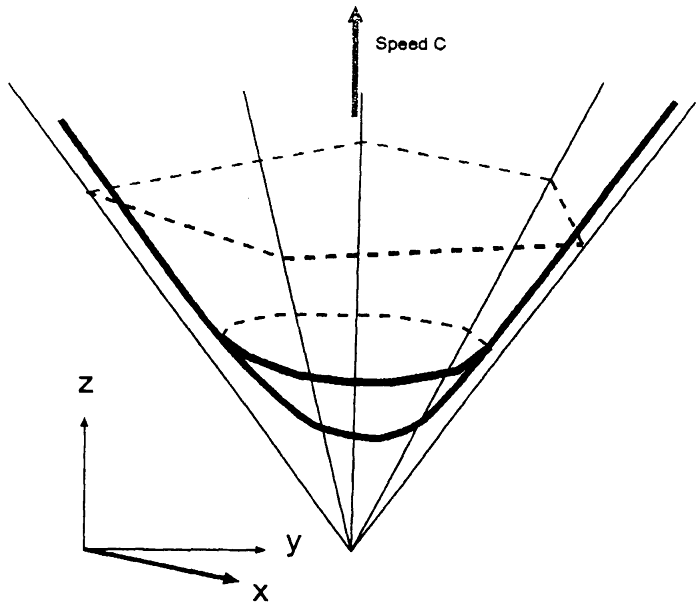

Figure 3: The pyramidal traveling wave V

If u_0 satisfies

$$\lim_{R\to+\infty} \sup_{x^2+y^2+z^2\geq R^2} |u_0(x,y,z) - V(x,y,z)| = 0,$$

it also satisfies (10). Thus the theorem also asserts that a pyramidal traveling wave V is asymptotically stable globally in space if a given fluctuation decays at infinity. The asymptotic stability is valid for a weaker condition (10). This means that V is robust for fluctuations added on edges. Now V as in Theorem 1 can be called the pyramidal traveling wave associated with a pyramid $z = h(x,y)$, since it is uniquely determined.

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3 Preliminaries

Under the assumption (A1) and (A4), $\Phi(\mu)$ as in (1) satisfies

$$\Phi'(\mu) < 0 \qquad \text{for all } \mu \in \mathbb{R}, \tag{11}$$
$$\max\{|\Phi'(\mu)|, |\Phi''(\mu)|\} \le K_0 \exp(-\kappa_0|\mu|). \tag{12}$$

Here K_0 and κ_0 are some positive constants. See Fife and McLeod [4] for the proof.

From the assumptions on f there exists a positive constant δ_* $(0 < \delta_* < 1/4)$ with

$$-f'(s) > \beta \qquad \text{if } |s+1| < 2\delta_* \text{ or } |s-1| < 2\delta_*,$$

where

$$\beta \stackrel{\text{def}}{=} \frac{1}{2}\min\{-f'(-1), -f'(1)\} > 0.$$

Then for all $\delta \in (0, \delta_*)$ we have

$$-f'(s) > \beta \qquad \text{if } |s+1| < 2\delta \text{ or } |s-1| < 2\delta.$$

We state the uniqueness and stability of a two-dimensional V-form front in the two-dimensional plane. See Figure 4. Let $\widetilde{w}(\xi, \eta, t; \widetilde{w}_0)$ be the solution of

$$\begin{aligned} \widetilde{w}_t - \widetilde{w}_{\xi\xi} - \widetilde{w}_{\eta\eta} - s\widetilde{w}_\eta - f(\widetilde{w}) &= 0 \qquad \text{for } (\xi,\eta) \in \mathbb{R}^2,\ t > 0, \\ w(\xi, \eta, 0) &= \widetilde{w}_0(\xi, \eta) \qquad \text{for } (\xi,\eta) \in \mathbb{R}^2 \end{aligned}$$

for a given bounded $\widetilde{w}_0 \in C^1(\mathbb{R}^2)$.

Theorem 3 (Two-dimensional traveling V-form fronts [17],[18]) *For any $s \in (k, +\infty)$, there exists unique $v_*(\xi, \eta; s)$ that satisfies*

$$-(v_*)_{\xi\xi} - (v_*)_{\eta\eta} - s(v_*)_\eta - f(v_*) = 0 \qquad \textit{for } (\xi,\eta) \in \mathbb{R}^2,$$
$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} \left| v_*(\xi,\eta) - \Phi\left(\frac{k}{s}\left(\eta - \frac{\sqrt{s^2-k^2}}{k}|\xi|\right)\right)\right| = 0. \tag{13}$$

One has

$$\Phi\left(\frac{k}{s}\left(\eta - \frac{\sqrt{s^2-k^2}}{k}|\xi|\right)\right) < v_*(\xi,\eta) \qquad \textit{for } (\xi,\eta) \in \mathbb{R}^2, \tag{14}$$
$$\inf_{-1+\delta \le v_*(\xi,\eta) \le 1-\delta} \left(-(v_*)_\eta(\xi,\eta)\right) > 0 \qquad \textit{for all } \delta \in (0, \delta_*). \tag{15}$$

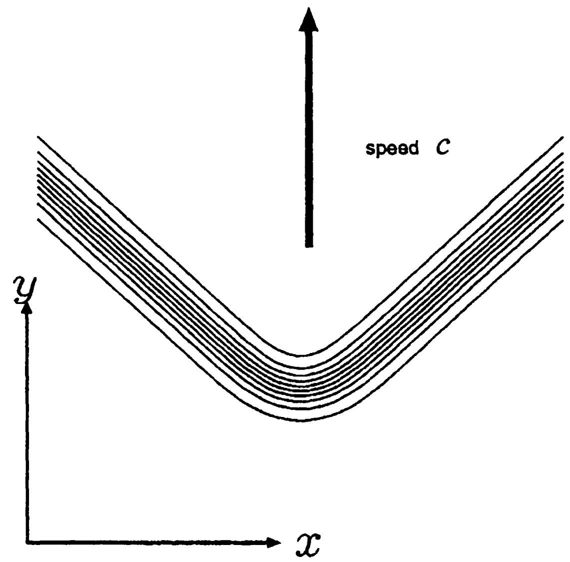

Figure 4: Contour lines of a two-dimensional V-form wave $v_*(x,y)$ ([17]).

The following convergence

$$\lim_{t\to+\infty} \|w(\xi,\eta,t) - v_*(\xi,\eta)\|_{L^\infty(\mathbb{R}^2)} = 0$$

holds true for any bounded function $\widetilde{w}_0 \in C^1(\mathbb{R}^2)$ *with*

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} |\widetilde{w}_0(\xi,\eta) - v_*(\xi,\eta)| = 0.$$

See also Hamel, Monneau and Roquejoffre [7, 8]. This v_* can be called the two-dimensional traveling V-form front associated with (13) since it is uniquely determined. We call the η-axis the traveling direction of $v_*(\xi,\eta;s)$. This theorem asserts the asymptotic stability of v_* for any fluctuation that decays at infinity.

Now we explain why we can take any $c \in (k,+\infty)$ and why we should use $\tan\theta = \frac{\sqrt{c^2-k^2}}{k}$. A planar traveling front travels with speed k to the vertical direction. Then towards the z-axis it travels faster. The speed c and the angle θ should satisfy $\tan\theta = \frac{\sqrt{c^2-k^2}}{k}$ as in Figure 5. If θ goes to $\pi/2$, a two-dimensional V-form front travels with $+\infty$. If θ goes to zero, a two-dimensional V-form front travels with k. Thus we can take any $c \in (k,+\infty)$.

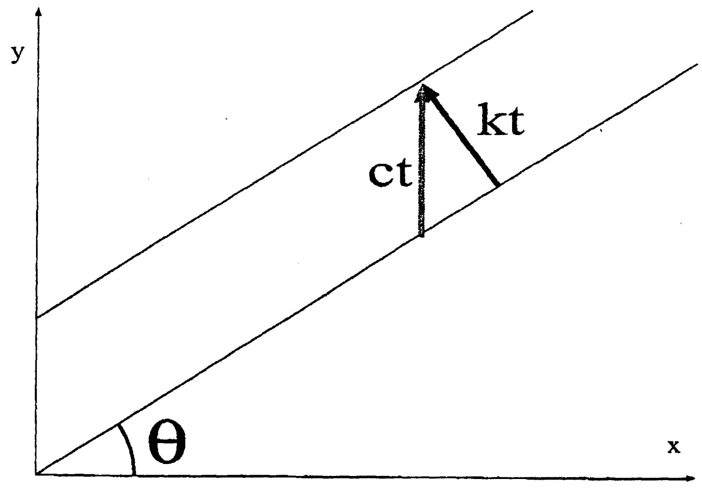

Figure 5: For a two-dimensional V-form front one has $\tan\theta = \frac{\sqrt{c^2-k^2}}{k}$.

A pyramidal traveling front V converges to two-dimensional traveling V-form fronts on the edges at infinity, that it inherits the stability property of v_* and that V is asymptotically stable.

Now $\overline{v}$ is called a supersolution if and only if

$$\mathcal{L}[\overline{v}] = -\overline{v}_{xx} - \overline{v}_{yy} - \overline{v}_{zz} - c\overline{v}_z - f(\overline{v}) \geq 0 \quad \text{in } \mathbb{R}^3.$$

Then one has

$$w(\boldsymbol{x}, t; \overline{v}) \leq \overline{v}(\boldsymbol{x}) \qquad \text{in } \mathbb{R}^3,\ t > 0.$$

A subsolution can be defined similarly, that is, $\underline{v}$ is called a subsolution if and only if

$$\mathcal{L}[\underline{v}] = -\underline{v}_{xx} - \underline{v}_{yy} - \underline{v}_{zz} - c\underline{v}_z - f(\underline{v}) \leq 0 \quad \text{in } \mathbb{R}^3.$$

Then one has

$$w(\boldsymbol{x}, t; \underline{v}) \geq \underline{v}(\boldsymbol{x}) \qquad \text{in } \mathbb{R}^3,\ t > 0.$$

For $\varphi(x, y) \in C^\infty(\mathbb{R}^2)$ we put

$$\nabla\varphi(x,y) \stackrel{\text{def}}{=} \begin{pmatrix} D_1\varphi(x,y) \\ D_2\varphi(x,y) \end{pmatrix} \qquad |\nabla\varphi(x,y)| = \sqrt{D_1\varphi(x,y)^2 + D_2\varphi(x,y)^2}.$$

Here $D_1\varphi(x,y) = \varphi_x(x,y)$ and $D_2\varphi(x,y) = \varphi_y(x,y)$. For $\alpha > 0$, $\varepsilon_1 > 0$

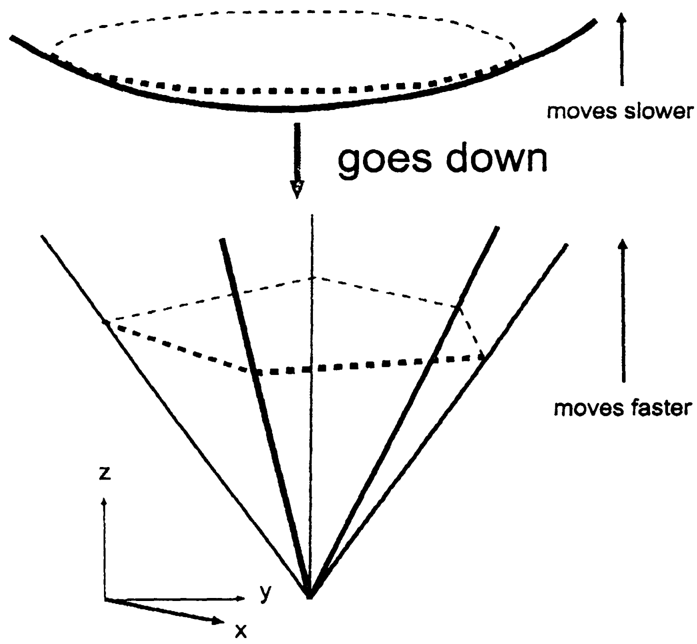

Figure 6: A supersolution U

and $\varphi \in C^\infty(\mathbb{R}^2)$ we put

$$U(x,y,z) \stackrel{\text{def}}{=} \Phi\left(\frac{z-\frac{1}{\alpha}\varphi(\alpha x,\alpha y)}{\sqrt{1+|\nabla\varphi(\alpha x,\alpha y)|^2}}\right)+\varepsilon_1\left(\frac{c}{\sqrt{1+|\nabla\varphi(\alpha x,\alpha y)|^2}}-k\right). \quad (16)$$

Lemma 1 ([21]) *For some positive-valued function $\varphi(x,y) \in C^\infty(\mathbb{R}^2)$ with $|\nabla\varphi| < \tau$ the following holds true. For sufficiently small ε_1, say $\varepsilon_1 \in (0,\varepsilon_1^*)$, there exists $\alpha_0(\varepsilon_1)$ so that U given by (16) satisfies*

$$\mathcal{L}[U] > 0, \quad \underline{v} < U \qquad \text{in } \mathbb{R}^3$$

for any $\alpha \in (0,\alpha_0(\varepsilon_1))$.

See [21] for the construction of φ and the definitions of ε_1^* and $\alpha_0(\varepsilon_1)$. Now we explain intuitively why U becomes a supersolution if $\alpha > 0$ is small enough.

For $0 < \alpha < 1$ we shift up and expand the graph of $z = \varphi(x, y)$ and obtain the graph of

$$z = \frac{1}{\alpha}\varphi(\alpha x, \alpha y).$$

If $\alpha > 0$ goes to zero, it becomes very flat like a plane. If we take $\alpha > 0$ smaller and smaller, the contour surface $\{\boldsymbol{x} \in \mathbb{R}^3 \,|\, U(\boldsymbol{x}) = 0\}$ becomes flatter and flatter like a plane. Then it should moves upwards with the speed k, since k is the speed of a planar traveling wave. We are now using the moving coordinate with speed c. The assumption $c > k$ implies that the contour surface $\{\boldsymbol{x} \in \mathbb{R}^3 \,|\, U(\boldsymbol{x}) = 0\}$ moves downwards with speed $c - k$ in the the moving coordinate. This gives an intuitive explanation of $w(\boldsymbol{x}, t; U)$ is decreasing in $t > 0$, that is, U is a supersolution as in Lemma 1.

In [21] V is defined by

$$V(x, y, z) \stackrel{\text{def}}{=} \lim_{t\to\infty} w(x, y, z, t; \underline{v}) \tag{17}$$

for any $(x, y, z) \in \mathbb{R}^3$. By Sattinger [20, Theorem 3.6], $w(x, y, z, t; \underline{v})$ is monotone increasing in $t > 0$ for each $(x, y, z) \in \mathbb{R}^3$.

Let U be as in (16) under the assumption of Lemma 1. We fix ε and α later. We write it by U though it depends on ε and α for simplicity. We have

$$\underline{v}(x, y, z) < V(x, y, z) < U(x, y, z) \qquad \text{in } \mathbb{R}^3.$$

Hereafter we set $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$. We have $\varphi(0, 0) > 0$. We get

$$\lim_{\alpha\to 0} \inf_{|\boldsymbol{x}|\le R} U(\boldsymbol{x}) \ge 1 \tag{18}$$

for any given $R > 0$. We have

$$U_z(x, y, z) = \frac{1}{\sqrt{1 + |\nabla\varphi(\alpha x, \alpha y)|^2}}\Phi'\left(\frac{z - \frac{1}{\alpha}\varphi(\alpha x, \alpha y)}{\sqrt{1 + |\nabla\varphi(\alpha x, \alpha y)|^2}}\right).$$

4 Uniqueness and stability

A pyramidal traveling front V converges to two-dimensional V-form fronts on edges at infinity. We write the explicit form of the two-dimensional V-form front on each edge.

For each j $(1 \leq j \leq n)$ we consider a plane perpendicular to an edge $\Gamma_j = S_j \cap S_{j+1}$. Then the cross section of $z = \max\{h_j(x,y), h_{j+1}(x,y)\}$ in this plane has a V-form front. Let E_j be the two-dimensional V-form front as in Theorem 3 associated with the cross section of $z = \max\{h_j(x,y), h_{j+1}(x,y)\}$. We write the precise definition of E_j later.

The direction of Γ_j is given by

$$\boldsymbol{\nu}_j \times \boldsymbol{\nu}_{j+1} = \frac{1}{\sqrt{q_j^2 + \tau^2 p_j^2}} \begin{pmatrix} B_{j+1} - B_j \\ A_j - A_{j+1} \\ \tau(A_j B_{j+1} - A_{j+1} B_j) \end{pmatrix}.$$

We note that the z-component is positive.

Now we define

$$p_j \stackrel{\text{def}}{=} A_j B_{j+1} - A_{j+1} B_j > 0, \quad q_j \stackrel{\text{def}}{=} \sqrt{(A_{j+1} - A_j)^2 + (B_{j+1} - B_j)^2} > 0.$$

for $1 \leq j \leq n$. We put $A_{n+1} \stackrel{\text{def}}{=} A_1$, $B_{n+1} \stackrel{\text{def}}{=} B_1$ and thus

$$p_n = A_n B_1 - A_1 B_n > 0, \quad q_n = \sqrt{(A_1 - A_n)^2 + (B_1 - B_n)^2} > 0.$$

The traveling direction of a two-dimensional V-form wave E_j is given by

$$\begin{aligned} &\frac{\boldsymbol{\nu}_{j+1} - \boldsymbol{\nu}_j}{|\boldsymbol{\nu}_{j+1} - \boldsymbol{\nu}_j|} \times (\boldsymbol{\nu}_j \times \boldsymbol{\nu}_{j+1}) \\ &= \frac{1}{q_j} \begin{pmatrix} A_j - A_{j+1} \\ B_j - B_{j+1} \\ 0 \end{pmatrix} \times \frac{1}{\sqrt{q_j^2 + \tau^2 p_j^2}} \begin{pmatrix} B_{j+1} - B_j \\ A_j - A_{j+1} \\ \tau(A_j B_{j+1} - A_{j+1} B_j) \end{pmatrix} \\ &= \frac{1}{q_j \sqrt{\tau^2 p_j^2 + q_j^2}} \begin{pmatrix} \tau(B_j - B_{j+1}) p_j \\ \tau(A_{j+1} - A_j) p_j \\ q_j^2 \end{pmatrix}. \end{aligned}$$

Let s_j be the speed of E_j. Let $2\theta_j$ $(0 < \theta_j < \pi/2)$ be the angle between S_j and S_{j+1}. Then we have

$$s_j \sin\theta_j = k.$$

The angle between $\boldsymbol{\nu}_j$ and $|\boldsymbol{\nu}_{j+1} - \boldsymbol{\nu}_j|^{-1}(\boldsymbol{\nu}_{j+1} - \boldsymbol{\nu}_j) \times (\boldsymbol{\nu}_j \times \boldsymbol{\nu}_{j+1})$ equals $\pi/2 - \theta_j$.

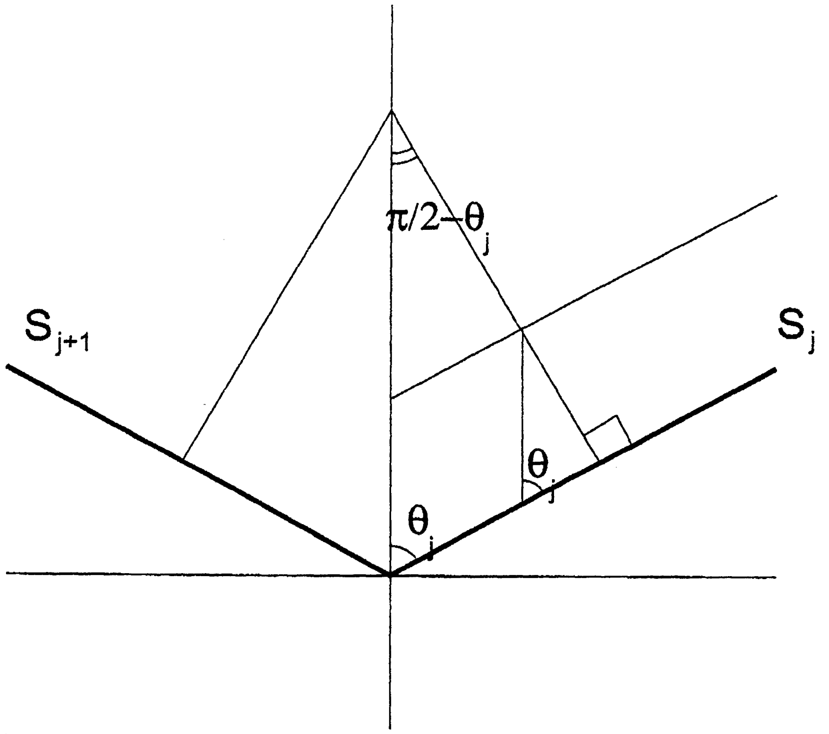

Figure 7: The angle between surfaces S_j and S_{j+1}

We get

$$\sin\theta_j = \frac{\sqrt{\tau^2 p_j^2 + q_j^2}}{q_j\sqrt{1+\tau^2}}$$

and thus

$$s_j = \frac{cq_j}{\sqrt{\tau^2 p_j^2 + q_j^2}}.$$

The speed of E_j toward the z-axis equals

$$\frac{\sqrt{\tau^2 p_j^2 + q_j^2}}{q_j} s_j = k\sqrt{1+\tau^2} = c,$$

which coincides with the speed of V. Since we are now using the moving coordinate, this fact suggests that E_j satisfies $\mathcal{L}(E_j) = 0$. We will check this later. We use the following change of variables

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_j \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \quad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = R_j^T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where R_j^T is the transposed matrix of R_j. Here we set

$$R_j \stackrel{\text{def}}{=} \begin{pmatrix} \dfrac{A_j - A_{j+1}}{q_j} & \dfrac{\tau(B_j - B_{j+1})p_j}{q_j\sqrt{\tau^2p_j^2 + q_j^2}} & \dfrac{B_j - B_{j+1}}{\sqrt{\tau^2p_j^2 + q_j^2}} \\ \dfrac{B_j - B_{j+1}}{q_j} & \dfrac{\tau(A_{j+1} - A_j)p_j}{q_j\sqrt{\tau^2p_j^2 + q_j^2}} & \dfrac{A_{j+1} - A_j}{\sqrt{\tau^2p_j^2 + q_j^2}} \\ 0 & \dfrac{q_j}{\sqrt{\tau^2p_j^2 + q_j^2}} & -\dfrac{\tau p_j}{\sqrt{\tau^2p_j^2 + q_j^2}} \end{pmatrix}.$$

and

$$(R_j)^T = \begin{pmatrix} \dfrac{A_j - A_{j+1}}{q_j} & \dfrac{B_j - B_{j+1}}{q_j} & 0 \\ \dfrac{\tau(B_j - B_{j+1})p_j}{q_j\sqrt{\tau^2p_j^2 + q_j^2}} & \dfrac{\tau(A_{j+1} - A_j)p_j}{q_j\sqrt{\tau^2p_j^2 + q_j^2}} & \dfrac{q_j}{\sqrt{\tau^2p_j^2 + q_j^2}} \\ \dfrac{B_j - B_{j+1}}{\sqrt{\tau^2p_j^2 + q_j^2}} & \dfrac{A_{j+1} - A_j}{\sqrt{\tau^2p_j^2 + q_j^2}} & -\dfrac{\tau p_j}{\sqrt{\tau^2p_j^2 + q_j^2}} \end{pmatrix}.$$

Now we define E_j as

$$E_j(x, y, z) \stackrel{\text{def}}{=} v_*\Biggl(\frac{(A_j - A_{j+1})x + (B_j - B_{j+1})y}{q_j}, \frac{\tau(B_j - B_{j+1})p_jx + \tau(A_{j+1} - A_j)p_jy + q_j^2z}{q_j\sqrt{\tau^2p_j^2 + q_j^2}}; \frac{cq_j}{\sqrt{\tau^2p_j^2 + q_j^2}}\Biggr)$$

Then after calculations we obtain

$$\mathcal{L}[E_j] = - (v_*)_{\xi\xi}(\xi, \eta; s_j) - (v_*)_{\eta\eta}(\xi, \eta; s_j) - s_j(v_*)_\eta(\xi, \eta; s_j) - f(v_*(\xi, \eta; s_j)) = 0$$

in $\mathbb{R}^3$. Thus for each j $(1 \le j \le n)$ $E_j(\boldsymbol{x})$ satisfies (3). We call E_j a planar V-form front associated with an edge Γ_j.

We put

$$Q_j \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^3 \mid \operatorname{dist}(\boldsymbol{x}, \Gamma) = \operatorname{dist}(\boldsymbol{x}, \Gamma_j)\} \qquad \text{for } 1 \le j \le n.$$

Then we have

$$\mathbb{R}^3 = \cup_{j=1}^{n} Q_j.$$

We define

$$\widehat{E}(\boldsymbol{x}) \stackrel{\text{def}}{=} \max_{1\le j\le n} E_j(\boldsymbol{x}).$$

Since E_j is strictly monotone decreasing in z for each j, $\widehat{E}$ is also strictly monotone decreasing in z. It satisfies

$$\underline{v}(\boldsymbol{x}) < \widehat{E}(\boldsymbol{x}) < V(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^3$$

and

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} \left| \widehat{E}(\boldsymbol{x}) - \underline{v}(\boldsymbol{x}) \right| = 0. \tag{19}$$

A pyramidal traveling front is uniquely determined as a combination of two-dimensional V-form fronts.

Corollary 4 ([22]) *Let h be as in (7) and let V be the pyramidal traveling wave associated with $z = h(x, y)$, that is, V satisfies (3) and (9). If (3) has a solution v with*

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R} \left| v(\boldsymbol{x}) - \widehat{E}(\boldsymbol{x}) \right| = 0,$$

then one has $v \equiv V$.

Thus a three-dimensional traveling wave is uniquely determined as a combination of two-dimensional V-form waves.

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