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# Characteristic classes of symplectomorphism groups as discrete groups

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# 1 Problem

$(M,\omega)$ : closed symplectic manifold, $\dim = 2n$

Symp $M$: symplectomorphism group, $\mathrm{Symp}^\delta M$: with discrete topology

**Problem 1** *(widely open problem)*

$$H^*(\mathrm{BSymp}\, M) = ?$$
$$H^*(\mathrm{BSymp}^\delta M) = H^*(\mathrm{Symp}^\delta M) = ?$$

$M = \Sigma_g$ closed oriented surface $g \geq 2$ :

$$\mathrm{Symp}\,\Sigma_g \overset{\text{Moser}}{\simeq} \mathrm{Diff}^+\Sigma_g \overset{\text{Earle-Eells}}{\simeq} \mathcal{M}_g \Rightarrow H^*(\mathrm{BSymp}\,\Sigma_g) \cong H^*(\mathrm{Diff}\,\Sigma_g) \cong H^*(\mathcal{M}_g)$$

but $H^*(\mathrm{Symp}^\delta\Sigma_g)$ completely different

**joint work with D. Kotschick**

# 2 Gel'fand-Fuks cohomology theory

$\mathfrak{a}_n$: Lie algebra of formal vector fields on $\mathbb{R}^n$

$$\mathfrak{a}_n = \{\sum_i f_i \frac{\partial}{\partial x_i};\, f_i \in \mathbb{R}[[x_1, \cdots, x_n]]\}$$

$\mathrm{B}\Gamma_n$: Haefliger classifying space for codimension $n$ foliations

$H^*_{GF}(\mathfrak{a}_n, \mathrm{O}(n)) \longrightarrow H^*(\mathrm{B}\Gamma_n; \mathbb{R})$ Gel'fand-Fuks cohomology

**Theorem 2 (Gel'fand-Fuks)**

$$H^*_{GF}(\mathfrak{a}_n),\ H^*_{GF}(\mathfrak{a}_n, \mathrm{O}(n))$$

**finite dimensional**

Vey found a basis. In particular, the class with the lowest degree:

$$H_{GF}^{2n+1}(\mathfrak{a}_n, \mathrm{O}(n)) \ni h_1 c_1^n$$

close relation with the **Godbillon-Vey** class:

$$\mathrm{GV} \in H^{2n+1}(\mathrm{B}\Gamma_n; \mathbb{R})$$

case of $n = 1$ :

$\mathcal{F}$ : cod. 1 foliation on $X$ defined by 1-form $\theta \Rightarrow d\theta = \eta \wedge \theta$ for some $\eta$

$$\eta \wedge d\eta \text{ is a closed 3-form and } \mathrm{GV}(\mathcal{F}) = [\eta \wedge d\eta] \in H^3(X; \mathbb{R})$$

**Theorem 3 (Thurston)** *There is a one parameter family $\mathcal{F}_t$ of cod. 1 foliations on $S^3$ such that*

$$\mathrm{GV}(\mathcal{F}_t) = t \in H^3(S^3; \mathbb{R}) \cong \mathbb{R}$$

essential use of hyperbolic geometry

# 3 Gel'fand-Fuks cohomology of formal Hamiltonian vector fields

$\mathfrak{ham}_{2n}$: Lie algebra of formal Hamiltonian vector fields on $(\mathbb{R}^{2n}, \omega)$

$\mathrm{B}\Gamma_{2n}^{\omega}$: Haefliger classifying space for codimension $2n$ transversely symplectic foliation

$H_{GF}^*(\mathfrak{ham}_{2n}, \mathrm{U}(n)) \longrightarrow H^*(\mathrm{B}\Gamma_{2n}^{\omega}; \mathbb{R})$ Gel'fand-Fuks cohomology

very difficult open question (**mystery!**):

$H_{GF}^*(\mathfrak{ham}_{2n})$ infinitely generated ?

Gel'fand-Kalinin-Fuks: (cohomology is **bigraded**)

$$H_{GF}^*(\mathfrak{ham}_{2n}, \mathrm{Sp}(2n, \mathbb{R}))_{(\leq 0)} \cong \mathbb{R}[\omega, p_1, p_2, \cdots, p_n]/\text{Bott vanishing}$$

$$\omega^k p_1^{k_1} \cdots p_n^{k_n} = 0$$

for $k + k_1 + 2k_2 + \cdots + nk_n > n$

For the case $n = 1$, they also found

$$H^*_{GF}(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{(k)} = 0 \text{ for } k = 1, 2, 3$$

But, $\exists$ **exotic class!** in $H^7_{GF}(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{(4)}$ **Gel'fand-Kalinin-Fuks class** (1972)

In 1999, Metoki found another exotic class in $H^9_{GF}(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{(7)}$ **Metoki class**

On the other hand, **Perchik** (1975) obtained a formula for the Euler characteristics

$$\chi(H^*_{GF}(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{(k)})$$

$\Rightarrow$ $\exists$ 57 − 4(−1) more exotic classes! He could not prove $\dim H^*_{GF}(\mathfrak{ham}_2) = \infty$

**Conjecture 4 (Folklore)**

$$\dim H^*_{GF}(\mathfrak{ham}_{2n}, \mathrm{Sp}(2n, \mathbb{R})) = \infty, \ \dim H^*_{GF}(\mathfrak{div}_n, \mathrm{SL}(n, \mathbb{R})) = \infty$$

$\mathfrak{div}_n$: Lie algebra of formal divergence free vector fields on $\mathbb{R}^n$, $\mathfrak{ham}_2 = \mathfrak{div}_2$

# 4 Foliated cohomology

$(X, \mathcal{F})$: codimension $n$ foliated manifold

**Definition 5 (foliated cohomology)**

$\Omega^*_{\mathcal{F}}(X) := \Gamma(X, \Lambda^*(T^*_{\mathcal{F}})) = \Omega^*(X)/(\textit{forms vanishing on leaves})$

$$H^*_{\mathcal{F}}(X) = H^*(\Omega^*_{\mathcal{F}}(X))$$

wild object, hard to compute

# 5 Foliated cohomology in the symplectic case

$$\mathfrak{ham}_{2n} \cong \mathbb{R}[[x_1, \cdots, x_n; y_1, \cdots, y_n]]/\mathbb{R} \cong \prod_{k=1}^{\infty} S^k(\mathbb{R}^{2n}, \omega)$$

$$\mathfrak{ham}_{2n}^0 = \mathfrak{ham}_{2n} \text{ without constant term}$$

Kontsevich: transversely symplectic codimension $2n$ foliated manifold $(X, \mathcal{F})$

$$H^*_{GF}(\mathfrak{ham}_{2n}^0, \mathrm{U}(n)) \longrightarrow H^*_{\mathcal{F}}(X) \xrightarrow{\wedge\, \omega^n} H^{*+2n}(X; \mathbb{R})$$

$$H^*_{GF}(\mathfrak{ham}_{2n}^0, \mathrm{U}(n)) \longrightarrow H^{*+2n}(\mathfrak{ham}_{2n}, \mathrm{U}(n)) \longrightarrow H^{*+2n}(\mathrm{B}\Gamma\omega_{2n}; \mathbb{R})$$

$\Rightarrow$ for any symplectic manifold $(M, \omega)$

$$H^*_{GF}(\mathfrak{ham}_{2n}^0, \mathrm{U}(n)) \longrightarrow H^*(\mathrm{Symp}^\delta(M, \omega); \mathbb{R})$$

Furthermore

$$\mathfrak{ham}_{2n}^1 = \mathfrak{ham}_{2n}^0 \text{ without linear term} \Rightarrow H^*_{GF}(\mathfrak{ham}_{2n}^0, \mathrm{Sp}(2n, \mathbb{R})) \cong H^*_{GF}(\mathfrak{ham}_{2n}^1; \mathbb{R})^{\mathrm{Sp}}$$

$\Rightarrow$ obtain homomorphisms

$$H^*_{GF}(\mathfrak{ham}_{2n}^1; \mathbb{R})^{\mathrm{Sp}} \longrightarrow H^{*+2n}(\mathfrak{ham}_{2n}, \mathrm{Sp}(2n, \mathbb{R})))$$

$$H^*_{GF}(\mathfrak{ham}_{2n}^1; \mathbb{R})^{\mathrm{Sp}} \longrightarrow H^*(\mathrm{Symp}(M, \omega); \mathbb{R})$$

in the case of $(\Sigma_g, \omega)$, what is

$$H^*_{GF}(\mathfrak{ham}_2^1)^{\mathrm{Sp}} \longrightarrow H^{*+2}_{GF}(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R})))?$$

$$H^*_{GF}(\mathfrak{ham}_2^1)^{\mathrm{Sp}} \longrightarrow H^*(\mathrm{Symp}(\Sigma_g, \omega); \mathbb{R})?$$

detects Gel'fand-Kalinin-Fuks and Metoki classes?

**Answer: yes for GKF class**

$\exists$ unique element $\eta \in H^5(\mathfrak{ham}_2^1)^{\mathrm{Sp}}$ s.t.

GKF class $= \eta \wedge \omega \in H^7(\mathfrak{ham}_2, \mathrm{Sp}(2,\mathbb{R}))$

merits of this approach:

(*) the stable cohomology

$$\lim_{n\to\infty} H^*_{GF}(\mathfrak{ham}_{2n}) \cong \mathbb{R}[\omega]$$

is uninteresting (Guillemin-Shnider), but the stable cohomology

$$\lim_{n\to\infty} H^*_{GF}(\mathfrak{ham}^1_{2n})^{\mathrm{Sp}}$$

seems to be highly non-trivial!

(**) $H^*_{GF}(\mathfrak{ham}^1_{2n})^{\mathrm{Sp}}$ is easier than $H^*_{GF}(\mathfrak{ham}_{2n})$ we are trying to decompose **Metoki class**, and possibly define new exotic classes

(***) similarly for the case of $\mathfrak{div}_n$

$$\exists\ \eta \wedge \text{volume form} \in H^{5+n}(\mathfrak{div}_n, \mathrm{SL}(n,\mathbb{R}))?$$

(****) we could try to prove

$$\dim H^*_{GF}(\mathfrak{ham}^1_{2n})^{\mathrm{Sp}} = \infty$$

# 6 The GV classes and foliated cohomology

The Godbillon-Vey classes

$$\mathrm{GV} = h_1 c_1^n \in H^{2n+1}_{GF}(\mathfrak{a}_n, \mathrm{O}(n))$$

are **NOT** stable classes, but they can be decomposed as

$$h_1 c_1^n = h_1 \wedge c_1^n \quad \text{where}$$
$$h_1 \in H^1_{GF}(\mathfrak{a}_n^0, \mathrm{O}(n))$$

is a **stable leaf invariant!** coming from the determinant of the holonomy

$\mathfrak{a}_n^0 = \mathfrak{a}_n$ without constant term

$(X, \mathcal{F})$: codimension $n$ foliated manifold $\Rightarrow H^1_{\mathcal{F}}(X) \ni h_1(\mathcal{F}) \overset{\wedge c_1^n}{\longmapsto} GV(\mathcal{F}) \in H^{2n+1}(X; \mathbb{R})$

So if $h_1(\mathcal{F}) = 0$, then $\mathrm{GV}(\mathcal{F}) = 0$ as well.

We have proved that the original GKF class can be decomposed into
a leaf invariant $\eta$ times $\omega$

# 7 Sketch of proof

Metoki gave a formula for GKF class , 15 lines (with about 100 terms!)

10 lines are **multiple of the symplectic form** $\omega$ others are not, but we expected that

the remaining terms will also be multiple of $\omega$ by adding suitable coboundaries

$\mathfrak{ham}^1_{2n}$ is a graded Lie algebra $\Rightarrow$

$H^*_{GF}(\mathfrak{ham}^1_{2n})^{\mathrm{Sp}}$ is bigraded and furthermore $C^*_{GF}(\mathfrak{ham}^1_{2n})^{\mathrm{Sp}}$ is a direct sum of

**finite subcomplexes**

We looked at the part $(H := (\mathbb{R}^{2n}, \omega))$

$C_{11}[32] = 0 \xrightarrow{\partial} C_{10}[30] \cong (\Lambda^{10} S^3 H)^{\mathrm{Sp}} \xrightarrow{\partial} C_9[28] \cong (\Lambda^8 S^3 H \otimes S^4 H)^{\mathrm{Sp}} \xrightarrow{\partial}$

$$C_8[26] \cong ((\Lambda^7 S^3 H \otimes S^5 H) \oplus (\Lambda^6 S^3 H \otimes \Lambda^2 S^4 H))^{\mathrm{Sp}} \xrightarrow{\partial}$$

$$\cdots \text{ more and more complicated terms } \cdots \xrightarrow{\partial}$$

$$C_2[14] \cong ((\Lambda^2 S^7 H) \oplus (S^3 H \otimes S^{11} H) \oplus \cdots))^{\mathrm{Sp}} \xrightarrow{\partial} C_1[12] \cong (S^{12} H)^{\mathrm{Sp}} \xrightarrow{\partial} C_0[10] = 0.$$

$\Rightarrow$ compute $H^5_{GF}(\mathfrak{ham}^1_{2n})^{\mathrm{Sp}}_{(20)}$

For $n = 1$, we found that rank is given by $0, 0, 0, 0, 4, 12, 9, 3, 1, 0$

$\Rightarrow \chi = -1$ and $H^5_{GF}(\mathfrak{ham}^1_2)^{\mathrm{Sp}}_{(20)} \cong \mathbb{R}$

# 8 Big open problem

**Problem 6** *Prove that the GKL class is* **geometrically** *non-trivial as an element of*

$$H^5(\mathrm{Symp}^\delta(\Sigma_g, \omega); \mathbb{R})$$

*or as an element of*

$$H^7(\mathrm{B}\Gamma_2^\omega; \mathbb{R}).$$

What kind of geometry should we use ? Probably hyperbolic geometry would be

**insufficient**