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PATH INTEGRATIONS FOR RELATIVISTIC SCHRÖDINGER OPERATORS AND BERNSTEIN FUNCTIONS

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1 Bernstein functions and Lévy subordinators

1.1 Introduction

This is a review article of [Hir09, HIL09, HS09]. It is proven that the Feynman-Kac type formula is a useful tool to investigate a strongly continuous one parameter semigroup generated by a self-adjoint elliptic operator. The Schrödinger operator with a vector potential $a=(a_1, a_2, a_3)$ and spin $1/2$ is given as a self-adjoint operator on $L^2(\mathbb{R}^3;\mathbb{C}^2)$ and it is defined by

$$h=\frac{1}{2}(\sigma\cdot(\mathrm{p}-a))^2+V, \tag{1.1}$$

where $\sigma=(\sigma_1, \sigma_2, \sigma_3)$ denotes 2×2 Pauli matrices satisfying $\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}I_2$ with the 2×2 identity I_2 and $V:\mathbb{R}^3\to\mathbb{R}$ is an external potential. The path integral representation of the semigroup e^{-th}, $t\geq 0$, is constructed through a Lévy process on a cádlág path space in [ALS83, HIL09]. Instead of this in [ARS91] a path integral representation for the relativistic Schrödinger operator

$$h_{\mathrm{rel}}=\sqrt{(\sigma\cdot(\mathrm{p}-a))^2+m^2}-m+V \tag{1.2}$$

is considered. In terms of the Schrödinger operator h, the relativistic Schrödinger operator h_{rel} can be expressed as

$$h_{\mathrm{rel}}=\sqrt{2h+m^2}-m+V.$$

We would like to extend h_{rel} to a more general form. The function

$$f(u)=\sqrt{2u+m^2}-m, \quad u\geq 0, \quad m\geq 0,$$

satisfies that

$$(1)\ f \in C^\infty((0,\infty)), \quad (2)\ f(u) \geq 0, \quad (3)\ (-1)^n \frac{d^n f}{du^n} \leq 0,\ n \geq 1.$$

In general a real-valued function] satisfying (1) - (3) is called a Bernstein function. So, in this article we will give a functional integral representation of general self-adjoint operators of the form

$$H^\Psi = \Psi\left(\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2\right) + V, \tag{1.3}$$

where Ψ is an arbitrary Bernstein function. Typical examples of Bernstein functions are $\Psi(u) = u^\alpha$, $0 \leq \alpha \leq 1$, and $\Psi(u) = \sqrt{2u + m^2} - m$. The cases we consider include not only the relativistic Schrödinger operator h_{rel} but also fractional Schrödinger operator

$$\left(\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2\right)^\alpha + V, \quad 0 \leq \alpha \leq 1. \tag{1.4}$$

By using a subordinator we will give the Feynman-Kac type formula of the semigroup e^{-tH^Ψ} for an arbitrary Bernstein function Ψ. This path integral representation can be also applied to study the spectral properties of models in quantum field theory. In particular it can be applied to study the Nelson model with a relativistic kinematic term and a relativistic Pauli-Fierz model.

1.2 Bernstein function and subordinators

We refer [BF73] for Bernstein functions.

Definition 1.1 (Bernstein function) Let

$$\mathscr{B} = \left\{ f \in C^\infty((0,\infty)) \,\middle|\, f(x) \geq 0 \text{ and } (-1)^n \left(\frac{d^n f}{dx^n}\right)(x) \leq 0 \text{ for all } n = 1, 2, \ldots, \right\}.$$

An element of $\mathscr{B}$ is called a *Bernstein function.* We also define the subclass

$$\mathscr{B}_0 = \left\{ f \in \mathscr{B} \,\middle|\, \lim_{u \to 0+} f(u) = 0 \right\}.$$

Bernstein functions are positive, increasing and concave. Examples of functions in $\mathscr{B}_0$ include $\Psi(u) = cu^\alpha$, $c \geq 0$, $\alpha \in (0,1]$, and $\Psi(u) = 1 - e^{-au}$, $a \geq 0$.

Let $\mathscr{L}$ be the set of Borel measures λ on $\mathbb{R} \setminus \{0\}$ such that

$$(1)\ \lambda((-\infty, 0)) = 0, \quad (2)\ \int_{\mathbb{R}\setminus\{0\}} (y \wedge 1)\lambda(dy) < \infty.$$

Note that each $\lambda \in \mathscr{L}$ satisfies that $\int_{\mathbb{R}\setminus\{0\}}(y^2 \wedge 1)\lambda(dy) < \infty$ so that λ is a Lévy measure. Denote $\mathbb{R}_+ = [0, \infty)$. We give the integral representation of Bernstein functions with vanishing right limits at the origin. It is well known that for each Bernstein function $\Psi \in \mathscr{B}_0$ there exists $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ such that

$$\Psi(u) = bu + \int_0^\infty (1 - e^{-uy})\lambda(dy). \tag{1.5}$$

Conversely, the right hand side of (1.5) is in $\mathscr{B}_0$ for each pair $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$. Thus the map $\mathscr{B}_0 \longrightarrow \mathbb{R}_+ \times \mathscr{L}$, $\Psi \mapsto (b, \lambda)$ is a one-to-one correspondence.

Next we consider a probability space $(\Omega_\nu, \mathscr{F}_\nu, \nu)$ given, with $\Omega_\nu \subset \mathbb{R}$, and the following special class of Lévy processes.

Definition 1.2 (Lévy subordinator) A random process $(T_t)_{t\geq 0}$ on $(\Omega_\nu, \mathscr{F}_\nu, \nu)$ is called a *(Lévy) subordinator* whenever

(1) $(T_t)_{t\geq 0}$ is a Lévy process starting at 0, i.e., $\nu(T_0 = 0) = 1$;

(2) T_t is almost surely non-decreasing in t.

Subordinators have thus independent and stationary increments, almost surely no negative jumps, and are of bounded variation. These properties also imply that they are Markov processes. Let $\mathscr{S}$ denote the set of subordinators on $(\Omega_\nu, \mathscr{F}_\nu, \nu)$. In what follows we denote expectation by $\mathbb{E}_m^x[\cdots] = \int \cdots dm^x$ with respect to the path measure m^x of a process starting at x.

Proposition 1.3 *Let $\Psi \in \mathscr{B}_0$ or, equivalently, a pair $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ be given. Then there exists a unique $(T_t)_{t\geq 0} \in \mathscr{S}$ such that*

$$\mathbb{E}_\nu^0[e^{-uT_t}] = e^{-t\Psi(u)}. \tag{1.6}$$

Conversely, let $(T_t)_{t\geq 0} \in \mathscr{S}$. Then there exists $\Psi \in \mathscr{B}_0$, i.e., a pair $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ such that (1.6) is satisfied.

In particular, (1.5) coincides with the Lévy-Khintchine formula for Laplace exponents of subordinators.

By the above there is a one-to-one correspondence between $\mathscr{B}_0$ and $\mathscr{S}$, or equivalently, between $\mathscr{B}_0$ and $\mathbb{R}_+ \times \mathscr{L}$. For clarity, we will use the notation T_t^Ψ for the Lévy subordinator associated with $\Psi \in \mathscr{B}_0$.

Example 1.4 (First hitting time) Since $\Psi(u) = \sqrt{2u + m^2} - m \in \mathscr{B}_0$ for $m \geq 0$, there exists $T_t^\Psi \in \mathscr{S}$ such that

$$\mathbb{E}_\nu^0[e^{-uT_t^\Psi}] = \exp\left(-t(\sqrt{2u + m^2} - m)\right).$$

This case is thus related with the one-dimensional 1/2-stable process and it is known that the corresponding subordinator T_t^Ψ can be represented as the first hitting time process $T_t^\Psi = \inf\{s > 0 \,|\, B_s + ms = t\}$ for one-dimensional Brownian motion $(B_t)_{t\geq 0}$.

1.3 Quantum field theory

The Nelson model: The Nelson model describes a linear interaction between quantum particles and a scalar field. Let $\mathscr{F} = \oplus_{n=0}^\infty L^2_{\rm sym}(\mathbb{R}^{3N})$, and $a^\dagger(k)$ and $a(k)$ denote the creation operator and the annihilation operator, respectively, which satisfy $[a(k), a^\dagger(k')] = \delta(k-k')$. The Hamiltonian of the Nelson model with kinetic term $\Psi(\mathrm{p}^2/2)$ is defined as a self-adjoint operator on

$$L^2(\mathbb{R}^3)\otimes\mathscr{F} \quad (\cong \int_{\mathbb{R}^3}^\oplus \mathscr{F}dx)$$

by

$$H_N = \big(\Psi(\mathrm{p}^2/2)+V\big)\otimes 1 + 1\otimes H_{\rm f} + \alpha\phi_{\hat\varphi}. \tag{1.7}$$

Here

$$H_{\rm f} = \int |k| a^\dagger(k)a(k)dk$$

is the free Hamiltonian on $\mathscr{F}$ and $\phi_{\hat\varphi}$ denotes a scalar field smeared by a cutoff function $\hat\varphi$ given by

$$\phi_{\hat\varphi} = \int_{\mathbb{R}^3}^\oplus \phi_{\hat\varphi}(x)dx,$$

where

$$\phi_{\hat\varphi}(x) = \frac{1}{\sqrt{2}}\int\left(a^\dagger(k)\frac{e^{-ikx}\hat\varphi(k)}{\sqrt{|k|}} + a(k)\frac{e^{+ikx}\hat\varphi(-k)}{\sqrt{|k|}}\right)dk.$$

Relativistic Pauli-Fierz model: Next let us introduce a relativistic Pauli-Fierz model. Let $\mathscr{F}_{\rm PF} = \oplus_{n=0}^\infty L^2_{\rm sym}(\mathbb{R}^{3n}\times\{-1,1\})$, and $a^\dagger(k,j)$ and $a(k,j)$, $j=1,2$, are the creation operator and the annihilation operator, respectively, which satisfy $[a(k,j), a^\dagger(k',j')] = \delta_{jj'}\delta(k-k')$. Relativistic Pauli-Fierz model describes a minimal coupling between quantum particles and a quantized radiation field. The Hamiltonian of the relativistic Pauli-Fierz model is defined as a self-adjoint operator on

$$L^2(\mathbb{R}^3)\otimes\mathscr{F}_{\rm PF} \quad (\cong \int_{\mathbb{R}^3}^\oplus \mathscr{F}_{\rm PF}dx)$$

by

$$H_{\rm PF} = \sqrt{(\mathrm{p}\otimes 1 - \alpha A_{\hat\varphi})^2 + m^2} - m + V\otimes 1 + 1\otimes H_{\rm f}^{\rm PF}. \tag{1.8}$$

Here

$$H_{\rm f}^{\rm PF} = \sum_{j=1,2} \int |k| a^\dagger(k,j) a(k,j) dk$$

is the free Hamiltonian on $\mathscr{F}_{\rm PF}$ and $A_{\hat\varphi}$ is a quantized radiation field smeared $\hat\varphi$ given by

$$A_{\hat\varphi} = \int_{\mathbb{R}^3}^{\oplus} A_{\hat\varphi}(x) dx,$$

where

$$A_{\hat\varphi}(x) = \frac{1}{\sqrt{2}} \sum_{j=1,2} \int e(k,j) \left(a^\dagger(k,j) \frac{e^{-ikx} \hat\varphi(k)}{\sqrt{|k|}} + a(k,j) \frac{e^{+ikx} \hat\varphi(-k)}{\sqrt{|k|}} \right) dk.$$

$e(k,j)$ denotes polarization vectors such that $k \cdot e(k,j) = 0$, $e(k,1) \cdot e(k,2) = 0$, $e(k,1) \times e(k,2) = k/|k|$ and $|e(k,j)| = 1$. The Hamiltonian (1.8) can be mathematically generalized as

$$\Psi \left(\frac{1}{2} ({\rm p} \otimes 1 - A_{\hat\varphi})^2 \right) + V \otimes 1 + 1 \otimes H_{\rm f}^{\rm PF} \tag{1.9}$$

by a Bernstein function Ψ.

It is a crucial issue to study the spectrum of $H_* = H_N, H_{\rm PF}$, since all the eigenvalue of these Hamiltonian with $\alpha = 0$ are embedded in the continuum. Thus in order to see the spectral properties of H_* but $\alpha \neq 0$ the regular perturbation theory [Kat76] can not be applied directly. One advantage to use a path integral representation of e^{-tH_*} is to be non-perturbative. We discuss quantum field models in [HS09] and [Hir09].

2 Path integrals

2.1 Generalized Schrödinger operators with no spin

Throughout we consider spinless Schrödinger operators for simplicity and we will use the following conditions on the vector potential.

Assumption 2.1 *The vector potential $a = (a_1, ..., a_d)$ is a vector-valued functions whose components a_μ, $\mu = 1, ..., d$, are real-valued functions. Furthermore, we consider the following regularity conditions:*

(A1) $a \in (L^2_{\rm loc}(\mathbb{R}^3))^3$.

(A2) $a \in (L^2_{\rm loc}(\mathbb{R}^3))^3$ *and* $\nabla \cdot a \in L^1_{\rm loc}(\mathbb{R}^3)$.

(A3) $a \in (L^4_{\rm loc}(\mathbb{R}^3))^3$ *and* $\nabla \cdot a \in L^2_{\rm loc}(\mathbb{R}^3)$.

Let $\partial_\mu : \mathscr{D}'(\mathbb{R}^3) \to \mathscr{D}'(\mathbb{R}^3)$, $\mu = 1, ..., d$, denote the μth derivative on the Schwartz distribution space $\mathscr{D}'(\mathbb{R}^3)$. Let $\mathrm{p}_\mu = -i\partial_\mu$ and $\mathrm{D}_\mu = \mathrm{p}_\mu - a_\mu$, $\mu = 1, ..., d$. Define the quadratic form

$$q(f, g) = \sum_{\mu=1}^{3} (\mathrm{D}_\mu f, \mathrm{D}_\mu g) \tag{2.1}$$

with domain $Q(q) = \{f \in L^2(\mathbb{R}^3) \mid \mathrm{D}_\mu f \in L^2(\mathbb{R}^3), \mu = 1, ..., d\}$. It can be seen that $Q(q)$ is complete with respect to the norm $\|f\|_q = \sqrt{q(f, f) + \|f\|^2}$ under Assumption (A1). By this q is a non-negative closed form and thus there exists a unique self-adjoint operator h satisfying $(hf, g) = q(f, g)$ for $f \in D(h)$ and $g \in Q(q)$ with domain $D(h) = \{f \in Q(q) \mid q(f, \cdot) \in L^2(\mathbb{R}^3)'\}$. The self-adjoint operator h is our main object in this section. We summarize some facts about the form core and operator core of h [LS81].

Proposition 2.2 *(1) Let Assumption (A1) hold. Then $C_0^\infty(\mathbb{R}^3)$ is a form core of h. (2) Let Assumption (A3) hold. Then $C_0^\infty(\mathbb{R}^3)$ is an operator core for h.*

Note that in case (2) of Proposition 2.2,

$$hf = \frac{1}{2}\mathrm{p}^2 f - a \cdot \mathrm{p} f + \left(-\frac{1}{2} a \cdot a - (\mathrm{p} \cdot a)\right) f.$$

Let $\Psi \in \mathscr{B}_0$ and take Assumption (A1). Whenever V is bounded we call

$$H^\Psi = \Psi(h) + V \tag{2.2}$$

generalized Schrödinger operator with vector potential a. Note that $\Psi \geq 0$ and $\Psi(h)$ is defined through the spectral projection of the self-adjoint operator h.

Theorem 2.3 *Take $\Psi \in \mathscr{B}_0$.*

(1) Let Assumption (A3) hold. Then $C_0^\infty(\mathbb{R}^3)$ is an operator core of $\Psi(h)$.

(2) Let Assumption (A1) hold. Then $C_0^\infty(\mathbb{R}^3)$ is a form core of $\Psi(h)$.

2.2 Singular magnetic fields

Before constructing a functional integral representation of e^{-th}, we extend stochastic integration to a class including $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ functions since the vector potentials we consider may be more singular.

Let $(B_t)_{t\geq 0}$ denote d-dimensional Brownian motion starting at $x \in \mathbb{R}^3$ on standard Wiener space $(\Omega_P, \mathscr{F}_P, dP^x)$. Let f be a $\mathbb{C}^3$-valued Borel measurable function on $\mathbb{R}^3$ such that

$$\mathbb{E}_P^x\left[\int_0^t |f(B_s)|^2 ds\right] < \infty. \tag{2.3}$$

Then the stochastic integral $\int_0^t f(B_s)\cdot dB_s$ is defined as a martingale and the Itô isometry $\mathbb{E}_P^x\left[\left|\int_0^t f(B_s)\cdot dB_s\right|^2\right] = \mathbb{E}_P^x\left[\int_0^t |f(B_s)|^2 ds\right]$ holds. However, vector potentials a under Assumption 2.1 do not necessarily satisfy (2.3). As we show next, a stochastic integral can indeed be defined for a wider class of functions than (2.3), and then $\int_0^t f(B_s)\cdot dB_s$ will be defined as a local martingale instead of a martingale. This extension will allow us to derive a functional integral representation of e^{-th} with $a \in (L^2_{\rm loc}(\mathbb{R}^3))^3$.

Consider the following class of vector valued functions on $\mathbb{R}^3$.

Definition 2.4 We say that $f = (f_1, ..., f_d) \in \mathscr{E}_{\rm loc}$ if and only if for all $t \geq 0$

$$P^x\left(\int_0^t |f(B_s)|^2 ds < \infty\right) = 1. \tag{2.4}$$

Let $R_n(\omega) = n \wedge \inf\left\{t \geq 0 \,\middle|\, \int_0^t |f(B_s(\omega))|^2 ds \geq n\right\}$ be a sequence of stopping times with respect to the natural filtration $\mathscr{F}_t^P = \sigma(B_s, 0 \leq s \leq t)$. Define

$$f_n(s,\omega) = f(B_s(\omega))1_{\{R_n(\omega)>s\}}. \tag{2.5}$$

Each of these functions satisfies $\int_0^\infty |f_n(s,\omega)|^2 ds = \int_0^{R_n} |f_n(s,\omega)|^2 ds \leq n$. In particular, we have $\mathbb{E}_P^x\left[\int_0^t |f_n|^2 ds\right] < \infty$ and thus $\int_0^t f_n \cdot dB_s$ is well defined. Moreover, it can be seen that

$$\int_0^{t\wedge R_m} f_n(s,\omega)\cdot dB_s = \int_0^t f_m(s,\omega)\cdot dB_s \tag{2.6}$$

for $m < n$.

Definition 2.5 Let $f \in \mathscr{E}_{\rm loc}$. We define the integral

$$\int_0^t f(B_s)\cdot dB_s := \int_0^t f_n(s,\omega)\cdot dB_s, \quad 0 \leq t \leq R_n. \tag{2.7}$$

This definition is consistent with (2.6).

$\mathscr{E}_{\rm loc}$ has properties below:

(1) Let $f \in \mathscr{E}_{\text{loc}}$. Suppose that a sequence of step functions f_n, $n = 1, 2, \ldots$, satisfies $\int_0^t |f_n(B_s) - f(B_s)|^2 ds \to 0$ in probability as $n \to \infty$. Then

$$\lim_{n\to\infty} \int_0^t f_n(B_s) \cdot dB_s = \int_0^t f(B_s) \cdot dB_s \quad \text{in probability.}$$

(2) $(L^2_{\text{loc}}(\mathbb{R}^3))^3 \subset \mathscr{E}_{\text{loc}}$.

(3) Let $a \in (L^2_{\text{loc}}(\mathbb{R}^3))^3$ and $\nabla \cdot a \in L^1_{\text{loc}}(\mathbb{R}^3)$. Then

$$\left| \int_0^t a(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \nabla \cdot a(B_s) ds \right| < \infty \quad \text{almost surely.}$$

For $a \in (L^2_{\text{loc}}(\mathbb{R}^3))^3$ such that $\nabla \cdot a \in L^1_{\text{loc}}(\mathbb{R}^3)$, we denote

$$\int_0^t a(B_s) \circ dB_s = \int_0^t a(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \nabla \cdot a(B_s) ds.$$

Proposition 2.6 *Under Assumption (A2) we have*

$$(f, e^{-th} g) = \int_{\mathbb{R}^3} dx \mathbb{E}_P^x \left[\overline{f(B_0)} g(B_t) e^{-i \int_0^t a(B_s) \circ dB_s} \right]. \tag{2.8}$$

PROOF. See [Sim04, Theorem 15.5] and [HIL09]. **qed**

2.3 Path integral representation

Now we turn to constructing a functional integral representation for generalized Schrödinger operators including a vector potential term defined by (2.2).

A key element in our construction of a Feynman-Kac-type formula for e^{-tH^Ψ} is to make use of a Lévy subordinator.

Theorem 2.7 *Let $\Psi \in \mathscr{B}_0$ and $V \in L^\infty(\mathbb{R}^3)$. Under Assumption (A2) we have*

$$(f, e^{-tH^\Psi} g) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu} \left[\overline{f(B_0)} g(B_{T_t^\Psi}) e^{-i \int_0^{T_t^\Psi} a(B_s) \circ dB_s} e^{-\int_0^t V(B_{T_s^\Psi}) ds} \right]. \tag{2.9}$$

PROOF. We divide the proof into four steps. To simplify the notation, in this proof we drop the superscript Ψ of the subordinator.

(Step 1) Suppose $V = 0$. Then we claim that

$$(f, e^{-t\Psi(h)} g) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu} \left[\overline{f(B_0)} g(B_{T_t}) e^{-i \int_0^{T_t} a(B_s) \circ dB_s} \right]. \tag{2.10}$$

To prove (2.10) let E^h denote the spectral projection of the self-adjoint operator h. Then

$$(f, e^{-t\Psi(h)}g) = \int_{\mathrm{Spec}(h)} e^{-t\Psi(u)} d(f, E^h_u g). \tag{2.11}$$

By inserting identity (1.6) in (2.11) we obtain

$$(f, e^{-t\Psi(h)}g) = \int_{\mathrm{Spec}(h)} \mathbb{E}^0_\nu[e^{-T_t u}] d(f, E^h_u g) = \mathbb{E}^0_\nu\left[(f, e^{-T_t h}g)\right].$$

Then by the Feynman-Kac-Itô formula for e^{-th} we have

$$(f, e^{-t\Psi(h)}g) = \mathbb{E}^0_\nu\left[\int_{\mathbb{R}^3} dx \mathbb{E}^x_P\left[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\right]\right],$$

thus (2.10) follows.

(Step 2) Let $0 = t_0 < t_1 < \cdots < t_n$, $f_0, f_n \in L^2(\mathbb{R}^3)$ and assume that $f_j \in L^\infty(\mathbb{R}^3)$ for $j = 1, ..., n-1$. We claim that

$$\left(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})\Psi(h)} f_j\right) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}\left(\prod_{j=1}^n f_j(B_{T_{t_j}})\right) e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\right]. \tag{2.12}$$

For easing the notation write $G_j = f_j \prod_{i=j+1}^n e^{-(t_i - t_{i-1})\Psi(h)} f_i(B_{T_{t_i}})$. By (Step 1) the left hand side of (2.12) can be represented as

$$\int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}e^{-i\int_0^{T_{t_1 - t_0}} a(B_s)\circ dB_s} G_1(B_{T_{t_1 - t_0}})\right].$$

Let $\mathscr{F}^{\mathrm{P}}_t = \sigma(B_s, 0 \le s \le t)$ and $\mathscr{F}^\nu_t = \sigma(T_s, 0 \le s \le t)$ be the natural filtrations. An application of the Markov property of B_t yields

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})\Psi(h)} f_j\right) \\
&= \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} \mathbb{E}^0_\nu \mathbb{E}^{B_{T_{t_1}}}_P\left[f_1(B_0)e^{-i\int_0^{T_{t_2 - t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_2 - t_1}})\right]\right] \\
&= \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s}\right. \\
&\qquad \left.\mathbb{E}^0_\nu\left[\mathbb{E}^0_P\left[f_1(B_{T_{t_1}})e^{-i\int_{T_{t_1}}^{T_{t_2 - t_1} + T_{t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_1} + T_{t_2 - t_1}})\middle| \mathscr{F}^P_{T_{t_1}}\right]\right]\right].
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})\Psi(h)} f_j\right) \\
&= \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} \mathbb{E}^0_\nu\left[f_1(B_{T_{t_1}})e^{-i\int_{T_{t_1}}^{T_{t_2 - t_1} + T_{t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_1} + T_{t_2 - t_1}})\right]\right].
\end{aligned}$$

The right hand side above can be rewritten as

$$\int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}})\mathbb{E}^{T_{t_1}}_{\nu}\left[e^{-i\int_0^{T_{t_2-t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_2-t_1}})\right]\right].$$

Using now the Markov property of T_t we see that

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^n e^{-(t_j-t_{j-1})\Psi(h)} f_j\right) \\
&= \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}})\mathbb{E}^0_{\nu}\left[e^{-i\int_{T_{t_1}}^{T_{t_2}} a(B_s)\circ dB_s} G_2(B_{T_{t_2}})\middle| \mathscr{F}^{\nu}_{t_1}\right]\right] \\
&= \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}}) e^{-i\int_{T_{t_1}}^{T_{t_2}} a(B_s)\circ dB_s} G_2(B_{T_{t_2}})\right].
\end{aligned}$$

By the above procedure we obtain (2.12).
(Step 3) Suppose now that $0 \neq V \in L^\infty$ and it is continuous; we prove (2.9) for such V. Since H^Ψ is self-adjoint on $D(\Psi(h)) \cap D(V)$ the Trotter product formula holds:

$$(f, e^{-tH^\Psi} g) = \lim_{n\to\infty} (f, (e^{-(t/n)\Psi(h)} e^{-(t/n)V})^n g).$$

(Step 2) yields

$$\begin{aligned}
(f, e^{-tH^\Psi} g) &= \lim_{n\to\infty} \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\sum_{j=1}^n (t/n) V(B_{T_{tj/n}})}\right] \\
&= \text{r.h.s. } (2.12)
\end{aligned}$$

Here we used that since $s \mapsto B_{T_s(\tau)}(\omega)$ has *càdlàg* paths, $V(B_{T_s(\tau)}(\omega))$ is continuous in $s \in [0,t]$ for each (ω, τ) except for at most finite points. Therefore $\sum_{j=1}^n \frac{t}{n} V(B_{T_{tj/n}}) \to \int_0^t V(B_{T_s}) ds$ as $n \to \infty$ for each path and exists as a Riemann integral.
(Step 4) Suppose that $V \in L^\infty$ and $V_n = \phi(x/n)(V * j_n)$, where $j_n = n^3\phi(xn)$ with $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \phi \le 1$, $\int \phi(x)dx = 1$ and $\phi(0) = 1$. Then $V_n(x) \to V(x)$ almost everywhere. V_n is bounded and continuous, moreover $V_n(x) \to V(x)$ as $n \to \infty$ for $x \notin \mathscr{N}$, where the Lebesgue measure of $\mathscr{N}$ is zero. Thus for almost every $(\omega, \tau) \in \Omega_P \times \Omega_N$, the measure of $\{t \in [0,\infty) \mid B_{T_t(\tau)}(\omega) \in \mathscr{N}\}$ is zero. Hence

$$\begin{aligned}
&\int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V_n(B_{T_s}) ds}\right] \\
&\quad \to \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s}) ds}\right]
\end{aligned}$$

as $n \to \infty$. On the other hand, $e^{-t(\Psi(h)+V_n)} \to e^{-t(\Psi(h)+V)}$ strongly as $n \to \infty$, since $\Psi(h) + V_n$ converges to $\Psi(h) + V$ on the common domain $D(\Psi(h))$. Thus the theorem follows. **qed**

Let

$$\mathrm{E_g}[T] = \inf \mathrm{Spec}(T).$$

Corollary 2.8 (Diamagnetic inequality) *Let $\Psi \in \mathscr{B}_0$, $V \in L^\infty(\mathbb{R}^3)$, and Assumption (A2) hold. Then $|(f, e^{-tH^\Psi} g)| \le (|f|, e^{-t(\Psi(\mathrm{p}^2/2)+V)}|g|)$ and*

$$\mathrm{E_g}[\Psi(\mathrm{p}^2/2) + V] \le \mathrm{E_g}[H^\Psi].$$

PROOF. By Theorem 2.7 we have

$$|(f, e^{-tH^\Psi} g)| \le \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu} \left[|f(B_0)||g(B_{T_t^\Psi})| e^{-\int_0^t V(B_{T_s^\Psi})ds} \right].$$

Then the corollary follows. **qed**

2.4 Singular external potentials

By making use of the functional integral representation obtained in the previous subsection we can now also consider more singular external potentials. We show results without proofs. See [HIL09] for details.

Theorem 2.9 *Let Assumption (A2) hold.*

(1) Suppose $|V|$ is relatively form bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound b. Then $|V|$ is also relatively form bounded with respect to $\Psi(h)$ with a relative bound not larger than b.

(2) Suppose $|V|$ is relatively bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound b. Then $|V|$ is also relatively bounded with respect to $\Psi(h)$ with a relative bound not larger than b.

Corollary 2.10 *(1) Take Assumption (A2) and let V be relatively bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound strictly smaller than one. Then $\Psi(h) + V$ is self-adjoint on $D(\Psi(h))$ and bounded from below. Moreover, it is essentially self-adjoint on any core of $\Psi(h)$. (2) Suppose furthermore (A3). Then $C_0^\infty(\mathbb{R}^3)$ is an operator core of $\Psi(h) + V$.*

PROOF. (1) By (2) of Theorem 2.9, V is relatively bounded with respect to $\Psi(h)$ with a relative bound strictly smaller than one. Then the corollary follows by the Kato-Rellich theorem. (2) follows from Theorem 2.3. **qed**

Theorem 2.9 also allows $\Psi(h) + V$ to be defined in form sense. Let $V = V_+ - V_-$ where $V_+ = \max\{V, 0\}$ and $V_- = \min\{-V, 0\}$. Theorem 2.9 implies that whenever V_- is form bounded to $\Psi(\mathrm{p}^2/2)$ with a relative bound strictly smaller than one, it is also form bounded with respect to $\Psi(h)$ with a relative bound strictly smaller than one. Moreover assume that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. We see that under Assumption (A1), $Q(\Psi(h)) \cap Q(V_+)$ is dense. Define the quadratic form

$$\mathrm{q}(f, f) := (\Psi(h)^{1/2} f, \Psi(h)^{1/2} f) + (V_+^{1/2} f, V_+^{1/2} f) - (V_-^{1/2} f, V_-^{1/2} f) \tag{2.13}$$

on $Q(\Psi(h)) \cap Q(V_+)$. By the KLMN Theorem [RS78] q is a semibounded closed form. We denote the self-adjoint operator associated with (2.13) by $\Psi(h) \dot{+} V_+ \dot{-} V_-$.

Now we are in a position to extend Theorem 2.7 to potentials expressed as form sums.

Theorem 2.11 *Take Assumption (A2). Let $V = V_+ - V_-$ be such that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and V_- is infinitesimally small with respect to $\Psi(\frac{1}{2}\mathrm{p}^2)$ in form sense. Then the functional integral representation given by Theorem 2.7 also holds for $\Psi(h) \dot{+} V_+ \dot{-} V_-$.*

2.5 Ψ-Kato class potential and hypercontractivity

In this section we give a meaning to Kato class for potentials V relative to Ψ and extend generalized Schrödinger operators with vector potential to such V. Recall that for given $\Psi \in \mathscr{B}_0$, the random process

$$X_t : \Omega_P \times \Omega_\nu \ni (\omega, \tau) \mapsto B_{T_t^\Psi(\tau)}(\omega) \tag{2.14}$$

is called subordinated Brownian motion with respect to the subordinator $(T_t^\Psi)_{t \geq 0}$. It is a Lévy process whose properties are determined by the pair (b, λ) in (1.5). Its characteristic function is $\mathbb{E}^{0,0}_{P \times \nu}[e^{iuX_t}] = e^{-t\Psi(u^2/2)}$.

Assumption 2.12 *Let $\Psi \in \mathscr{B}_0$ be such that $\int_0^\infty e^{-t\Psi(u^2/2)} du < \infty$, for all $t > 0$.*

Under Assumption 2.12 we define

$$p_t(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-ixu} e^{-t\Psi(u^2/2)} du \tag{2.15}$$

and $\Pi_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x) dt$. Let $\|f\|_{l^1(L^\infty)} = \sum_{\alpha \in \mathbb{Z}^3} \sup_{x \in C_\alpha} |f(x)|$, where C_α denotes the unit cube centred at $\alpha \in \mathbb{Z}^3$.

Assumption 2.13 *Let p_t be such that* $\sup_{t>0} \|1_{\{|x|>\delta\}} p_t\|_{l^1(L^\infty)} < \infty$.

Note that Assumption 2.13 is satisfied if p_t is spherically symmetric and radially non-increasing. The next facts can be proven in the same way as Theorem III.1 in [CMS90].

Proposition 2.14 *Let $V \geq 0$. Under Assumptions 2.12 and 2.13 the following three properties are equivalent:*

(1) $\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^3} \int_0^t \mathbb{E}^{x,0}_{P\times\nu}[V(X_s)]ds = 0,$

(2) $\lim_{\lambda\to\infty} \sup_{x\in\mathbb{R}^3} \left((\Psi(\mathrm{p}^2/2)+\lambda)^{-1}V\right)(x) = 0,$

(3) $\lim_{\delta\downarrow 0} \sup_{x\in\mathbb{R}^3} \int_{|x-y|<\delta} \Pi_1(x-y)V(y)dy = 0.$

Take Assumptions 2.12 and 2.13. Write $V = V_+ - V_-$ in terms of its positive and negative parts. The Ψ*-Kato class* is defined as the set of potentials V for which V_- and $1_C V_+$ with every compact subset $C \subset \mathbb{R}^3$ satisfy any of the three equivalent conditions in Proposition 2.14. Here 1_C denotes the indicator function on C.

By (3) of Proposition 2.14 we can derive explicit conditions defining Ψ-Kato class using the relation of the Lévy measure of the subordinator with the associated Bernstein function. In the case $d = 3$, we have

$$\Pi_\lambda(x) = \frac{1}{2\pi^2|x|}\int_0^\infty \frac{r\sin r}{|x|^2\left(\lambda+\Psi\left(\frac{r^2}{2|x|^2}\right)\right)}dr.$$

Lemma 2.15 *Let $V \geq 0$ and $\Psi \in \mathscr{B}_0$. Suppose that V satisfies (1) of Proposition 2.14. Then* $\sup_{x\in\mathbb{R}^3}\mathbb{E}^{x,0}_{P\times\nu}\left[e^{\int_0^t V(X_s)ds}\right] < \infty$ *for $t \geq 0$.*

The next result says that we can define a Feynman-Kac semigroup for Ψ-Kato class potentials.

Theorem 2.16 *Let $\Psi \in \mathscr{B}_0$, V belong to Ψ-Kato class and let Assumption (A2) hold. Consider*

$$\mathrm{U}_t f(x) = \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-i\int_0^{T_t^\Psi} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s^\Psi})ds} f(B_{T_t})\right].$$

Then U_t is a strongly continuous symmetric semigroup. In particular, there exists a self-adjoint operator K^Ψ bounded from below such that $\mathrm{U}_t = e^{-tK^\Psi}$.

Proof. Let $V = V_+ - V_-$. Hence by Lemma 2.15 we have

$$\|\mathrm{U}_t f\|^2 \leq C_t\|e^{-t\Psi(\mathrm{p}^2/2)}f\|^2 \leq C_t\|f\|^2,$$

where $C_t = \sup_{x\in\mathbb{R}^3}\mathbb{E}^{x,0}_{P\times\nu}[e^{2\int_0^t V_-(X_s)ds}]$. Thus U_t is a bounded operator from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. In the same fashion as in Step 2 of the proof of Theorem 2.7 we conclude that the semigroup property $\mathrm{U}_t\mathrm{U}_s = \mathrm{U}_{t+s}$ holds for $t, s \geq 0$. We check strong continuity of U_t in t; it suffices to show weak continuity. Let $f, g \in C_0^\infty(\mathbb{R}^3)$. Then we have

$$(f, \mathrm{U}_t g) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}e^{-\int_0^t V(B_{T_s})ds}\right].$$

Since $T_t(\tau) \to 0$ as $t \to 0$ for each $\tau \in \Omega_\nu$, the dominated convergence theorem gives $(f, \mathrm{U}_t g) \to (f, g)$.

Finally we check the symmetry property $\mathrm{U}_t^* = \mathrm{U}_t$. By a limiting argument it is enough to show this for $a \in (C_b^2(\mathbb{R}^3))^3$. Let $\widetilde{B}_s = \widetilde{B}_s(\omega,\tau) = B_{T_t(\tau)-s}(\omega) - B_{T_t(\tau)}(\omega)$. Then for each $\tau \in \Omega_\nu$, $\widetilde{B}_s \stackrel{\mathrm{d}}{=} B_s$ with respect to dP^x. (Here $Z \stackrel{\mathrm{d}}{=} Y$ denotes that Z and Y are identically distributed.) Thus we have

$$(f, \mathrm{U}_t g) = \mathbb{E}^{0,0}_{P\times\nu}\left[\int_{\mathbb{R}^3} dx \overline{f(x-\widetilde{B}_{T_t})}e^{-i\int_0^{T_t} a(x+\widetilde{B}_s-\widetilde{B}_{T_t})\circ d\widetilde{B}_s}e^{-\int_0^t V(x+\widetilde{B}_{T_s}-\widetilde{B}_{T_t})ds}g(x)\right].$$

Here we changed the variable x to $x - \widetilde{B}_{T_t}$. Then in $L^2(\Omega_P, dP^0)$ we have that

$$\int_0^{T_t} a(\widetilde{B}_s - \widetilde{B}_{T_t}) \circ d\widetilde{B}_s = -\int_0^{T_t} a(B_s) \circ dB_s.$$

Since $\widetilde{B}_{T_t} \stackrel{\mathrm{d}}{=} -B_{T_t}$ and $\widetilde{B}_{T_s} - \widetilde{B}_{T_t} \stackrel{\mathrm{d}}{=} B_{T_t - T_s}$, we have

$$(f, \mathrm{U}_t g) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_{T_t})}e^{+i\int_0^{T_t} a(B_s)\circ dB_s}e^{-\int_0^t V(B_{T_t-T_s})ds}g(x)\right].$$

Moreover, as $T_t - T_s \stackrel{\mathrm{d}}{=} T_{t-s}$ for $0 \leq s \leq t$, we obtain

$$(f, \mathrm{U}_t g) = \int_{\mathbb{R}^3} dx \overline{\mathbb{E}^{x,0}_{P\times\nu}\left[f(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}e^{-\int_0^t V(B_{T_s})ds}\right]}g(x) = (\mathrm{U}_t f, g).$$

The existence of a self-adjoint operator K^Ψ bounded from below such that $\mathrm{U}_t = e^{-tK^\Psi}$ is a consequence of the Hille-Yoshida theorem. This completes the proof. **qed**

Let V be in Ψ-Kato class and take Assumption (A2). We call K^Ψ given in Theorem 2.16 *generalized Schrödinger operator for* Ψ*-Kato class potentials.* We refer to the one-parameter operator semigroup e^{-tK^Ψ}, $t \geq 0$, as the Ψ*-Kato class generalized Schrödinger semigroup.* Put K_0^Ψ for the operator defined by K^Ψ with a replaced by 0.

Theorem 2.17 **(Hypercontractivity)** *Let V be a Ψ-Kato class potential and assume (A2) to hold. Then e^{-tK^Ψ} is a bounded operator from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$, for all $1 \leq p \leq q \leq \infty$. Moreover, $\|e^{-tK^\Psi}\|_{p,q} \leq \|e^{-tK_0^\Psi}\|_{p,q}$ holds for all $t \geq 0$.*

PROOF. By the Riesz-Thorin theorem it suffices to show that e^{-tK^Ψ} is bounded as an operator of (1) $L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$, (2) $L^1(\mathbb{R}^3) \to L^1(\mathbb{R}^3)$ and (3) $L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Since

$$|e^{-tK^\Psi} f(x)| \leq e^{-tK_0^\Psi} |f|(x), \tag{2.16}$$

we will prove (1)-(3) for $e^{-tK_0^\Psi}$. For simplicity we denote $\mathbb{E}^{x;0}_{P\times\nu} = \mathbb{E}^x$ and $\mathrm{P}_t = e^{-tK_0^\Psi}$, i.e., we have $\mathrm{P}_t f(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s)ds} f(X_t)]$. To consider (1), let $f \in L^\infty(\mathbb{R}^3)$. We have by Lemma 2.15,

$$\|\mathrm{P}_t f\|_\infty \leq \sup_{x\in\mathbb{R}^3} \left(\mathbb{E}^x[e^{-\int_0^t V(X_s)ds}]\right) \|f\|_\infty.$$

Thus (1) follows. To derive (2), let $0 \leq f \in L^1(\mathbb{R}^3)$ and $g \equiv 1 \in L^\infty(\mathbb{R}^3)$. Then $\mathrm{P}_t g \in L^\infty(\mathbb{R}^3)$ by (1) above. In the same way as in the proof of the symmetry of U_t in Theorem 2.16 it can be shown that

$$\int_{\mathbb{R}^3} dx f(x) \cdot \mathrm{P}_t g(x) = \int_{\mathbb{R}^3} dx \mathrm{P}_t f(x) \cdot g(x) = \int_{\mathbb{R}^3} dx \mathrm{P}_t f(x).$$

Since $\mathrm{P}_t f(x) \geq 0$, we have $\|\mathrm{P}_t f\|_1 \leq \|f\|_1 \|\mathrm{P}_t 1\|_\infty$. Taking any $f \in L^1(\mathbb{R}^3)$ and splitting it off as $f = \Re f_+ - \Re f_- + i(\Im f_+ - \Im f_-)$, we get $\|\mathrm{P}_t\|_1 \leq 4\|f\|_1 \|\mathrm{P}_t 1\|_\infty$. This gives (2).

Combining (1) and (2) with the Riesz-Thorin theorem we deduce that P_t is a bounded operator from $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$, for all $1 \leq p \leq \infty$. Moreover, the Markov property of $(X_t)_{t\geq 0}$ implies that P_t is a semigroup on $L^p(\mathbb{R}^3)$, for $1 \leq p \leq \infty$.

Finally we consider (3) with the diagram

$$L^1(\mathbb{R}^3) \xrightarrow{\mathrm{P}_t} L^2(\mathbb{R}^3) \xrightarrow{\mathrm{P}_t} L^\infty(\mathbb{R}^3). \tag{2.17}$$

Let $f \in L^2(\mathbb{R}^3)$. Then

$$\|\mathrm{P}_t f\|_\infty^2 \leq \mathbb{E}^x[e^{-2\int_0^t V(X_s)ds}] \mathbb{E}^x[|f(X_t)|^2] \leq C_t \int_{\mathbb{R}^3} dx |f(x+y)|^2 p_t(y) dy$$

by Lemma 2.15, where $C_t = \sup_{x\in\mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(X_s)ds}]$. Since $|p_t(y)| \leq \int_0^\infty e^{-t\Psi(u^2/2)} du < \infty$ by Assumption 2.12, with p_t in (2.15), it follows that

$$\|\mathrm{P}_t f\|_\infty \leq (C_t \|p_t\|_\infty)^{1/2} \|f\|_2. \tag{2.18}$$

Thus P_t is a bounded operator from $L^2(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$. Next, let $f \in L^1(\mathbb{R}^3)$ and $g \in L^2(\mathbb{R}^3)$. We have $\int_{\mathbb{R}^3} dx \mathrm{P}_t f(x) \cdot g(x) = \int_{\mathbb{R}^3} dx f(x) \cdot \mathrm{P}_t g(x)$. Then by (2.18) we obtain

$$\left| \int_{\mathbb{R}^3} dx \mathrm{P}_t f(x) \cdot g(x) \right| \leq \|\mathrm{P}_t g\|_\infty \|f\|_1 \leq C_t \|p_t\|_\infty \|g\|_2 \|f\|_1.$$

Since $g \in L^2(\mathbb{R}^3)$ is arbitrary, $\mathrm{P}_t f \in L^2(\mathbb{R}^3)$ and

$$\|\mathrm{P}_t f\|_2 \leq C_t \|p_t\|_\infty \|f\|_1 \tag{2.19}$$

follows, hence P_t is a bounded operator from $L^1(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Thus (2.17) holds.

By the semigroup property and (2.17) we have for $f \in L^1(\mathbb{R}^3)$,

$$\|\mathrm{P}_t f\|_\infty = \|\mathrm{P}_{t/2}\mathrm{P}_{t/2} f\|_\infty \leq (C_t \|p_{t/2}\|_\infty)^{1/2} \|\mathrm{P}_{t/2} f\|_2 \leq (C_t \|p_{t/2}\|_\infty)^{3/2} \|f\|_1.$$

The fact $\|e^{-tK^\Psi}\|_{p,q} \leq \|e^{-tK_0^\Psi}\|_{p,q}$ follows from (2.16). This completes the proof of the theorem. **qed**

3 Relativistic Schrödinger operators

Finally we consider the relativistic Schrödinger operator. We write

$$h_{\mathrm{rel}} = \sqrt{(\mathrm{p} - a)^2 + m^2} - m, \tag{3.1}$$

$$h_{\mathrm{rel}}(0) = \sqrt{\mathrm{p}^2 + m^2} - m. \tag{3.2}$$

Theorem 3.1 *Suppose Assumption (A3) and let V be relatively bounded with respect to $\sqrt{\mathrm{p}^2 + m^2}$ with relative bound strictly smaller than one. Then h_{rel} is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ and*

$$(f, e^{-th_{\mathrm{rel}}} g) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu} \left[\overline{f(B_0)} g(B_{T_t^\Psi}) e^{-i\int_0^{T_t^\Psi} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s^\Psi}) ds} \right], \tag{3.3}$$

where $T_t^\Psi = \inf\{s > 0 | B_s + ms = t\}$.

Proof. The essential self-adjointness follows from (2) of Corollary 2.10, and (3.3) from Theorem 2.11. **qed**

By Theorem 3.1 we also have the following energy comparison inequality.

Corollary 3.2 (Diamagnetic inequality) *Suppose the assumptions of Theorem 3.1. Then $|(f, e^{-th_{\mathrm{rel}}} g)| \leq (|f|, e^{-th_{\mathrm{rel}}(0)} |g|)$ and $\mathrm{E_g}[h_{\mathrm{rel}}(0)] \leq \mathrm{E_g}[h_{\mathrm{rel}}]$.*

Furthermore, by Theorem 2.17 we have the result below.

Corollary 3.3 (Hypercontractivity) *Let the assumptions of Theorem 3.1 and one of the three equivalent conditions in Proposition 2.14 with $\Psi(u) = \sqrt{2u + m^2} - m$ hold. Then $e^{-t(h_{\mathrm{rel}})}$ is a bounded operator from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ for all $1 \leq p \leq q \leq \infty$.*

A Feynman-Kac type formula for a relativistic Schrödinger operator with spin 1/2, $\sqrt{(\sigma\cdot(\mathrm{p}-a))^2+m^2}-m+V$, and its generalization $\Psi(\frac{1}{2}(\sigma\cdot(\mathrm{p}-a))^2)+V$ can be also constructed in [HIL09]. From this formula a diamagnetic inequality is also derived.

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