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LIMIT PERIOD FORMULA FOR SPECIAL CYCLES ON REAL HYPERBOLIC SPACES

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1. PRELIMINARY

1.1. Let G be a connected semisimple Lie group with finite center of non-compact type. We fix a Haar measure $\mathrm{d}g$ of G. Given a uniform lattice $\Gamma \subset G$ i.e., discrete subgroup such that $\Gamma\backslash G$ is compact, let $L^2(\Gamma\backslash G)$ be the Hilbert space of all the measurable functions $\phi : G \rightarrow \mathbb{C}$ such that $\phi(\gamma g) = \phi(g)$ for any $\gamma \in \Gamma$ with the finite L^2-norm

$$\int_{\Gamma\backslash G} |\phi(g)|^2 \,\mathrm{d}g < +\infty.$$

Then, the right regular action of G on $L^2(\Gamma\backslash G)$ yields a unitary representation of $(R_\Gamma, L^2(\Gamma\backslash G))$, which, by a fundamental theorem of Gelfand, Graev and Piatetsuki-Shapiro, is discretely decomposable to irreducible unitary representations of G with finite multiplicities:

$$\text{there exists a function } \hat{G} \ni \pi \mapsto m_\Gamma(\pi) \in \mathbb{N} \text{ s.t.}$$

$$L^2(\Gamma\backslash G) = \hat{\bigoplus}_{\pi\in\hat{G}} L^2(\Gamma\backslash G)_\pi,$$
$$L^2(\Gamma\backslash G)_\pi \cong \pi^{\oplus m_\Gamma(\pi)} \quad (\pi\text{-isotypic part})$$

Let K be a maximal compact subgroup of G and (τ, F_τ) an irreducible unitary representation of K. Then, the space of F_τ-valued π-automorphic forms on Γ defined by

$$\begin{aligned} L^2_\tau(\Gamma\backslash G)_\pi &\overset{\text{def}}{=} \mathrm{Hom}_K(F_\tau^\vee, L^2(\Gamma\backslash G)_\pi) \\ &\cong \{L^2(\Gamma\backslash G)_\pi \otimes_{\mathbb{C}} F_\tau\}^K \end{aligned}$$

becomes a Hilbert space in a natural way; it is of finite dimension

$$\dim_{\mathbb{C}} L^2_\tau(\Gamma\backslash G)_\pi = m_\Gamma(\pi)\,\mathrm{mult}_K(\tau^\vee, \pi).$$

1.2. Let H be a connected symmetric subgroup of G. Thus, there exists an involutive automorphism σ of G such that $H = (G^\sigma)^\circ$. We assume that σ is taken so that $\sigma(K) = K$. Then, $K_H = K \cap H$ is a maximal compact subgroup of H. Let (τ, F_τ) be an irreducible unitary representation of K. Since K_H is a symmetric subgroup of K, the trivial representation of K_H occurs in $\tau|K^H$ at most once, i.e., $\dim F_\tau^{K_H} \leqslant 1$.

Let $\mathfrak{L}_G^H$ be the set of uniform lattices $\Gamma \subset G$ such that $\sigma(\Gamma) = \Gamma$. For each $\Gamma \in \mathfrak{L}_G^H$, the intersection $\Gamma_H = \Gamma \cap H$ is a uniform lattice of H.

Fix a Haar measure $\mathrm{d}h$ of H. Given $\Gamma \in \mathfrak{L}_G^H$, $\pi \in \hat{G}$ and $\tau \in \hat{K}$, consider the map

$$L^2_\tau(\Gamma\backslash G)_\pi \quad \ni \phi \longrightarrow \phi^H \overset{\text{def}}{=} \int_{\Gamma_H\backslash H} \phi(h)\,\mathrm{d}h \in \quad F_\tau^{K_H}$$

and set

$$\mathbb{P}_\tau(\Gamma)_\pi \stackrel{\text{def}}{=} \sum_{\phi \in \mathcal{B}_\tau(\Gamma)_\pi} \|\phi^H\|^2,$$

where $\mathcal{B}_\tau(\Gamma)_\pi$ is an orthonormal basis of $L^2_\tau(\Gamma\backslash G)_\pi$. It is easy to see that $\mathbb{P}_\tau(\Gamma)_\pi$ is independent of the choice of $\mathcal{B}_\tau(\Gamma)_\pi$.

1.3. In this note, we are interested in the asymptotic behavior of $\mathbb{P}_\tau(\Gamma)_\pi$ (with fixed π and τ) when '$\Gamma \to \{e\}$'. To make the meaning of '$\Gamma \to \{e\}$'more exact, we introduce the notion of a tower of lattices. A sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ is called a tower if

(1) Γ_n is uniform lattice in G
(2) $\Gamma_{n+1} \subset \Gamma_n$, $[\Gamma_n : \Gamma_{n+1}] < +\infty$
(3) Γ_n is normal in Γ_0
(4) $\bigcap \Gamma_n = \{e\}$

A tower $\{\Gamma_n\}$ in G is said to be H-admissible if $\Gamma_n \in \mathfrak{L}^H_G$ for all n. Then, for a given tower of H-admissible uniform lattices in G, we have some speculation on the limiting behaviour of $\mathbb{P}_\tau(\Gamma_n)_\pi$ as $n \to \infty$; we report a partial result obtained for a particular symmetric pair (G, H).

2. Speculations

2.0.1. *Group case.* Let G_0 be a connected semisimple Lie group with finite center, and $\{\Gamma_{0,n}\}$ a tower of uniform lattices in G_0. Let $\hat{G}_{0,\mathrm{d}}$ be the equivalence classes of irreducibel unitary representations with square integrable matrix coefficients. Then, for any $\pi_0 \in \hat{G}_{0,\mathrm{dis}}$, the formal degree of π_0 is the number $d(\pi_0)$ such that

$$\int_G (\pi_0(g)v_1|v_2)\,\overline{(\pi(g)w_1|w_2)}\,dg = \frac{(v_1|w_1)\,\overline{(v_2|w_2)}}{d(\pi_0)} \quad \text{for any} \quad v_1, v_2, w_1, w_2 \in \mathcal{H}_{\pi_0}.$$

For convenience, set $d(\pi_0) = 0$ for $\pi_0 \in \hat{G}_0 - \hat{G_{0,\mathrm{d}}}$. Then, the limit multiplicity formula proved by DeGeorge-Wallach [5] asserts

$$\lim_{n\to\infty} \frac{m_{\Gamma_{0,n}}(\pi_0)}{\mathrm{vol}(\Gamma_{0,n}\backslash G_0)} = d(\pi_0), \qquad \pi_0 \in \hat{G}, \tag{2.1}$$

which was extended to a tower of non-uniform lattices by L.Clozel and G. Savin.

This result is reformulated in our framework as follows. Fix a maximal compact subgroup $K_0 \subset G_0$. Then, $K = K_0 \times K_0$ is a maximal compact subgroup of $G = G_0 \times G_0$. For $\pi_0 \in \hat{G}_0$ and $\tau_0 \in \hat{K}_0$, set $\pi = \pi_0 \boxtimes \check{\pi}_0$ and $\tau = \tau_0 \boxtimes \check{\tau}_0$.

If $\Gamma \subset G$ is of the form $\Gamma_0 \times \Gamma_0$ with $\Gamma_0 \subset G_0$ a uniform lattice, then

$$L^2_\tau(\Gamma\backslash G)_\pi \cong L^2_{\tau_0}(\Gamma_0\backslash G_0)_{\pi_0} \hat{\boxtimes} L^2_{\check{\tau}_0}(\Gamma_0\backslash G_0)_{\check{\pi}_0}.$$

If $H = \Delta G_0$ is the diagonal subgroup of G, then,

$$\mathbb{P}_\tau(\Gamma)_\pi = \frac{\mathrm{mult}_{K_0}(\tau_0^\vee, \pi_0)}{\dim \tau_0}\, m_{\Gamma_0}(\pi_0).$$

Given a tower of uniform lattices $\{\Gamma_{0,n}\}$ in G_0, the direct products $\Gamma_n = \Gamma_{0,n} \times \Gamma_{0,n}$ affords an H-admissible tower in G and the limit multiplicity formula (2.1) is equivalent to

$$\lim_{n\to\infty} \frac{\mathbb{P}_\tau(\Gamma_n)_\pi}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} = \frac{\mathrm{mult}_{K_0}(\tau_0^\vee, \pi_0)}{\dim \tau_0} d(\pi_0).$$

2.1. Limit period formula.

2.1.1. *Problem.* The group case suggests that the main term of $\mathbb{P}_\tau(\Gamma)_\pi$ as $\Gamma \to \{e\}$ should be $\mathrm{vol}(\Gamma_H\backslash H)$. Now, we raise the following question:
Let $(\pi, \mathcal{H}_\pi) \in \hat{G}$ and $(\tau, F_\tau) \in \hat{K}$ be such that the condition (2.2) is satisfied. Let $\{\Gamma_n\}$ be an H-admissible tower in G. Does the limit

$$\lim_{n\to\infty} \frac{\mathbb{P}_\tau(\Gamma_n)_\pi}{\mathrm{vol}(\Gamma_n \cap H\backslash H)}$$

exists? If exists, what is the limit value? □

If the limiting value is non zero, we infer that $\mathbb{P}_\tau(\Gamma_n)_\pi$ is non vanishing for sufficiently large n, which in turn yields a new proof of the existence of a realization of π in the space $L^2(\Gamma_n\backslash G)$.

We put a remark here. Let $\Gamma \in \mathfrak{L}_G^H$, $\pi \in \hat{G}$ and $\tau \in \hat{K}$. The non-vanishing of $\mathbb{P}_\tau(\Gamma)_\pi$ imposes the following restriction on the data (Γ, π, τ).

- $m_\Gamma(\pi) \neq 0$;
- The (local) compatibility condition of π and τ :

$$\exists \ell \in (\mathcal{H}_\pi^{-\infty})^H,\ \exists \theta \in (\mathcal{H}_\pi^{\infty}[\tau])^{H\cap K} \text{ s.t. } \ell(\theta) \neq 0, \tag{2.2}$$

in particular,

$$F_\tau^{H\cap K} \neq \{0\}, \qquad (\mathcal{H}_\pi^{-\infty})^H \neq \{0\}$$

Here, $\mathcal{H}_\pi^\infty$ denotes the space of C^∞-vectors of π, $\mathcal{H}_\pi^\infty[\tau]$ the τ-isotypic part of $\mathcal{H}_\pi^\infty$ and $\mathcal{H}_\pi^{-\infty}$ the space of distribution vectors of π.

2.1.2. *Relative discrete series of $H\backslash G$.* Let G, H be as in 1.2. An irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G is called to be H-spherical if $(\mathcal{H}_\pi^{-\infty})^H \neq 0$; π is called to be a relative discrete series representation of $H\backslash G$ if $\mathcal{L}_\pi \neq 0$. Here, $(\mathcal{H}_\pi^{-\infty})^H$ is the space of H-invariant distribution vectors of π, and $\mathcal{L}_\pi$ is the space of all those $\ell \in (\mathcal{H}_\pi^{-\infty})^H$ such that

$$\exists v \in \mathcal{H}_\pi^\infty \quad \text{s.t.} \quad \int_{H\backslash G} |\ell(\pi(g)v)|^2\, dg < +\infty$$

We denote by $\hat{G}^H$ the set of equivalence classes of all H-spherical irreducible unitary representations of G and by $\hat{G}_{\mathrm{d}}^H$ the subset of $\hat{G}^H$ of those classes containing a relative discrete series.

2.1.3. *Formal degree.* We define an analogue of formal degree as follows. Let $\pi \in \hat{G}^H_{\mathrm{d}}$ and $\tau \in \hat{K}$ are such that

$\Diamond_1$ $\dim \mathcal{L}_\pi = 1$ (multiplicit one condition).

$\Diamond_2$ $\mathrm{mult}_K(\tau, \pi) = 1$.

$\Diamond_3$ $(\exists \ell \in \mathcal{L}_\pi)(\exists \theta \in \mathcal{H}_\pi^\infty[\tau]^{K_H})(\ell(\theta) \neq 0)$ (*cf.* (2.2)).

Then, there exists $d_\tau^{H\backslash G}(\pi)$ such that

$$\int_{H\backslash G} \ell(\pi(g)v) \cdot \overline{\ell(\pi(g)w)}\, \mathrm{d}g = \frac{d_\tau^{H\backslash G}(\pi)^{-1}}{\dim \tau} \frac{|\ell(\theta)|^2}{\|\theta\|^2} \cdot (v|w)_\pi, \quad \forall v, w \in \mathcal{H}_\pi^\infty.$$

Note that the number $d_\tau^{H\backslash G}(\pi)$ is independent of the choice of (ℓ, θ).

2.1.4. *Limit period formula.* Now, from the experience of the group case, we pose the following.

Conjecture : Let $\pi \in \hat{G}^H_{\mathrm{d}}$ and $\tau \in \hat{K}$ be such that the conditions $(\Diamond)_i$ $(i = 1, 2, 3)$ in 1.4.3 are satisfied. Let $\{\Gamma_n\}$ be an H-admissible tower in G. Then,

$$\lim_{n\to\infty} \frac{\mathbb{P}_\tau(\Gamma_n)_\pi}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} = d_\tau^{H\backslash G}(\pi). \tag{2.3}$$

For $\pi \in \hat{G} - \hat{G}^H_{\mathrm{d}}$ and $\tau \in \hat{K}$, the same limit should be zero. □

Note that this conjecture is compatible with the group case.

3. Results

We consider the case

$$G = \mathrm{SO}_0(d, 1), \qquad (d \geqslant 2),$$
$$H = \mathrm{SO}_0(d-p, 1) \times \mathrm{SO}(p), \qquad (1 \leqslant p < [d/2]),$$

and report a partial result to the conjecture for some π and for an H-admissible tower of congruence subgroups of G.

3.1. Setting. Let F be an algebraic number field such that $F/\mathbb{Q}$ is totally real and $n_F = [F : \mathbb{Q}]$ is greater than 1. We enumerate all the embeddings of F to $\mathbb{R}$ as $\iota_\nu : F \hookrightarrow \mathbb{R}$ $(1 \leqslant \nu \leqslant n_F)$. Let V be an F-vector space of dimension $d + 1\, (\geqslant 2)$ and Q a non-degenerate F-quadratic form on V. Define G be the restriction of scalars from F to $\mathbb{Q}$ of the orthogonal $\mathrm{O}(Q)$ of the quadratic space (V, Q). Thus, for a $\mathbb{Q}$-algebra A,

$$\mathsf{G}(A) = \{g \in \mathrm{GL}(V \otimes_{\mathbb{Q}} A) |\, Q \circ g = Q\}.$$

For each ν, let $V^{(\nu)} = V \otimes_{F,\iota_\nu} \mathbb{R}$ and $Q^{(\nu)}$ the $\mathbb{R}$-quadratic form on $V^{(\nu)}$ induced by Q. From now on, we suppose

$$\mathrm{sgn}(Q^{(1)}) = (d+, 1-),$$
$$\mathrm{sgn}(Q^{(\nu)}) = ((d+1)+, 0-), \qquad (2 \leqslant \nu \leqslant n_F)$$

Set $\tilde{G} = \mathrm{O}(Q^{(1)})$ and $G = \tilde{G}^{\circ}$. Then,

$$\begin{aligned} \mathsf{G}(\mathbb{R}) &\cong \tilde{G} \times \prod_{\nu=2}^{n_F} \mathrm{O}(Q^{(\nu)}) \quad \xrightarrow{\mathrm{pr}_1} \tilde{G} \\ \tilde{G} &\cong \mathrm{O}(d,1) \qquad \text{(real rank one)} \\ \mathrm{O}(Q^{(\nu)}) &\cong \mathrm{O}(d+1) \qquad \text{(compact)} \quad (\nu \geqslant 2) \end{aligned}$$

Let $U \subset V$ be an F-subspace such that $Q^{(\nu)}|U^{(\nu)} > 0$ for all ν. We suppose $p := \dim_F(U) \in [1, [d/2]-1]$. Set

$$\mathsf{H} = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{Stab}_{\mathrm{O}(Q)}(U))$$

and

$$H = \mathrm{pr}_1 \mathsf{H}(\mathbb{R})^{\circ} \qquad \subset G.$$

Thus, H is a connected symmetric subgroup of G such that

$$G \cong \mathrm{SO}_0(d,1), \quad H \cong \mathrm{SO}_0(d-p,1) \times \mathrm{SO}(p).$$

Let $\mathfrak{L}$ be an $\mathfrak{o}_F$-lattice in V such that $\mathfrak{L} = (\mathfrak{L} \cap U) \oplus (\mathfrak{L} \cap U^{\perp})$. Let $\mathfrak{a} \subset \mathfrak{o}_F$ an $\mathfrak{o}_F$-ideal. Set

$$\begin{aligned} \tilde{\Gamma}_{\mathfrak{L}}(\mathfrak{o}_F) &= \mathrm{GL}(\mathfrak{L}) \cap \mathsf{G}(\mathbb{Q}) \quad \hookrightarrow \mathsf{G}(\mathbb{R}), \\ \tilde{\Gamma}_{\mathfrak{L}}(\mathfrak{a}) &= \{\gamma \in \tilde{\Gamma}_{\mathfrak{L}}(\mathfrak{o}_F) | \, \gamma v - v \in \mathfrak{a}\mathfrak{L} \, (\forall v \in \mathfrak{L})\}, \\ \Gamma_{\mathfrak{L}}(\mathfrak{a}) &= \mathrm{pr}_1(\tilde{\Gamma}_{\mathfrak{L}}(\mathfrak{a})) \cap G \end{aligned}$$

Then, $\Gamma_{\mathfrak{L}}(\mathfrak{a})$ is a uniform lattice of G belonging to $\mathfrak{L}_G^H$. If $\{\mathfrak{a}_n\}$ is a sequence of $\mathfrak{o}_F$-ideals such that $\mathfrak{a}_{n+1} \subset \mathfrak{a}_n$ and such that the distance from 0 to $\mathfrak{a}_n - \{0\}$ in $F \otimes_{\mathbb{Q}} \mathbb{R}$ tends $+\infty$ with n. Then, $\Gamma_n = \Gamma_{\mathfrak{L}}(\mathfrak{a}_n)$ is an H-admissible tower in G.

We fix a maximal compact subgroup $K \cong \mathrm{SO}(d)$ of G such that $K \cap H$ is maximally compact in H. The unitary dual $\hat{K}$ is parametrized by the set of dominant integral weights, which are δ-tuples

$$[l_1, l_2, \ldots, l_\delta] \in (\mathbb{Z}/2)^{\delta}, \quad (\delta = [d/2])$$

such that

$$\begin{aligned} &l_1 \geqslant \ldots \geqslant l_\delta \geqslant 0 \quad (d \text{ :odd}) \\ &l_1 \geqslant \ldots \geqslant l_{\delta-1} \geqslant |l_\delta| \quad (d: \text{ even}). \end{aligned}$$

We remark that $(\tau_\lambda)^{H \cap K} \neq 0$ if and only if

$$\lambda = [l_1, \ldots, l_p, 0, \ldots, 0].$$

Let ( ) be the bilinear form on $V^{(1)}$ associated with $Q^{(1)}$:

$$(v, w) = 2^{-1}\{Q^{(1)}(v+w) - Q^{(1)}(v) - Q^{(1)}(w)\}.$$

We may suppose that K is the stabilizer in G of a vector $v_0 \in V^{(1)}$ such that $Q^{(1)}(v_0) = -1$, $v_0 \perp U^{(1)}$. Thus, the tangent space of G/K at the origin $o = eK$ is identified with the orthogonal complement of v_0 in the natural way: $T_o(G/K) \cong (v_0)^{\perp}$. Then, the restriction $(\, , \,)|v_0^{\perp}$ is a positive definite bilinear form, which propagates a G-invariant metric on G/K. The associated Riemannian volume form is denoted by $\mathrm{d}\mu_{G/K}$. Fix the Haar measure $\mathrm{d}k$

with the total volume 1. Then, we fix the Haar measure $\mathrm{d}g$ of G in such a way that the quotient $\mathrm{d}g/\mathrm{d}k$ coincides with $\mathrm{d}\mu_{G/K}$. We fix a Haar measure $\mathrm{d}h$ of H by a similar construction.

3.2. The case $p = 1$ (i.e. $H \cong \mathrm{SO}_0(d-1,1)$). Let $P = MAN$ be a minimal parabolic subgroup of $G = \mathrm{SO}_0(d,1)$. Then,

$$M \cong \mathrm{SO}(d-1), \quad A \cong \mathbb{R}_{>0}.$$

For any $s \in \mathbb{C}$, the K-spherical principal series $\pi_0(s)$ is defined to be the representation of G (unitarily) induced from the character $1_M \otimes e^s \otimes 1_N$ of P:

$$\pi_0(s) = \mathrm{Ind}_P^G(1_M \otimes e^s \otimes 1_N).$$

The following properties of $\pi_0(s)$ is known:
♠1 $\pi_0(s)|_{K=\mathrm{SO}(d)} \cong \bigoplus_{l\in\mathbb{N}} \tau_{[l,0,\dots,0]}$.
♠2 $\pi_0(s)$ is irreducible unitarizable iff

$$s \in \sqrt{-1}\mathbb{R} \cup (-\rho,\rho) \qquad (\text{where}\rho = \tfrac{d-1}{2}).$$

♠3 $\pi_0(s)$ $(\mathrm{Re}(s) > 0)$ is reducible iff

$$s = \rho + k, \quad \exists k \in \mathbb{N} = \{0,1,\dots\}.$$

♠4 For $k \in \mathbb{N}$, $\pi_0(\rho + k)$ has a unique irreducible $(\mathfrak{g}, K)$-submodule

$$\delta_k = \bigoplus_{l\geqslant k+1} \tau_{[l,0,\dots,0]} \hookrightarrow \pi_0(\delta + k).$$

♠5 Set $\delta_{-1} = \pi_0(\rho - 1)$ if $d \geqslant 4$. Then

$$\hat{G}_{\mathrm{d}}^H = \begin{cases} \{\delta_k |\, k \in \mathbb{N}\}, & d = 2,3, \\ \{\delta_k |\, k \in \mathbb{N}\} \cup \{\delta_{-1}\}, & d \geqslant 4. \end{cases}$$

Theorem 1. *Let $\{\mathfrak{a}_n\}$ be a sequence of $\mathfrak{o}_F$-ideals such that $\mathfrak{a}_{n+1} \subset \mathfrak{a}_n$ and such that the Euclidean distance from 0 to the lattice points $\mathfrak{a}_n - \{0\}$ in $F \otimes_{\mathbb{Q}} \mathbb{R}$ tends infinity with n. set $\Gamma_n = \Gamma_L(\mathfrak{a}_n)$.*

(1) *If*

$$\pi = \delta_k, \qquad \tau = \tau_{[k+1,0,\dots,0]}, \qquad (k \in \mathbb{N}),$$

then

$$\lim_{n\to\infty} \frac{\mathbb{P}_\tau(\Gamma_n)_\pi}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\rho + k + 1/2)}{\Gamma(\rho + k)} = d_\tau^{H\backslash G}(\pi) \tag{3.1}$$

(2) *If $\pi \in \hat{G} - \hat{G}_{\mathrm{d}}^H$, then for any $\tau \in \hat{K}$,*

$$\lim_{n\to\infty} \frac{\mathbb{P}_\tau(\Gamma_n)_\pi}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} = 0$$

Remark:
(i) δ_k is integrable (i.e. $\hookrightarrow L^1(H\backslash G)$) if and only if $k \geqslant 1$
(ii) The first identity in (3.1) for $k = 0$ has been proved by a geometric technique ([1]).
(iii) (3.1) is true even for $\pi = \delta_{-1}$, if we assume the existence of "spectral gap" at δ_{-1} along $\pi_0(s)$,i.e.,

$$(\exists\epsilon > 0)(\forall n \in \mathbb{N})$$
$$[\,(m_{\Gamma_n}(\pi_0(s)) \neq 0,\ |s| < \rho - 1) \implies (|s| \leqslant \rho - 1 - \epsilon)\,]$$

This is a consequence of Arthur's conjecture (*cf.* [3], [2]).

Corollary 2. *Let* $k \in \mathbb{N}$ *and* $\tau = \tau_{[l,0,\ldots,0]}$. *Let* $\{\Gamma_n\}$ *be as in Theorem 1.*
(1) *There exists* $n \in \mathbb{N}$ *and* $\phi : G \longrightarrow F_\tau$ *satisfying*

$$\phi(\gamma g k) = \tau(k)^{-1}\phi(g), \qquad \forall\gamma \in \Gamma_n,\ \forall k \in K$$
$$C_{\mathfrak{g}}\phi = 2k(k+\rho)\,\phi \qquad (C_{\mathfrak{g}}\text{: Casimir operator}),$$
$$\int_{\Gamma_n \cap H\backslash H} \phi(h)\,dh \neq 0.$$

(2) $m_{\Gamma_n}(\delta_k) \neq 0$ *if* n *is large enough.*

Remark : This is not new. Indeed, for $k > 0$, this is a special case of [10], and for $k = 0$, this may be deduced from [7].

3.3. The case $p > 1$ **(i.e.** $H \cong \mathrm{SO}_0(d-p,1)\times\mathrm{SO}(p)$**).** Let $\pi_{p-1}(s) = \mathrm{Ind}_P^G(\xi_{p-1}\otimes e^s\otimes 1_N)$ ($s \in \mathbb{C}$) be the non-unitary principal series with

$$\xi_{p-1} : M = \mathrm{SO}(d-1) \longrightarrow \mathrm{GL}_{\mathbb{R}}(\wedge^{p-1}\mathbb{R}^{d-1}).$$

The following properties are known.
♣$_1$ $\pi_{p-1}(s)$ is irreducible unitarizable iff

$$s \in \sqrt{-1}\mathbb{R} \cup (-\rho_p, \rho_p) \qquad (\text{where } \rho_p = \tfrac{d-1}{2} - p + 1).$$

♣$_2$ $\pi_{p-1}(s)$ $(\mathrm{Re}(s) > 0)$ is reducible iff

$$[\,s = \rho_p\,] \quad \text{or} \quad [\,s = \rho + k, \quad \exists k \in \mathbb{N} = \{0, 1, \ldots\}\,].$$

♣$_3$ $\pi_{p-1}(\rho_p)$ contains a unique irreducible $(\mathfrak{g}, K)$-submodule $\delta^{(p)} \hookrightarrow \pi_{p-1}(\rho_p)$.
For $k \in \mathbb{N}$, $\pi_{p-1}(\rho + k)$ has a unique irreducible $(\mathfrak{g}, K)$-submodule $\delta_k^{(p)} \hookrightarrow \pi_{p-1}(\delta + k)$.
♣$_4$ $\{\delta^{(p)}\} \cup \{\delta_k^{(p)} | k \in \mathbb{N}\} \subset \hat{G}_{\mathrm{d}}^H$.

We remark that $\hat{G}_{\mathrm{d}}^H$ is *not* exhausted by $\delta^{(p)}$ and $\delta_k^{(p)}$.

Theorem 3. *Let* $\{\mathfrak{a}_n\}$ *and* $\Gamma_n = \Gamma(\mathfrak{a}_n)$ *be as in Theorem 1. Suppose the existence of "spectral gap" at* $\delta^{(p)}$ *along* $\pi_{p-1}(s)$ *,i.e.,*

$$(\exists\epsilon > 0)(\forall n \in \mathbb{N})$$
$$[\,(m_{\Gamma_n}(\pi_{p-1}(s)) \neq 0,\ |s| < \rho_p) \implies (|s| \leqslant \rho_p - \epsilon)\,]$$

Then, for $\pi = \delta^{(p)}$ *and* $\tau : K = \mathrm{SO}(d) \rightarrow \mathrm{GL}(\wedge^p\mathbb{R}^d)$, *we have the formula:*

$$\lim_{n\to\infty} \frac{\mathbb{P}_\tau(\Gamma_n)_\pi}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} = \frac{1}{\pi^{p/2}}\frac{\Gamma(\rho_p + 1/2)}{\Gamma(\rho_p)} = d_\tau^{H\backslash G}(\pi) \tag{3.2}$$

Remark : (i) Although we do not settle the case for $\delta_k^{(p)}$'s yet, we expect a similar formula.
(ii) $\delta^{(p)}$ is *not* integrable (on $H\backslash G$).
(iii) Theorem is true under a weaker hypothesis

$$(\exists \epsilon > 0)(\forall n \in \mathbb{N})$$
$$[\,(\mathbb{P}_{\tau_p}(\Gamma_n)_{\pi_{p-1}(s)} \neq 0,\ |s| < \rho_p) \implies (|s| \leqslant \rho_p - \epsilon)\,].$$

(iv) The first identity of (3.2) was conjectured by Bergeron in a geometric form (explained below). His method may yields a proof of the formula under a spectral gap hypothesis for Hodge-Laplacian on p-forms.

3.4. Application to geometry. Let $G = \mathrm{SO}_0(d,1)$ and $H \cong \mathrm{SO}_0(d-p,1) \times \mathrm{SO}(p)$ with $1 \leqslant p < [d/2]$. Given a torsion free lattice $\Gamma \in \mathfrak{L}_G^H$, we have a $(d-p)$-dimensional cycle

$$C_H^\Gamma = \Gamma_H\backslash H/K_H \overset{\iota}{\hookrightarrow} \Gamma\backslash G/K$$

on $\Gamma\backslash G/K$. Then, the harmonic Poincare dual form ω_H^Γ of C_H^Γ is defined by

$$\begin{aligned}[C_H^\Gamma] \in \mathrm{H}_{d-p}(\Gamma_H\backslash H/K_H;\mathbb{Z}) \overset{\iota_*}{\to} \mathrm{H}_{d-p}(\Gamma\backslash G/K;\mathbb{Z}) \to\ & \mathrm{H}^{d-p}(\Gamma\backslash G/K;\mathbb{R})^\vee \\ \overset{\mathrm{PD}}{\cong}\ & \mathrm{H}^p(\Gamma\backslash G/K;\mathbb{R}) \\ \cong\ & \{\,\text{harmonic } p\text{-forms}\} \ni \omega_H^\Gamma,\end{aligned}$$

where PD is the Poincaré duality map. The L^2-norm of ω_H^Γ is defined as

$$\|\omega_H^\Gamma\|^2 = \int_{\Gamma\backslash G/K} \omega_H^\Gamma \wedge *\omega_H^\Gamma,$$

where $*$ is the Hodge $*$-operator of $\Gamma\backslash G/K$

Proposition 4. *Let $\{\Gamma_n = \Gamma_L(\mathfrak{a}_n)\}$ be as in Theorem 1. Suppose the 'H-spectral gap hypothesis'*

$$(\exists \epsilon > 0)(\forall n \in \mathbb{N})$$
$$[\,(\mathbb{P}_{\tau_p}(\Gamma_n)_{\pi_{p-1}(s)} \neq 0,\ |s| < \rho_p) \implies (|s| \leqslant \rho_p - \epsilon)\,].$$

is true if $p > 1$. Then,

$$\lim_{n\to\infty} \frac{\|\omega_H^{\Gamma_n}\|^2}{\mathrm{vol}(C_H^{\Gamma_n})} = \frac{1}{\pi^{p/2}} \frac{\Gamma(\rho_p + 1/2)}{\Gamma(\rho_p)}. \tag{3.3}$$

Remark :
(1) The form ω_H^Γ is explicitly constructed as a residue of the analytic continuation of some Poincaré series ([7], [8]).

(2) The formula (3.3) for $p = 1$ is proved by a geometric method [1]. The unconditional validity of (3.3) for $p > 1$ is also conjectured by [1].

4. A few words on proofs

Following [11] (where the case $G = \mathrm{U}(p,q)$, $H = \mathrm{U}(p-1,q) \times \mathrm{U}(1)$ is discussed), we prove Theorem 2 by showing the two inequalities :
(1)
$$\limsup_{n\to\infty} \frac{\mathbb{P}_\tau(\Gamma_n)_\pi}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} \leqslant \frac{1}{\pi^{p/2}} \frac{\Gamma(\rho_p + 1/2)}{\Gamma(\rho_p)}.$$
To prove this, we follow the argument used by [9] in the proof of the limit multiplicity formula.

(2)
$$\liminf_{n\to\infty} \frac{\mathbb{P}_\tau(\Gamma_n)_\pi}{\mathrm{vol}(\Gamma_n \cap H\backslash H)} \geqslant \frac{1}{\pi^{p/2}} \frac{\Gamma(\rho_p + 1/2)}{\Gamma(\rho_p)}.$$
This part is accomplished by a form of relative trace formula.

5. Remarks

- Similarly, we can treat the cases :
  - $G = \mathrm{U}(p,q)$, $H = \mathrm{U}(p-1,q) \times \mathrm{U}(1)$
  - $G = \mathrm{SO}_0(p,q)$, $H = \mathrm{SO}_0(p-1,q)$
  - $G = \mathrm{U}(n,1)$, $H = \mathrm{U}(n-p,1) \times \mathrm{U}(p)$ $(1 \leqslant p < n)$
- We expect the same method works at least when the split rank of $H\backslash G$ is 1.
- The following (naive) question seems natural. For $S \subset \hat{G}$, set
$$\mu_\tau^H(\Gamma; S) = \sum_{\pi\in S} \mathbb{P}_\tau(\Gamma)_\pi.$$
Does the measure
$$S \mapsto \frac{\mu_\tau^H(\Gamma_n; S)}{\mathrm{vol}(\Gamma_n \cap H\backslash H)},$$
approximate the spectral measure (Plancherel measure) of the decomposition of $L^2(H\backslash G;\tau)$? By extending the argument in [11], we already have a regorus result on this observation for the case $(G,H) = (\mathrm{U}(p,q), \mathrm{U}(p-1,q))$ ([12]).

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