数理解析研究所講究録
第1652巻 2009年 44-54

Interesting Variants of the Josephus Problem. -How high school students can discover theorems using computer algebra systems.-

宮寺良平
関西学院高等部*
RYOHEI MIYADERA
KWANSEI GAKUIN

峰松大介
関西学院大学†
DAISUKE MINEMATSU
KWANSEI GAKUIN

松井啓史
関西学院高等部‡
HIROSHI MATSUI
KWANSEI GAKUIN

山内俊幸
関西学院高等部§
TOSHIYUKI YAMAUCHI
KWANSEI GAKUIN

内藤昌宗
関西学院高等部¶
MASAKAZU NAITO
KWANSEI GAKUIN

巽創
関西学院高等部||
SOH TATSUMI
KWANSEI GAKUIN

井上貴文
関西学院高等部**
TAKAHUMI INOUE
KWANSEI GAKUIN

1 Introduction

This article has 2 purposes. One is to present new theorems of variants of the famous Josephus Problem. This will be interesting for mathematicians who study discrete mathematics. Another is to present a method for high school students to discover theorems of mathematics using computer algebra systems, and we are going to use the research on the Josephus problem as an example of the method. This will be interesting for mathematics teachers who are looking for new methods of education.

We have studied two variants of the Josephus problem. In the first variant, two processes of elimination intersect each other, and we have discovered interesting theorems on the self-similarity of the graph that is produced by the variant.

*miyadera1272000@yahoo.co.jp
†comp.mine@gmail.com
‡hiroshi_m27@yahoo.co.jp
§toshi _20_9@yahoo.co.jp
¶mailadress-naninisiyo@ezweb.ne.jp
||momonja@wm.pdx.ne.jp
**paul-inoue1218@k.vodafone.ne.jp

In the second variant the process of elimination moves on a line, and we have discovered some interesting facts.

We have already introduced the research by high school students in [1], but in this article we are going to present a very effective method to carry out the research by high school students who use computer algebra systems.

2 The traditional Josephus problem.

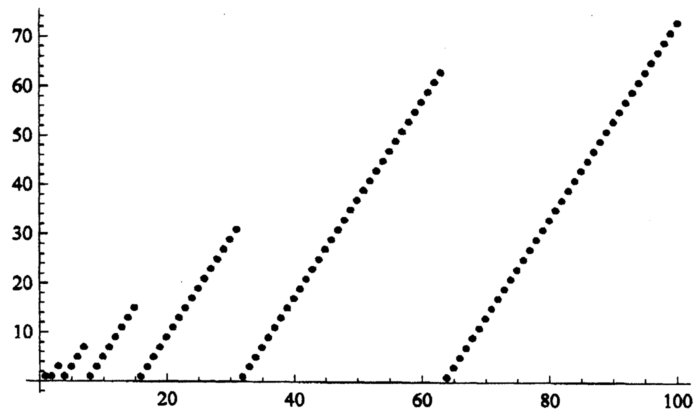
In our method we usually begin the research with an introduction of interesting problems.
This time the students studied the traditional Josephus Problem.

According to a legend, Josephus was the leader of Jewish rebels trapped by the Romans. His subordinates preferred suicide to surrender, so they decided to form a circle and eliminate every other person until no one was left. Josephus wanted to live, so he calculated where to stand and managed to be the last person. If there were n persons, where did Josephus stand? We denote by $J(n)$ the position of the surviver.

Theorem 1
$J(2m) = 2J(m) - 1$ *and* $J(2m+1) = 2J(m) + 1$.

This is a well known formula. See [2].
Since $J(1) = 1$, by Theorem 1 we can calculate $J(n)$ for any natural number n.

Example 1
The graph of the list {J(n) , n = 1, 2, 3, ..., 100}.



As you can see, the graph of the function $J(n)$ is very simple. In Examples 3, 4, 5, 9, 10, 11, 12 we will find that the graphs of the variants are very different from this one.

3 A Josephus problem with an intersection

After students study the original problem, the teacher usually asks them to modify the problem.
This time students proposed the following variant of the Josephus problem with an intersection.

In this variant of the Josephus Problem two persons are to be eliminated at the same time, but the two processes of elimination go in different directions. Suppose that there are n-persons. Then the first process of elimination starts with the 1st person and the 2nd, 4th person,...are to be eliminated. The second process starts with the n-th person, and the $(n-1)$-th, $(n-3)$-th person, ... are to be eliminated. We suppose that the first process comes first and the second process second at every stage. We denote the position of the surviver by $JI(n)$.

When students propose a good problem, then they usually begin to make a program using Mathematica.

Example 2

In our method we usually make a Mathematica program to study the problem that we have.

This is a Mathematica function to calculate $JI(m)$. It is based on a very simple algorithm.

```
JI[m_] := Block[t, p, q, u, v,
t = Range[m];
p = t; q = t; Do[
p = RotateLeft[p, 1];
u = First[p]; p = Rest[p];
q = Drop[q, Position[q, u][[1]]];
If[Length[p] == 1, Break[],];
q = RotateRight[q, 1];
v = Last[q]; q = Drop[q, -1];
p = Drop[p, Position[p, v][[1]]];
If[Length[q] == 1, Break[],],
n, 1, Ceiling[m/2]]; p[[1]]];
```

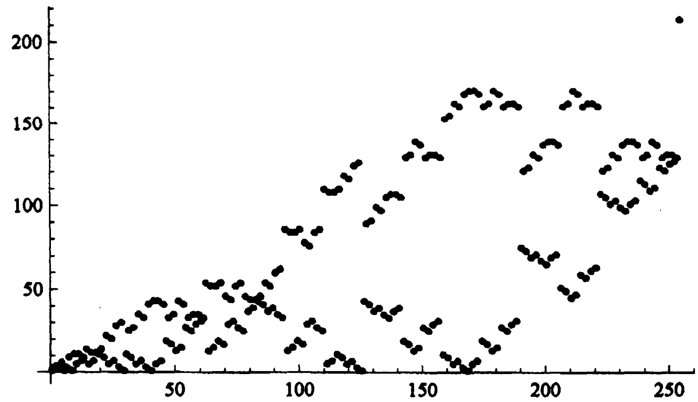
Remark 1

Note that this program is very simple, and it takes little time to make.

As to the Mathematica program for discrete mathematics see [3].

Example 3

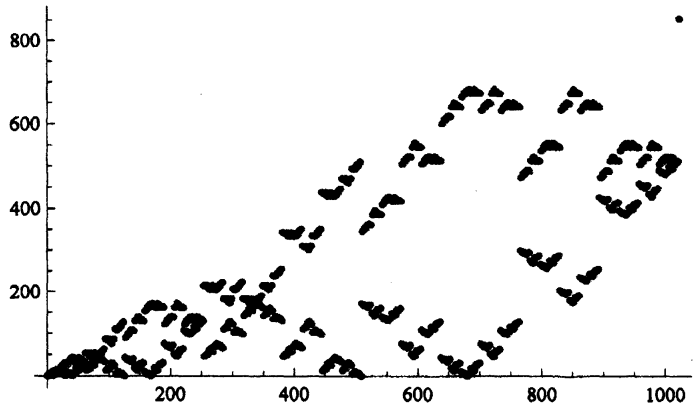
The graph of the list {JI(n) , n = 2, 3, ..., 256}. The horizontal coordinate is for the number of people involved in the game, and the vertical coordinate is for the position of the survivor. For example by $JI(256) = 214$ *we have the point* $(256, 214)$ *in the graph.*



The mathematical structure of the graph in Example 1 is quite clear. On the other hand there seems to be no mathematical structure in the graph of Example 3.

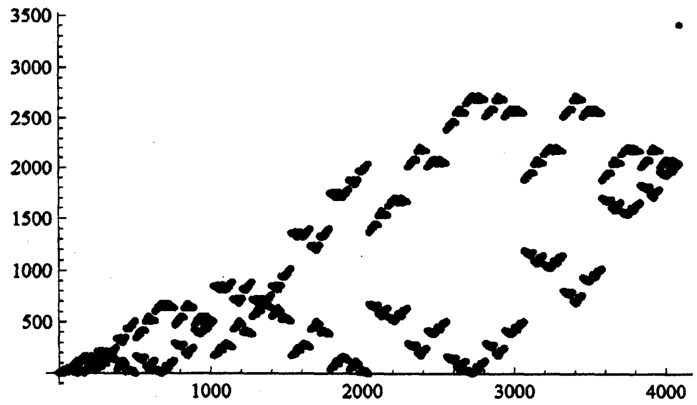
Example 4

The graph of the list {JI(n) , n = 2, 3, ..., 1024}.



Example 5

The graph of the list {JI(n) , n = 2, 3, ..., 4096}.



If we compare the graphs in Example 3,4 and 5, we can discover a very interesting fact. That is the existence of self-similarity, but you need to get recursive relations for $JI(n)$ to prove the existence of self-similarity. We used the program in Example 2 to get the recursive relations in Theorem 2.

Theorem 2

(1) $JI(8n) = 4JI(2n) - 1 - \lfloor JI(2n)/(n+1) \rfloor$.

(2) $JI(8n+1) = 8n + 5 - 4JI(2n)$.

(3) $JI(8n+2) = 4JI(2n) - 3 - \lfloor JI(2n)/(n+2) \rfloor$

(4) $JI(8n+3) = 8n + 7 - 4JI(2n)$.

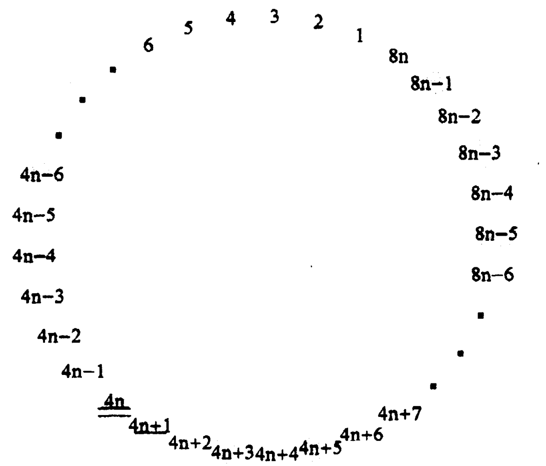
(5) $JI(8n+4) = 8n + 8 - 4JI(2n+1) + \lfloor JI(2n+1)/(n+2) \rfloor$.

(6) $JI(8n+5) = 4JI(2n+1) - 1$.

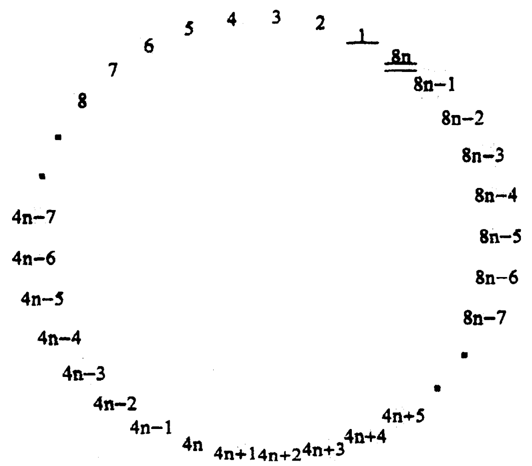
(7) $JI(8n+6) = 8n + 10 - 4JI(2n+1) + \lfloor (JI(2n+1)/(n+2) \rfloor$.

(8) $JI(8n+7) = 4JI(2n+1) - 3$.

Proof (1) We suppose that there are 8n persons. The first process begins to eliminate them, starting with the 2nd person, while the second process starts with the (8n-1)-th person. When the two processes have eliminated 4n persons, 4n persons remain. See Figure 1.
Figure 1.



After this, the two processes are going to intersect each other.
When 6n persons are eliminated, 2n persons remain. See Figure 2.
Figure 2.



Since there are 2n persons remaining, the value $JI(8n)$ depends on the value of $JI(2n)$. Let $JI(2n) = k$. If $k \leq n$, then by the above graph, it is easy to see that
$JI(8n) = 4JI(2n) - 1$.
If $k \geq n+1$, then by the above graph, it is easy to see that
$JI(8n) = 4JI(2n) - 2$.
We have proved (1) of Theorem 2.
Similarly we can prove (2), (3), (4), (5), (6), (7) and (8) of Theorem 2. We are going to omit the proofs of these. ∎

Example 6

You can make a Mathematica function based on Theorem 2 to calculate $JI(m)$.
$JI[m\_] := JI[m] = Block[n, h, h = Mod[m, 8]; n = (m - h)/8;$
$Which[h == 0, 4JI[2n] - 1 - Floor[JI[2n]/(n+1)], h == 1,$
$8n + 5 - 4JI[2n], h == 2, 4JI[2n] - 3 - Floor[JI[2n]/(n+2)],$
$h == 3, 8n + 7 - 4JI[2n], h == 4,$
$8n + 8 - 4JI[2n+1] + Floor[JI[2n+1]/(n+2)], h == 5,$
$4JI[2n+1] - 1, h == 6,$
$8n + 10 - 4JI[2n+1] + Floor[JI[2n+1]/(n+2)], h == 7,$
$4JI[2n+1] - 3]]$

Remark 2

We made the Mathematica function JI[n] using the recursive relations in Theorem 2, but we used this Mathematica function to check if the recursive relations are correct. It is a very complicated job to find recursive relations and it is quite easy to make mistakes, but Mathematica can make the job a lot easier.

Now we are going to prove the existence of self-similarity in the graph of $JI(n)$.
For any $x = (x_1, x_2)$, $y = (y_1, y_2)$ we define $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, and $d(x, A) = \inf_{y \in A} d(x, y)$. We define the distance between two subsets of R^2 by $\delta(A, B) = \mathrm{Max}(\sup_{x \in A} d(x, B), \sup_{y \in B} d(x, A))$.

We define $R_{s,h,K} = \{(sn+h, JI(sn+h)); n \le K\}$ and $S_{s,h,K} = \{(sn+h, sn+h-JI(sn+h)); n \le K\}$ for natural numbers s, h and K with $h < s$.

Theorem 3

$\lim_{K \to \infty} \frac{\delta(R_{2,1,K}, S_{2,0,K})}{K} = 0.$

Proof If we divide even numbers into 4 groups of integers, then by Theorem 2 we have

$$\begin{aligned}
S_{2,0,4K} &= \{(2n, 2n - JI(2n)); n \le 4K\} \\
&= \{(8n, 8n - JI(8n)); n \le K\} \cup \{(8n+2, 8n+2 - JI(8n+2)); n \le K-1\} \\
&\cup \{(8n+4, 8n+4 - JI(8n+4)); n \le K-1\} \cup \{(8n+6, 8n+6 - JI(8n+6)); n \le K-1\} \\
&= \{(8n, 8n - (4JI(2n) - 1 - \lfloor \tfrac{JI(2n)}{n+1} \rfloor)); n \le K\} \cup \{(8n+2, 8n+2 - (4JI(2n) - 3 - \lfloor \tfrac{JI(2n)}{n+2} \rfloor)); n \le K-1\} \\
&\cup \{(8n+4, 8n+4 - (8n+8 - 4JI(2n+1) + \lfloor \tfrac{JI(2n+1}{n+2} \rfloor))); n \le K-1\} \\
&\cup \{(8n+6, 8n+6 - (8n+10 - 4JI(2n+1) + \lfloor (\tfrac{JI(2n+1}{n+2} \rfloor)); n \le K-1\} \\
&= \{(8n, 8n+1 + \lfloor \tfrac{JI(2n)}{n+1} \rfloor - 4JI(2n)); n \le K\} \cup \{(8n+2, 8n+5 + \lfloor \tfrac{JI(2n)}{n+2} \rfloor) - 4JI(2n)); n \le K-1\} \\
&\cup \{(8n+4, 4JI(2n+1) - 4 - \lfloor \tfrac{JI(2n+1)}{n+2} \rfloor); n \le K-1\} \cup \{(8n+6, 4JI(2n+1) - 4 - \lfloor (\tfrac{JI(2n+1)}{n+2} \rfloor); n \le K-1\}. \qquad (1)
\end{aligned}$$

If we divide odd numbers into 4 groups of integers, then by Theorem 2 we have

$$\begin{aligned}
R_{2,1,4K} &= \{(2n+1, JI(2n+1)); n \le 4K\} = \{(8n+1, JI(8n+1)); n \le K\} \cup \{(8n+3, JI(8n+3)); n \le K-1\} \\
&\cup \{(8n+5, JI(8n+5)); n \le K-1\} \cup \{(8n+7, JI(8n+7)); n \le K-1\} \\
&= \{(8n+1, 8n+5 - 4JI(2n)); n \le K\} \cup \{(8n+3, 8n+7 - 4JI(2n))); n \le K-1\} \\
&\cup \{(8n+5, 4JI(2n+1) - 1); n \le K-1\} \cup \{(8n+7, 4JI(2n+1) - 3); n \le K-1\}. \qquad (2)
\end{aligned}$$

Now we are going to compare the first term of (1) and the first term of (2).
Let $A = \{(8n, 8n+1+\lfloor\frac{JI(2n)}{n+1}\rfloor - 4JI(2n)); n \leq K\}$ and
$B = \{(8n+1, 8n+5-4JI(2n)); n \leq K\}$. It is clear that
$\lim_{K\to\infty} \frac{\delta(A,B)}{K} = 0$,
since natural numbers 1,5 and $\lfloor\frac{JI(2n)}{n+1}\rfloor$ are relatively small compared to K when K is very large. We can do the same thing for the second, third and fourth terms of (1) and (2), and hence we have

$\lim_{K\to\infty} \frac{\delta(R_{2,1,4K}, S_{2,0,4K})}{K} = 0$.
Since K is an arbitrary natural number, we can finish the proof. ∎

Remark 3
By Theorem 3 $R_{2,1,K}$ can be said to be very similar to $S_{2,0,K}$ as subsets of R^2 when the number K is large. We express this fact by $R_{2,1,K} \sim S_{2,0,K}$.

Theorem 4
$R_{2,0,K} \sim S_{2,1,K}$.

Proof We are going to omit the proof of this theorem, since we can prove this by the same method that we used in Theorem 3. ∎

Theorem 5
$R_{1,0,4K} \sim 4R_{1,0,K}$ for any natural number K , and hence there is a self-similarity in the graph of $JI(n)$.

Proof By Theorem 2 it is clear that

$$R_{8,0,K-1} \sim 4R_{2,0,K-1} \sim R_{8,2,K-1},\ R_{8,1,K-1} \sim 4S_{2,0,K-1} \sim R_{8,3,K-1}, \tag{3}$$

$$R_{8,4,K-1} \sim 4S_{2,1,K-1} \sim R_{8,6,K-1} \ \ \textit{and} \ \ R_{8,5,K-1} \sim 4R_{2,1,K-1} \sim R_{8,7,K-1}. \tag{4}$$

By Theorem 3 and Theorem 4 we have
$4R_{2,0,K-1} \sim 4S_{2,1,K-1}$ and $4R_{2,1,K-1} \sim 4S_{2,0,K-1}$, and hence by (3) and (4)

$$R_{8,0,K-1} \sim R_{8,4,K-1} \ \ \textit{and} \ \ R_{8,1,K-1} \sim R_{8,5,K-1}. \tag{5}$$

By (3), (4) and (5) we have

$$R_{8,0,K-1} \sim R_{8,2,K-1} \sim R_{8,4,K-1} \sim R_{8,6,K-1} \tag{6}$$

and

$$R_{8,1,K-1} \sim R_{8,3,K-1} \sim R_{8,5,K-1} \sim R_{8,7,K-1}. \tag{7}$$

It is clear that

$$R_{8,1,K-1} \subset R_{8,1,K}. \tag{8}$$

Since
$R_{1,0,8K} = R_{8,0,K} \cup R_{8,1,K-1} \cup R_{8,2,K-1} \cup R_{8,3,K-1} \cup R_{8,4,K-1} \cup R_{8,5,K-1} \cup R_{8,6,K-1} \cup R_{8,7,K-1}$,
by (6), (7) and (8) we have

$$R_{1,0,8K} \sim (R_{8,0,K} \cup R_{8,5,K-1}). \tag{9}$$

By Theorem 2

$$R_{8,0,K} \sim 4R_{2,0,K} \tag{10}$$

and

$$R_{8,5,K-1} \sim 4R_{2,1,K-1}. \tag{11}$$

By the definition of $R_{1,0,2K}$

$$R_{1,0,2K} \sim (R_{2,0,K} \cup R_{2,1,K-1}). \tag{12}$$

Therefore by (9), (10), (11) and (12) we have $R_{1,0,8K} \sim 4R_{1,0,2K}$, , and hence there is a self-similarity in the graph of $JI(n)$. ∎

4 An unsolved problem of the Josephus Problem with an intersection.

Example 7

The list of the sequence $\{JI(n), n = 1, 2, 3, ..., 62\}$*is*
$\{1, 3, 4, 3, 6, 1, 3, 9, 1, 11, 5, 11, 7, 9, 14, 5, 12, 7, 12, 11, 14, 9, 22, 5, 20, 7, 28, 3, 30, 1, 11, 25, 9, 27, 5, 35, 7, 33, 3, 41, 1, 43, 5, 43, 7, 41, 19, 33, 17, 35, 13, 43, 15, 41, 27, 33, 25, 35, 29, 35, 31\}$.
We denote this sequence modulo 2 by $JI(\mod 2)$.
Then $JI(\mod 2)$ *is*
$\{1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$.

We can find a very beautiful pattern if we arrange them as the followings.
$\{1, 1\}$, $\{1, 0, 1, 0\}$, $\{1, 1, 1, 1, 1, 1, 1, 1\}$, $\{1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0\}$,
$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$.
The pattern is almost obvious, but we have not managed to prove the existence of this pattern for $JI(\mod 2)$.
The only thing we can prove easily is the fact that $JI(n)(\mod 2)$ *is odd for any odd number* n. *This is direct from Theorem 2.*

5 Linear Josephus Problem

This is another variant of the Josephus problem. Let n and r be natural numbers. We put n players on a line. We start with the 1st player and move from left to right removing every rth player. We change the direction when we reach the end of the line. Then we begin removing every r th player again. We denote by $JLr(n)$ the position of the last one remaining.
Although this problem had been studied in [5], the students presented this by themselves without knowing [5].

Example 8

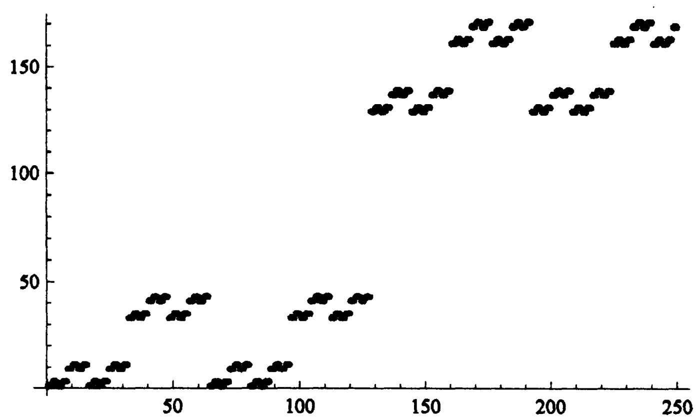
Let $n = 12$ *and* $r = 2$. *We have 12 players, and we are going to remove every second player. When we remove 2,4,6,8,10,12, players 1,3,5,7,9,11 remain. See Figure 3.*
Figure 3

1 2 3 4 5 6 7 8 9 10 11 12

Once we have reached the right end of the line, we move in the opposite direction removing 9,5,1. Then we change the direction again, and remove 7. Then we change the direction again, and remove 3. 11 is the last remaining player. Therefore $JL2(12) = 11$.

Example 9

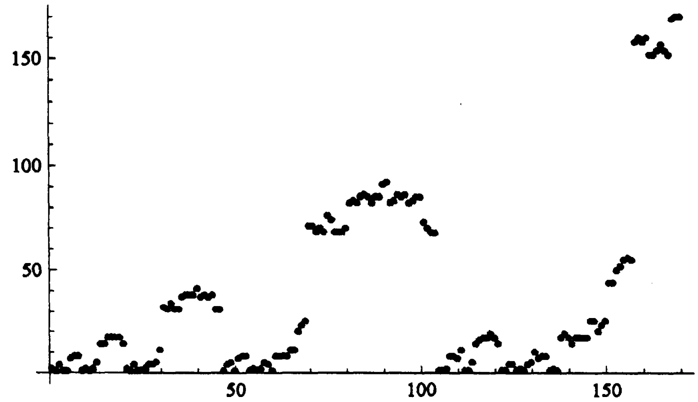
The graph of the list {JL2(n) , $n = 1$*, 2, 3, ..., 256}. This is quite beautiful. The self-similarity of the graph of* $JL2(n)$ *is studied in [5].*



Example 10

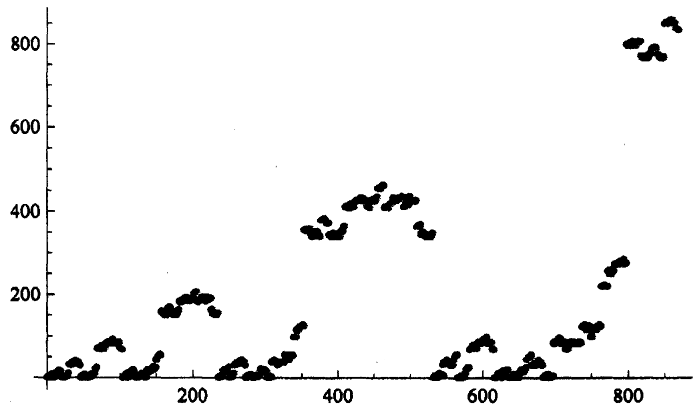
The graph of the list {JL3(n) , $n = 1$*, 2, 3, ..., 170}.*

This graph is complicated, and it looks like the graphs in Example 3, 4 and 5.



Example 11

The graph of the list {JL3(n) , $n = 1$*, 2, 3, ..., 870}*



The existence of the similarity of graphs in Example 10 and 11 seems to be obvious, but we have not proved it.

6 An overview of the research by high school students.

Here we are going to talk about an effective way for high school students to do mathematical research using computer algebra systems.

First, the students study some well known problems, and after that they are asked to modify them. Once they manage to present interesting modifications of the problems, then they begin to study them using computer algebra systems.

We are using the computer algebra system Mathematica. Since it has many mathematical functions, it is usually far easier to make a computer program by using Mathematica than by using general-purpose languages such as C, Java, Basic or Pascal. See Example 2.

Many people say that with computer algebra systems such as Mathematica you can make a program for a mathematical problem in less than a fifth of the amount of time you use with general-purpose languages.

Mathematica is very good at making many kinds of graphics. Graphics are very useful for the research of mathematics, and it is often the case that a good graphical representation can present some hidden structures of the problem. See Example 3,4, 5, 9, 10, 11. The graphs in these examples show the existence of self-similarity.

After we make a lot of data using computer algebra systems, we look for some formulas and patterns. Usually, general-purpose languages are a lot faster in calculations, but computer algebra system is usually enough for students to find important patterns.

A good graphing function in a computer algebra system can reduce the number of errors in the program. In our study of the Josephus problem we made a program that gave us graphical representations, and by looking at them we could find our programming errors.

In our research we used complicated recursive relations, but by using Mathematica we could make correct recursive relations in a short time. See Example 6.

In spite of the advantage of computer algebra systems, general-purpose languages have their strong points. One is the speed of simple repetitive calculation, and the other is the fact that anyone can use the program made by these languages.

In the case of our research we sometimes use C and Java when we need the speed of calculation. We also make Java applets to show our research to the people who do not have Mathematica.

The use of general-purpose languages is important from the viewpoint of education. The ability to use such languages can benefit the students in the future.

The research of mathematics can give students a very good chance to use their skill in general-purpose languages. There are a lot of Java applets on the web all around the world, but there are very few applets that deal with original research. For example, we can find many applets of the traditional Josephus Problem, but we could not find any applets that deal with variants of the Josephus Problem. The students on our team are making a lot of applets for new variants of the Josephus Problem, and they can feel the joy of creating new things.

Acknowledgements

Contributions from Satoshi Hashiba, Tomo Hamada and Takuto Hieda. Although they were not the primary authors, their contributions were significant. We would like to thank Mr. Harrison Gray for helping us to prepare this article.

References

[1] R. Miyadera, S.Hashiba and D.Minematsu, "How high school students can discover original ideas of mathematics using Mathematics", Mathematica in education and research 11 (3), 2006.

[2] R. Graham, D. Knuth and O. Patashnik, "Concrete Mathematics", Addison-Wesley Publishing Company (1994).

[3] R. Cowen and J. Kennedy, "Discovering Mathematics with Mathematica", Erudition Books, 2001.

[4] R.Miyadera, T.Hashiba, D.Minematsu, H. Matsui, T. Yamauchi, Y. Nakagawa, T. Inoue,"A Mathematical Research Project by High School and Undergraduate students (数式処理システムを利用した高校生や大学初年級の生徒による数学研究)", Journal of Japan Society for Symbolic and Algebraic Computation (数式処理通信) 112, No.4,2006.

[5] Chris Groer, "Mathematics of Survival", American Mathematical Monthly 110 ,2003.