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Genelarized Whittaker functions of degenerate principal series

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Abstract

In the theory of modular forms, modular forms with weights are important objects. For automorphic forms on $SL(2,\mathbb{R})$, the notion of weights are translated to characters of $SO(2)$. Hence for general cases, K-types of admissible representations can be seen as a generalization of weights of corresponding automorphic forms. In this paper, we consider degenerate principal series representations and define a class of their K-types which are called strongly spherical (Definition 3.2). And we give a characterization of generalized Whittaker functions with strongly spherical K-types of degenerate principal series representation (Theorem 5.2). The contents in this paper will appear with concrete proofs in [2].

1 Notation and preliminaries

In this section we give a quick review of some definitions and well known facts in the representation theory of Lie groups.

Let G be a connected real semisimple Lie group, K a maximal compact subgroup and θ the associated Cartan involution. Throughtout this paper we assume that G is split over $\mathbb{R}$ and has a complexification $G_{\mathbb{C}}$. The differentiation of θ is also written by same symbol. The associated Cartan decomposition of Lie algebra $\mathfrak{g}$ of G is denoted by $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{s}$. Here $\mathfrak{k}$ and $\mathfrak{s}$ are eigenspaces of θ with eigenvalues 1 and -1 respectively.

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Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{s}$ and Σ the root system of $(\mathfrak{g}, \mathfrak{a})$. Its Weyl group W is isomorphic with $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Fix a positive system Σ^+ of Σ and denote the set of simple roots by $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Let $\mathfrak{n}$ be the sum of the root space $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X$ for any $H \in \mathfrak{a}\}$ for $\alpha \in \Sigma^+$, i.e., $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$. Then we have an Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = KAN$ where $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$. Also we define $\bar{\mathfrak{n}} = \oplus_{\alpha\in\Sigma^+}\mathfrak{g}_{-\alpha}$ and $\bar{N} = \exp \bar{\mathfrak{n}}$. Let us denote the Killing form on $\mathfrak{g}$ by B. For $\lambda \in \mathfrak{a}^*$, we take $H_\lambda \in \mathfrak{a}$ satisfying the equations $\lambda(H) = B(H_\lambda, H)$ for any $H \in \mathfrak{a}$. We introduce an inner product $\langle\ ,\ \rangle$ on $\mathfrak{a}^*$ defined by $\langle\mu, \nu\rangle = B(H_\mu, H_\nu)$ for $\mu, \nu \in \mathfrak{a}^*$.

We denote the centralizer of A in K by M. Then a minimal parabolic subgroup P is defined by $P = MAN$. Let $\Theta \subset \Pi$ be a finite subset and define the parabolic subgroup P_Θ associated to Θ as follows. Let $\mathfrak{a}_\Theta = \{H \in \mathfrak{a} \mid \alpha(H) = 0$ for any $\alpha \in \Theta\}$ and $\mathfrak{a}_\Theta^\perp$ the orthgonal complement of $\mathfrak{a}_\Theta$ in $\mathfrak{a}$ with respcet to the Killing form. Furthermore let $\mathfrak{n}_\Theta = \bigoplus_{\alpha\in\Sigma^+\setminus\mathrm{span}(\Theta)} \mathfrak{g}_\alpha$ and $\mathfrak{m}_\Theta = \mathfrak{a}_\Theta^\perp \oplus \bigoplus_{\alpha\in\Sigma^+\cap\mathrm{span}(\Theta)} \mathfrak{g}_\alpha$. Then we can define the parabolic subalgebra associated to Θ by $\mathfrak{p}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta \oplus \mathfrak{n}_\Theta$. Let $L_\Theta = Z_G(\mathfrak{a}_\Theta)$, $K_\Theta = L_\Theta \cap K$ and $M_\Theta = K_\Theta \exp(\mathfrak{m}_\Theta \cap \mathfrak{s})$. Then we can define the parabolic subgroup assocated to Θ by $P_\Theta = M_\Theta A_\Theta N_\Theta$. If $\Theta = \emptyset$, the parabolic subgroup $P_\emptyset = M_\emptyset A_\emptyset N_\emptyset$ equals to the minimal parabolic subgroup $P = MAN$ defined above.

We write $\mathfrak{g}_\mathbb{C}$, $\mathfrak{k}_\mathbb{C}$ etc. as the complexfications of $\mathfrak{g}$, $\mathfrak{k}$ etc. Let $U(\mathfrak{g})$, $U(\mathfrak{k})$ etc. be the universal enveloping algebras of complexfications of $\mathfrak{g}$, $\mathfrak{k}$, etc. Also let $Z(\mathfrak{g})$, $Z(\mathfrak{k})$ be the centers of universal enveloping algebras $U(\mathfrak{g})$, $U(\mathfrak{k})$ respectively. As it is well-known, there is an inclusion

$$Z(\mathfrak{g}) \subset U(\mathfrak{a}) \oplus \bar{\mathfrak{n}}_\mathbb{C} U(\mathfrak{g}).$$

Let $\sigma\colon Z(\mathfrak{g}) \to U(\mathfrak{a})$ be the projection map along this decomposition. Put $\rho = \mathrm{tr}(\mathrm{Ad}|_\mathfrak{n}) \in \mathfrak{a}_\mathbb{C}^*$, then we can define the ρ-shfted map $\sigma'\colon Z(\mathfrak{g}) \to U(\mathfrak{a})$ by $\sigma'(X)(\lambda) = \sigma(X)(\lambda - \rho)$ for $X \in Z(\mathfrak{g})$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. It is well known that this map gives an algebra isomorphism

$$\sigma'\colon Z(\mathfrak{g}) \longrightarrow U(\mathfrak{a})^W,$$

which is called Harish-Chandra isomorphism. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$, we can define a character of $Z(\mathfrak{g})$ by

$$\begin{array}{rccc} \chi_\lambda\colon & Z(\mathfrak{g}) & \longrightarrow & \mathbb{C} \\ & X & \longmapsto & \sigma'(X)(\lambda). \end{array}$$

For $C^\infty(G, E)$, the space of smooth functions from G to a finite dimentional vector space E, we can consider natural actions of G and $\mathfrak{g}$ by left (right) translations and left (right) derivations, i.e.,

$$L_g f(x) = f(g^{-1}x), \qquad R_g f(x) = f(xg), \tag{1.1}$$

$$L_X f(x) = \frac{d}{dt} L_{(\exp tX)} f(x)|_{t=0}, \qquad R_X = \frac{d}{dt} R_{(\exp tX)} f(x)|_{t=0}, \tag{1.2}$$

where $x, g \in G$, $X \in \mathfrak{g}$ and $f \in C^\infty(G, E)$.

Let (π, E) be a continuous representation of G where E is a Hausdorff locally convex complete topological vector space. We write the space of K-finite vectors of E by E_K.

2 Poisson transform on vector bundle.

The Poisson transform is a continuous G-homomorphism from a spherical principal series representation to the space of right K-invariant functions on G. As a generalization of this, we will define a vector-valued Poisson transform and determine its image.

Let (τ, V_τ) be an irreducible unitary representation of K and λ an element of $\mathfrak{a}_{\mathbb{C}}^*$. Then we consider the induced representation $\pi_{\tau,\lambda}$ realized as follows. The rerpresentation space is

$$\mathcal{H}_{\tau,\lambda}^\infty = \{f \in C^\infty(G, V_\tau) \mid f(gman) = \tau(m)^{-1} a^{\lambda-\rho} f(g) \text{ for } (m, a, n, g) \in M \times A \times N \times G\}$$

and G acts on this space by left translation, i.e., $\pi_{\tau,\lambda}(g) f(x) = L_g f(x) = f(g^{-1}x)$ for $f \in \mathcal{H}_{\tau,\lambda}^\infty$ and $g \in G$. This is an admissible representation of G with infinitesimal character χ_λ. Also we denote the space of K-finite vectors of $\mathcal{H}_{\tau,\lambda}^\infty$ by $H_{\tau,\lambda}$ which becomes a $(\mathfrak{g}_{\mathbb{C}}, K)$-module naturally.

Also we consider another induced representation. The representation space is

$$C_\tau^\infty(G/K; \chi_\lambda) = \left\{ f \in C^\infty(G, V_\tau) \mid \begin{array}{c} f(gk) = \tau(k)^{-1} f(g), \ (k, g) \in K \times G, \\ R_X f = \chi_\lambda(X) f \text{ for } X \in Z(\mathfrak{g}) \end{array} \right\}$$

and G acts on this space by left translation. We denote the space of its K-finite vectors by $C_\tau^\infty(G/K; \chi_\lambda)_K$.

We define the generalized Harish-Chandra C-function as follows,

$$C(\lambda, \tau) = \int_{\overline{N}} \tau(k(\overline{n})) e^{-(\lambda+\rho) H(\overline{n})} \, d\overline{n}.$$

Here $g = k(g)\exp H(g)n(g)$ for $k(g) \in K, H(g) \in \mathfrak{a}$ and $n(g) \in N$. It is known that this integral is absolutely convergent by the operator norm of $\mathrm{End}(V_\tau)$ in $\{\lambda \in \mathfrak{a}_\mathbb{C}^* \mid \mathrm{Re}\langle\lambda,\alpha\rangle > 0 \text{ for any } \alpha \in \Sigma^+\}$. It is meromorphically continued in all $\mathfrak{a}_\mathbb{C}^*$ (cf. [4]).

Since M is the finite abelian group, V_τ can be decomposed as the direct sum of 1-dimentional repsesentations of M. Therefore we can take a basis $\{v_1,\ldots,v_l\}$ of V_τ so that there exist 1-dimentional representation σ_i $(i = 1,\ldots,l)$ of M such that $\tau(m)v_i = \sigma_i(m)v_i$ $(i = 1,\ldots,l)$ for $m \in M$. Also we take the dual basis $\{v_1^*,\ldots,v_l^*\}$ of $V_\tau^* = \mathrm{Hom}_\mathbb{C}(V_\tau,\mathbb{C})$, i.e., each v_i satisfies $v_i^*(v_j) = \delta_{ij}$ for $i,j = 1,\ldots,l$. We regard V_τ^* as a representation space of M by the contragradient representation.

Definition 2.1 (Poisson transform). *We define the G-homomorphism $\mathcal{P}_{\tau,\lambda}$ from $\mathcal{H}_{\tau,\lambda}^\infty$ to $C_\tau^\infty(G/K;\chi_\lambda)$ by*

$$\begin{array}{cccc} \mathcal{P}_{\tau,\lambda}\colon & \mathcal{H}_{\tau,\lambda}^\infty & \longrightarrow & C_\tau^\infty(G/K;\chi_\lambda) \\ & f & \longmapsto & \int_K \tau(k)f(gk)\,dk \end{array}$$

This is called the Poisson transform.

We see that $\mathcal{P}_{\tau,\lambda}$ gives a bijection between the K-finite subspaces for generic $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

Theorem 2.2. *We put following assumptions.*

1. *$\lambda \in \mathfrak{a}_\mathbb{C}^*$ is regular and dominant, i.e.,*

$$2\frac{\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle} \notin \{0,-1,-2,\ldots\} \text{ for any } \beta \in \Sigma^+.$$

2. *The determinant of $C(\tau,\lambda) \in \mathrm{End}(V_\tau)$ is nonzero.*

Then $\mathcal{P}_{\tau,\lambda}$ gives a $(\mathfrak{g}_\mathbb{C},K)$-isomorphism,

$$\mathcal{P}_{\tau,\lambda}\colon H_{\tau,\lambda} \xrightarrow{\sim} C_\tau^\infty(G/K;\chi_\lambda)_K.$$

Remark 2.3. *This theorem is first proved by An Yang [5] in more general settings. However Yang put a stronger assumption*

$$2\frac{\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle} \notin \mathbb{Z} \text{ for any } \beta \in \Sigma.$$

This is too strong for our purpose in this paper. Therefore we need a refined theorem under the weaker condition as above.

3 Strongly spherical K-types and vector valued Poisson transforms of degenerate principal series representations

Our purpose of this note is to give a charactarization of the vector-valued generalized Whittaker functions of degenerate principal series. To do this, we need the Poisson transforms on degenerate principal series representations. Hence we need to restrict the vector-valued Poisson transform to degenerate principal series representations and determine their images.

Take a finite subset $\Theta \subset \Pi$ and let P_Θ be the corresponding parabolic subgroup of G. For $\lambda \in (\mathfrak{a}_\Theta^*)_\mathbb{C}$, we define a character λ_Θ of $\mathfrak{p}_\Theta$ by

$$\begin{array}{rccc} \lambda_\Theta: & \mathfrak{p}_\Theta & \longrightarrow & \mathbb{C} \\ & X+H & \longmapsto & \lambda(H), \end{array}$$

where $X \in \mathfrak{m}_\Theta + \mathfrak{n}_\Theta$ and $H \in \mathfrak{a}_\Theta$. We take a character Λ_Θ of P_Θ whose differentiation is λ_Θ. Then we define a degenerate principal series representation of G as follows. The representation space is $C^\infty(G/P_\Theta;\Lambda_\Theta) = \{f \in C^\infty(G) \mid f(gp) = \Lambda_\Theta(p)f(g)$ for $p \in P_\Theta, g \in G\}$. The action of G on this space is defined by left translation. We denote the space of K-finte vectors of $C^\infty(G/P_\Theta;\Lambda_\Theta)$ by $H_{\Theta,\lambda}$.

Definition 3.1 (annihilator ideal). *We define a left ideal of $U(\mathfrak{g})$ by*

$$J_\Theta(\lambda) = \sum_{X \in (\mathfrak{p}_\Theta)_\mathbb{C}} U(\mathfrak{g})(X - \lambda_\Theta(X))$$

and also define a two-sided ideal

$$I_\Theta(\lambda) = \bigcap_{g \in G} \mathrm{Ad}(g) J_\Theta(\lambda).$$

This two-sided ideal $I_\Theta(\lambda)$ is studied by H. Oda and T. Oshima in [3] and they give explicit generators of $I_\Theta(\lambda)$. This ideal is very important tool to investigate $C^\infty(G/P_\Theta;\Lambda_\Theta)$, because we can show that for any $X \in I_\Theta(\lambda)$ and $f \in C^\infty(G/P_\Theta;\Lambda_\Theta)$, we have $R_X f = 0$, i.e., $I_\Theta(\lambda)$ is the annihilator ideal of $C^\infty(G/P_\Theta;\Lambda_\Theta)$. Also it is known that $I_\Theta(\lambda)$ is the annihilator of the generalized Verma module $U(\mathfrak{g})/J_\Theta(\lambda)$.

We define the notion of strongly spherical K-types.

Definition 3.2 (Strongly spherical K-type). *Let (τ, V_τ) be a irreducible unitary representation of K such that $\dim \mathrm{Hom}_K(V_\tau, H_{\Theta,\lambda}) \neq 0$. We call this represntation τ a strongly spherical K-type of $H_{\Theta,\lambda}$ if the dimension of $V_\tau^{\mathfrak{m}_\Theta \cap \mathfrak{k}} = \{v \in V_\tau \mid \tau(X)v = 0$ for $X \in \mathfrak{m}_\Theta \cap \mathfrak{k}\}$ is equal to 1.*

Remark 3.3. *If $\Theta = \emptyset$, i.e., P_Θ is minimal parabolic subgroup, this condition says V_τ is 1-dimensional because $\mathfrak{m}_\Theta$ is trivial. On the other hand, if $(K, M_\Theta \cap K)$ is a symmetric pair, it is easy to see that every irreducible unitary representation of K is strongly spherical.*

For these strongly shperical K-types, we can consider vector valued Poisson transform of degenerate principal series. And we can determine its image. For an irreducible representation (τ, V_τ) of K, we define a space

$$C_\tau^\infty(G/K; I_\Theta(\lambda)) =$$
$$\{f \in C^\infty(G, V_\tau) \mid f(gk) = \tau(k^{-1})f(g), R_X f = 0 \text{ for } g \in G, k \in K, X \in I_\Theta(\lambda)\}.$$

This is a G-representation by the left translation.

Theorem 3.4. *We use the notations as above. For $\lambda \in (\mathfrak{a}_\Theta^*)_{\mathbb{C}}$, we assume that*

1. *$\lambda + \rho$ is regular and dominant.*

2. $\det C(\tau, \lambda + \rho) \neq 0$.

Let (τ, V_τ) be a strongly shperical K-type of $H_{\Theta,\lambda}$. Then the restirction of $\mathcal{P}_{\tau,\lambda}$ to $H_{\Theta,\lambda}$ gives a following $(\mathfrak{g}_{\mathbb{C}}, K)$-isomorphism,

$$\begin{array}{rccc} \mathcal{P}_{\Theta,\lambda}: & H_{\Theta,\lambda} & \longrightarrow & C_\tau^\infty(G/K; I_\Theta(\lambda))_K \\ & \phi & \longmapsto & \int_K \tau(k)\phi(gk)\, dk. \end{array}$$

Here we note that we can see $\mathfrak{a}_\Theta^* \subset \mathfrak{a}^*$ by the Killing form B.

Proof. By the assumption, we have the $(\mathfrak{g}_{\mathbb{C}}, K)$-isomorphism

$$\begin{array}{rccc} \mathcal{P}_{\tau,\lambda}: & H_{\tau,\lambda} & \longrightarrow & C_\tau^\infty(G/K; \chi_\lambda)_K \\ & \phi & \longmapsto & \int_K \tau(k)\phi(gk)\, dk. \end{array}$$

Since $H_{\Theta,\lambda}$ is a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule of $H_{\tau,\lambda}$, we have

$$\mathcal{P}_{\tau,\lambda}(H_{\Theta,\lambda}) \subset C_\tau^\infty(G/K; I_\Theta(\lambda))_K.$$

Here we notice that since it is easy to show that $\sum_{X\in Z(\mathfrak{g})} U(\mathfrak{g})(X-\chi_\lambda(X)) \subset I_\Theta(\lambda)$, we have $C^\infty_\tau(G/K; I_\Theta(\lambda)) \subset C^\infty_\tau(G/K;\chi_\lambda)$. It remains to show that $H_{\Theta,\lambda} \supset \mathcal{P}^{-1}_{\tau,\lambda}(C^\infty_\tau(G/K; I_\Theta(\lambda))_K)$. To show this, we take an arbitrary element $u \in C^\infty_\tau(G/K; I_\Theta(\lambda))$. We can see $\lambda \in (\mathfrak{a}^*_\Theta)_\mathbb{C}$ as an element of $\mathfrak{a}^*_\mathbb{C}$, hence we denote this by $\lambda_\Theta \in \mathfrak{a}^*_\mathbb{C}$. We define a character of the Borel subalgebra of $\mathfrak{g}_\mathbb{C}$, $\mathfrak{b} = \mathfrak{a}_\mathbb{C} + \mathfrak{n}_\mathbb{C}$ as follows,

$$\begin{array}{rccc} \lambda_\mathfrak{b}: & \mathfrak{b} & \longrightarrow & \mathbb{C} \\ & H+X & \longmapsto & \lambda(H) \end{array}$$

where $H \in \mathfrak{a}_\mathbb{C}$ and $X \in \mathfrak{n}_\mathbb{C}$. We define a left ideal of $U(\mathfrak{g})$ by $J(\lambda_\mathfrak{b}) = \sum_{X\in\mathfrak{b}} U(\mathfrak{g})(X-\lambda_\mathfrak{b}(X))$. Then for any $X \in J(\lambda_\mathfrak{b})$ and $f \in H_{\tau,\lambda}$ we have $R_X f = 0$. Hence $\mathcal{P}^{-1}_{\tau,\lambda}u$ satisfies that $R_X\mathcal{P}^{-1}_{\tau,\lambda}u = 0$ for any $J_\Theta(\lambda)$ because $J_\Theta(\lambda) = I_\Theta(\lambda) + J(\lambda_\mathfrak{b})$ by the result of Oda and Oshima (Theorem 3.12 in [3]). This implies that there exists a representation σ of M_Θ which satisfies that $\mathrm{Hom}_{M_\Theta\cap K}(\sigma,\tau) \neq \{0\}$ and differentiation of σ is trivial. And $\mathcal{P}^{-1}_{\tau,\lambda}u \in C^\infty\text{-}\mathrm{Ind}^G_{P_\Theta}(\sigma\otimes e^{-\lambda}\otimes 1_{N_\Theta})$. However since $\dim V_\tau^{\mathfrak{m}_\Theta\cap\mathfrak{k}} = 1$, σ must be equal to $\Lambda_\Theta|_{M_\Theta}$. □

4 Maximal globalization

The vector-valued Poisson transform gives a $(\mathfrak{g}_\mathbb{C}, K)$-isomorphism from the degenerate principal series $H_{\Theta,\lambda}$ to $C^\infty_\tau(G/K; I_\Theta(\lambda))_K$ if τ is a strongly spherical K-type of $H_{\Theta,\lambda}$. Furthermore, we see that this $(\mathfrak{g}_\mathbb{C}, K)$-isomorphism extends to the continuous G-isomorphism.

Let X be an admissible $(\mathfrak{g}_\mathbb{C}, K)$-module with finite length. We consider the space of $(\mathfrak{g}_\mathbb{C}, K)$-homomorphisms from the dual $(\mathfrak{g}_\mathbb{C}, K)$-module X^* to $C^\infty(G)$, $\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(X^*, C^\infty(G))$ where G acts on $C^\infty(G)$ by left translation. Since $C^\infty(G)$ has a unifromly covergent topology and X^* has a countably many basis, we can define the complete locally convex topology on $\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(X^*, C^\infty(G))$. On the other hand, G can also act on $C^\infty(G)$ by right translation. This action is continuous on the topology of $\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(X^*, C^\infty(G))$. the space of K-finite elements of $\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(X^*, C^\infty(G))$ can be identified with $(X^*)^* \cong X$ by the evaluation at the origin, i.e., for $I \in \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(X^*, C^\infty(G))$, $X^* \ni v \mapsto I(v)(e) \in \mathbb{C}$ is a linear form of X^*. Hence $\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(X^*, C^\infty(G))$ is a continuous G representation and its K-finite subspace is X, i.e., $\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(X^*, C^\infty(G))$ is a globalization of X. This is called the maximal globalization [1].

Let us return to our setting. In the previous section we see that there is a

$(\mathfrak{g}_{\mathbf{C}}, K)$-isomorphism

$$\begin{array}{rccl} \mathcal{P}_{\Theta,\lambda}\colon & H_{\Theta,\lambda} & \longrightarrow & C^\infty_\tau(G/K; I_\Theta(\lambda))_K \\ & \phi & \longmapsto & \int_K \tau(k)\phi(gk)\,dk. \end{array}$$

This $(\mathfrak{g}_{\mathbf{C}}, K)$-isomorphism can be extend to G-isomorphism as follows. If (τ, V_τ) is a strongly spherical K-type of $(\mathfrak{g}_{\mathbf{C}}, K)$-module $H_{\Theta,\lambda}$, it is multiplicity free by definition. We fix a K-projection $p_\tau\colon H_{\Theta,\lambda} \to V_\tau$. We define a K-embedding $\iota_\tau\colon V_\tau^* \hookrightarrow H_{\Theta,\lambda}^*$ as the dual map of p_τ.

Theorem 4.1. *We assume that*

1. $\lambda_\Theta + \rho$ *is regular and dominant.*

2. $\det C(\lambda + \rho, \tau) \neq 0$.

Let (τ, V_τ) *be a strongly shperical* K*-type of* $H_{\Theta,\lambda}$*. Then we have the following topological* G*-isomorphism.*

$$\begin{array}{rccl} \Phi\colon & \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(H^*_{\Theta,\lambda}, C^\infty(G)) & \longrightarrow & C^\infty_\tau(G/K; I_\Theta(\lambda)) \\ & I & \longmapsto & \sum_{i=1}^l I(\iota_\tau(v_i^*))(g)v_i \end{array}.$$

5 Generalized Whittaker models

Finally we give the main theorem of this note. We can give a characterization of vector-valued generalized Whittaker functions as solutions of system of differential equations which comes from $I_\Theta(\lambda)$.

Let U be a closed subgroup of N and (η, V_η) an irreducible unitary representation of U. We consider a representation of G induced from η. The representation space is

$$C^\infty_\eta(U\backslash G) = \{f\colon G \to V_\eta^\infty \text{ smooth} \mid f(ug) = \eta(u)f(g) \text{ for all } u \in U, g \in G\}.$$

Here V_η^∞ stands for the space of smooth vectors of V_η. We note that V_η^∞ has a Hausdorff complete locally convex topology and we can define the derivation of $f\colon G \to V_\eta^\infty$ by the convergence on the topology of V_η^∞.

Definition 5.1 (Generalized Whittaker model). *Let* X *be an admissible* $(\mathfrak{g}_{\mathbf{C}}, K)$*-module with finite length. Let* U *be a closed subgroup of* N *and* (η, V_η) *an* *irreducible unitary representation of U. We consider the space of $(\mathfrak{g}_{\mathbb{C}}, K)$-homorphisms from X to $C_\eta^\infty(U\backslash G)$,*

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X, C_\eta^\infty(U\backslash G)).$$

If $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X, C_\eta^\infty(U\backslash G)) \neq \{0\}$, *we say X has generalized Whittaker models.*

We consider genelarized Whittaker models of $H_{\Theta,\lambda}$. Let (τ, V_τ) be a strongly spherical K-type of $H_{\Theta,\lambda}$. Take a irreducible unitary representation (η, V_η) of N. For the algebraic tensor product $V_\eta^\infty \otimes V_\tau$, we can define a natural topology comes from V_η^∞ because V_τ is finte dimensional. Hence we can consider the following space of smooth functions from G to $V_\eta^\infty \otimes V_\tau$,

$$C_{\eta,\tau}^\infty(U\backslash G/K) =$$
$$\{f\colon G \to V_\eta^\infty \otimes V_\tau \text{ smooth} \mid f(ugk) = \eta(u)\otimes\tau(k^{-1})f(g) \text{ for } u \in U, g \in G, k \in K,\}.$$

Also we define

$$C_{\eta,\tau}(U\backslash G/K; I_\Theta(\lambda)) = \{f \in C_{\eta,\tau}^\infty(U\backslash G/K) \mid R_X f = 0 \text{ for } X \in I_\Theta(\lambda)\}.$$

As a colloraly of Theorem 4.1, we have the follwing characterization of genelarized Whittaker models.

Theorem 5.2. *We use the same notations as Theorem 4.1. We assume that*

1. $\lambda_\Theta + \rho$ *is regular and dominant.*

2. $\det C(\lambda + \rho, \tau) \neq 0$.

Let (τ, V_τ) be a strongly shperical K-type of $H_{\Theta,\lambda}$. Then we have the following linear isomorphism.

$$\begin{array}{rccc} \Phi: & \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(H_{\Theta,\lambda}^*, C_\eta^\infty(U\backslash G)) & \longrightarrow & C_{\eta,\tau}^\infty(U\backslash G/K; I_\Theta(\lambda)) \\ & I & \longmapsto & \sum_{i=1}^l I(\iota_\tau(v_i^*))(g)v_i \end{array}.$$

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