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SMITH SET FOR A NONGAP OLIVER GROUP

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1. Introduction

We study the Smith problem that two tangential representations are isomorphic or not for a smooth action on a homotopy sphere with exactly two fixed points ([11]). Two real G-modules U and V are called *Smith equivalent* if there exists a smooth action of G on a sphere Σ such that $S^G = \{x, y\}$ for two points x and y at which $T_x(\Sigma) \cong U$ and $T_y(\Sigma) \cong V$ as real G-modules which is a finite dimensional real vector space with a linear G-action. Let $Sm(G)$, called a Smith set, be the subset of the real representation ring $RO(G)$ of G consisting of the differences $U - V$ of real G-modules U and V which are Smith equivalent. In many groups, Smith equivalent modules are not isomorphic. Let $\mathcal{P}(G)$ be the set of subgroups of G of prime power order, possibly 1. We also define a subset $CSm(G)$ of $Sm(G)$ consisting of the differences $U - V \in Sm(G)$ of real G-modules U and V such that for the sphere Σ appearing in the definition of Smith equivalence of U and V satisfies that Σ^P is connected for every $P \in \mathcal{P}(G)$. For any $U - V \in CSm(G)$, G-modules U and V are $\mathcal{P}(G)$-matched pair, that is,

$$\mathrm{Res}^G_P U \cong \mathrm{Res}^G_P V$$

for any subgroup P of G of prime power order, possibly 1. Let $RO(G)$ be the real representation ring and we denote by $RO(G)_{\mathcal{P}(G)}$ the subset of $RO(G)$ consisting the differences of real $\mathcal{P}(G)$-matched pairs. Then $CSm(G)$ is a subset of $RO(G)_{\mathcal{P}(G)}$.

Proposition 1.1.

$$\begin{cases} 0 \in CSm(G) & \textit{if } G \textit{ is not of prime power order} \\ CSm(G) = \varnothing & \textit{if } G \textit{ is of prime power order.} \end{cases}$$

In this paper, we discuss the Smith problem for an Oliver nongap group. Throughout this paper we assume a group is finite.

2. $RO(G)_{\mathcal{P}(G)}$ and induced virtual modules

We denote by $\pi(G)$ the set of all primes dividing the order $|G|$ of G. For a prime p, we denote by $O^p(G)$, called the Dress subgroup of type p, the smallest normal subgroup of G with index a power of p:

$$O^p(G) = \bigcap_{L \trianglelefteq G, [G:L] = p^* \geq 1} L.$$

Note that $O^p(G) = G$ if $p \notin \pi(G)$. Let $\mathcal{L}(G)$ be the set of subgroups of G containing some Dress subgroup.

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Let

$$LO(G) := (RO(G)_{\mathcal{P}(G)})^{\mathcal{L}(G)} = \bigcap_{p\in\pi(G)} \ker(\mathrm{fix}^{O^p(G)}\colon RO(G) \to RO(G/O^p(G)) \cap RO(G)_{\mathcal{P}(G)}.$$

A group G is called *Oliver* if there is no series of subgroups

$$P \triangleleft H \triangleleft G$$

such that P and G/H are of prime power order and H/P is cyclic. An Oliver group can be characterized as a group having a one fixed action on a sphere ([2]). A group G is called *gap* if there is a real G-module W such that $V^{O^p(G)} = 0$ for any prime p and

$$\dim V^P > 2\dim V^H$$

for all pairs (P, H) of subgroups of G which satisfy that P is of prime power order and $P < H$. If G is a gap Oliver group, then $LO(G)$ is a subset of $CSm(G)$ ([8]). We remark that $CSm(G)$ is not a subset of $LO(G)$ in general (cf. [3]).

For an element not of prime power order, we call it an *NPP element.* We denote by a_G the number of real conjugacy classes of NPP elements of G.

Proposition 2.1. *$RO(G)_{\mathcal{P}(G)}$ is a free abelian subgroup of $RO(G)$ with rank a_G.*

For a complex G-module ξ we denote by $\overline{\xi}$ whose character is the complex conjugate of the character of ξ.

Proposition 2.2. *Let $p_1, p_2, \ldots, p_k$ be distinct primes each other and let $a_1, a_2, \ldots, a_k$ be positive integers. Put $G = C_{p_1^{a_1}} \times C_{p_2^{a_2}} \times \cdots \times C_{p_k^{a_k}}$, where $C_{p_j^{a_j}}$ is a cyclic group of order $p_j^{a_j}$. Then $RO(G)_{\mathcal{P}(G)}$ is spanned by the set of virtual real G-modules having characters as same as*

$$\bigotimes_j(\mathbb{C} - \xi_j) + \bigotimes_j(\mathbb{C} - \overline{\xi}_j),$$

where ξ_j's are irreducible complex $C_{p_j^{a_j}}$-modules or zero and two of them are nonzero at least. In particular the rank of $RO(G)_{\mathcal{P}(G)}$ is equal to $((\prod_j p_j^{a_j} - 1) - \sum_j(p_j^{a_j} - 1))/2$.

This proposition can be extend to nilpotent groups instead of cyclic groups.

Theorem 2.3. *Let $p_1, p_2, \ldots, p_k$ be distinct primes each other and P_j a nontrivial p_j-group for each j. Put $G = P_1 \times P_2 \times \cdots \times P_k$. Then the set of virtual real G-modules having characters as same as*

$$\bigotimes_j(\dim_{\mathbb{C}}(\xi_j)\mathbb{C} - \xi_j) + \bigotimes_j(\dim_{\mathbb{C}}(\xi_j)\mathbb{C} - \overline{\xi}_j),$$

where ξ_j's are irreducible complex P_j-modules or zero and two of them are nonzero at least, become a basis of $RO(G)_{\mathcal{P}(G)}$. In particular the rank of $RO(G)_{\mathcal{P}(G)}$ is equal to $((\prod_j q_j - 1) - \sum_j(q_j - 1))/2$, where q_j is the number of irreducible complex P_j-modules.

Theorem 2.4. *Let $p_1, p_2, \ldots, p_k$ be distinct primes each other, P a nontrivial p_1-group and C_j a nontrivial cyclic p_j-group for each $j \geq 2$. Put $G = P \times C_2 \times \cdots \times C_k$ which is an elementary group. Then $RO(G)_{\mathcal{P}(G)}$ is spanned by the set of virtual real G-modules $\mathrm{Ind}_E^G\eta$ for subgroups E and for virtual real E-modules η whose character is same as one of*

$$\bigotimes_j(\mathbb{C} - \xi_j) + \bigotimes_j(\mathbb{C} - \overline{\xi}_j),$$

where ξ_j's are 1-dimensional complex p_j-modules or zero and two of them are nonzero at least.

We denote by $\mathfrak{B}(G)$ the set of all virtual real G-modules as in Theorem 2.4 for an elementary group G.

$CSm(G)$ is a subset of

$$RO(G)^{\{G\}}_{\mathcal{P}(G)} = \ker(\mathrm{fix}^G\colon RO(G) \to RO(G/G)) \cap RO(G)_{\mathcal{P}(G)}.$$

For a nilpotent group G, by fixing $X_0 \in \mathfrak{B}(G)$, the set consisting of $X - X_0$ for $X \in \mathfrak{B}(G)$, $X \neq X_0$ spans $RO(G)^{\{G\}}_{\mathcal{P}(G)}$.

Artin's induction theorem gives the following.

Theorem 2.5. *The set*

$$\bigcup_C \{\mathrm{Ind}_C^G \eta \mid \eta \in \mathfrak{B}(C)\}$$

where C runs over all representative of conjugacy classes of cyclic subgroups of G not of prime power order spans the vector space $\mathbb{Q} \otimes_{\mathbb{Z}} RO(G)_{\mathcal{P}(G)}$ over the rational number field $\mathbb{Q}$. The set of differences of virtual modules of the above set spans $\mathbb{Q} \otimes_{\mathbb{Z}} RO(G)^{\{G\}}_{\mathcal{P}(G)}$.

The following theorem is related to Brauer's induction theorem.

Theorem 2.6. *An virtual G-module $RO(G)_{\mathcal{P}(G)}$ is described as a linear combination (with integer coefficients) of virtual modules of*

$$\bigcup_E \{\mathrm{Ind}_E^G \eta \mid \eta \in \mathfrak{B}(E)\}$$

where E runs over all representatives of conjugacy classes of elementary subgroups E of G. Furthermore, $RO(G)^{\{G\}}_{\mathcal{P}(G)}$ is described as a linear combination (with integer coefficients) of differences of the above virtual modules.

Let $\overline{\mathrm{NPP}}(G)$ be the set of all representatives of real conjugacy classes of NPP elements of G. For a normal subgroup N of G and $gN \in G/N$ we denote by $a_{G,N}(gN)$ the number of elements of $f_N^{-1}(gN)$, where $f_N\colon \overline{\mathrm{NPP}}(G) \to G/N$ is a mapping induced by a canonical epimorphism $G \to G/N$. It holds that

$$a_G = \sum_{gN \in G/N} a_{G,N}(gN).$$

For a normal subgroup N of G let

$$RO(G)^{\{N\}}_{\mathcal{P}(G)} = \ker(\mathrm{fix}^N\colon RO(G) \to RO(G/N)) \cap RO(G)_{\mathcal{P}(G)}.$$

We denote by G^{nil} the smallest normal subgroup of G by which a quotient group of G is nilpotent:

$$G^{\mathrm{nil}} = \bigcap_{p \in \pi(G)} O^p(G)$$

Proposition 2.7. *Let p be a prime and N a normal subgroup of G. The rank of $RO(G)^{\{N\}}_{\mathcal{P}(G)}$ is less than or equal to*

$$\sum_{gN \in G/N} \max(a_{G,N}(gN) - 1, 0).$$

The rank of $LO(G)$ is greater than or equal to

$$\sum_{gG^{\mathrm{nil}} \in G/G^{\mathrm{nil}})} \max(a_{G,G^{\mathrm{nil}}}(gG^{\mathrm{nil}}) - 1, 0)$$

and in particular if G/G^{nil} is a p-group then the equality holds.

Theorem 2.8 ([4, Morimoto]). *Let G be a finite group. $Sm(G) \subset RO(G)^{\{G^{\cap 2}\}}_{\mathcal{P}(G)}$ where $G^{\cap 2} = \cap_{[G:L]\leq 2} L$ is a normal subgroup of G.*

Therefore, if G/G^{nil} is an elementary abelian 2-group then $CSm(G) \subset LO(G)$.

Theorem 2.9. *Let N be a normal subgroup of G. Then $\mathbb{Q} \otimes_{\mathbb{Z}} RO(G)^{\{N\}}_{\mathcal{P}(G)}$ is spanned by the set of virtual modules $X - Y$ such that*

$$X, Y \in \bigcup_{C} \{\mathrm{Ind}_C^G \eta \mid \eta \in \mathfrak{B}(C)\}$$

with $\mathrm{fix}^N(X - Y) = 0$ *in $RO(G/N)$, where C runs over all representative of conjugacy classes of cyclic subgroups of G not of prime power order.*

Theorem 2.10. *Let N be a normal subgroup of G. An virtual G-module $RO(G)^{\{N\}}_{\mathcal{P}(G)}$ is described as a linear combination (with integer coefficients) of virtual modules $X - Y$ such that*

$$X, Y \in \bigcup_{E} \{\mathrm{Ind}_E^G \eta \mid \eta \in \mathfrak{B}(E)\}$$

with $\mathrm{fix}^N(X - Y) = 0$ *in $RO(G/N)$, where E runs over all representatives of conjugacy classes of elementary subgroups E of G.*

3. Weak gap condition

We say that a smooth G-manifold X satisfies the *weak gap condition* (WGC) if the conditions (WGC1)–(WGC4) all hold (cf. [5]).

(WGC1) $\dim X^P \geq 2 \dim X^H$ for every $P < H \leq G$, $P \in \mathcal{P}(G)$.

(WGC2) If $\dim X^P = 2 \dim X^H$ for some $P < H \leq G$, $P \in \mathcal{P}(G)$, then $[H : P] = 2$, $\dim X^H > \dim X^K + 1$ for every $H < K \leq G$, and X^H is connected.

(WGC3) If $\dim X^P = 2 \dim X^H$ for some $P < H \leq G$, $P \in \mathcal{P}(G)$, and $[H : P] = 2$, then X^H can be oriented in such a way that the map $g \colon X^H \to X^H$ is orientation preserving for any $g \in N_G(H)$.

(WGC4) If $\dim X^P = 2 \dim X^H$ and $\dim X^P = 2 \dim X^{H'}$ for some $P < H$, $P < H'$, $P \in \mathcal{P}(G)$, then the smallest subgroup $\langle H, H' \rangle$ of G containing H and H' is not a large subgroup of G.

A real G-module V is called *$\mathcal{L}(G)$-free* if $\dim V^H = 0$ for each $H \in \mathcal{L}(G)$, which amounts to saying that $\dim V^{O^p(G)} = 0$ for each prime $p \in \pi(G)$. For a finite group G, we define subgroups $WLO(G)$ of the free abelian group $LO(G)$ as follows.

$$WLO(G) = \{U - V \in LO(G) \mid U \text{ and } V \text{ both satisfy the weak gap condition}\}$$

A real G-module W is called *nonnegative* if (WGC1) holds for $X = W$.

We denote by $V(G)$ as

$$\mathbb{R}[G]_{\mathcal{L}(G)} = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \in \pi(G)} (\mathbb{R}[G] - \mathbb{R})^{O^p(G)}.$$

Theorem 3.2 in [2] implies the following proposition.

Proposition 3.1. *Let W be a real nonnegative G-module. For $X = W \oplus V(G)$,* (WGC2) *holds if G is a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and and* (WGC4) *holds if G is an Oliver group.*

Theorem 3.2. *For an Oliver group G, it holds that $WLO(G)$ is a subset of $CSm(G)$.*

More generally we obtain

Theorem 3.3. *Let G be an Oliver group and let $V_1, \dots, V_k$ be real G-modules satisfying that $V_i - V_j \in WLO(G)$. Then there exist a real G-module W and a smooth action on a sphere Σ such that $\Sigma^G = \{x_1, \dots, x_k\}$ and $V_i \oplus W$ is isomorphic to the tangential G-module $T_{x_i}(\Sigma)$ for any i.*

4. $LO(G)$ vs $WLO(G)$

In this section we consider the difference between $LO(G)$ and $WLO(G)$. Note that if G/G^{nil} is an elementary abelian 2-group then $WLO(G) \subset CSm(G) \subset LO(G)$.

We say that G is a gap group if G admits an $\mathcal{L}(G)$-free positive G-module V, that is, $\dim V^{O^p(G)} = 0$ for any prime $p \in \pi(G)$ and $\dim V^P > 2 \dim V^H$ for any pair (P, H) of subgroups of G with $P \in \mathcal{P}(G)$, $P < H$.

Theorem 4.1. *Let G be a group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. Suppose that for each $X \in LO(G)$ there are $\mathcal{L}(G)$-free nonnegative G-modules U and V such that $X = U - V$. For each subgroup K of G with $K > O^2(G)$, $[K : O^2(G)] = 2$, if all elements x of $K \smallsetminus O^2(G)$ of order 2 such that $C_K(x)$ is not a 2-group are not conjugate in K, then K is a gap group.*

Theorem 4.2. *Let G be an Oliver group. Let U and V be $\mathcal{L}(G)$-free nonnegative G-modules with $U - V \in RO(G)_{\mathcal{P}(G)}$. There are $\mathcal{L}(G)$-free G-modules X and Y such that they satisfy the weak gap condition and $U - V = X - Y$.*

Thus we have immediately the following theorem.

Theorem 4.3. *Let G be an Oliver group. Suppose that for each subgroup K of G with $K > O^2(G)$, $[K : O^2(G)] = 2$, if K is not a gap group then all elements x of $K \smallsetminus O^2(G)$ of order 2 such that $C_K(x)$ is not a 2-group are conjugate in G. Then $LO(G) \subset CSm(G)$. Furthermore, if G/G^{nil} is an elementary abelian 2-group then $LO(G) = CSm(G)$.*

If K is an Oliver group with $|K| \leq 2000$ and $[K : O^2(K)] = 2$, then K is a gap group or all elements x of $K \smallsetminus O^2(K)$ of order 2 such that $C_K(x)$ is not a 2-group are conjugate in K. We have still no example of a group G so that $WLO(G) \neq LO(G)$.

Let $H = D_{2p_1} \times D_{2p_2} \times \cdots \times D_{2p_r}$ be a direct product group of dihedral groups D_{2p_j}, where $p_1, \dots, p_r \geq 1$ are odd integers. Then $G \times H$ is a nongap group if G is a nongap group.

Theorem 4.4. *Let G be an Oliver group as in Theorem 4.3 and let H be as above. It holds that $LO(G \times H)$ is a subset of $CSm(G \times H)$. Furthermore if G/G^{nil} is an elementary abelian 2-group, then $CSm(G \times H) = LO(G \times H)$.*

5. Projective General Linear Groups

We note that $PGL(2, q)$ is isomorphic to the dihedral group D_6 for $q = 2$, the symmetric group S_4 for $q = 3$, the alternating group A_5 for $q = 4$, the symmetric group S_5 for $q = 5$, and nonsolvable for $q \geq 4$. The group $PGL(2, q)$ is isomorphic to $PSL(2, q)$ if q is a power of 2. If $q \geq 5$ is odd, $PGL(2, q)$ has a perfect subgroup $PSL(2, q)$ with index 2, which implies $[PGL(2, q) : O^2(PGL(2, q))] = 2$.

It is easy to see the rank of $LO(PGL(2, q))$. Note that $\operatorname{rank} LO(G) = \max(a_G - 1, 0)$ if G is a perfect group.

Proposition 5.1. *Suppose that q is odd.*

$$\operatorname{rank} LO(PGL(2,q)) = \begin{cases} 0 & q = 3,5,7 \\ a_{PGL(2,q)} - 1 & q = 9, 17 \\ a_{PGL(2,q)} - 2 & \text{otherwise} \end{cases}$$

Remark 5.2. *Suppose that q is an odd prime power integer.*

(1) *$PGL(2,q)$ is not a gap group if and only if $q = 3,5,7,9,17$.*
(2) *$PGL(2,q)$ is a Oliver group if and only if $q \geq 5$.*
(3) *$\operatorname{rank} LO(PGL(2,q)) = a_{C_{q+1}} - 1$ if $q = 9, 17$.*
(4) *$\operatorname{rank} LO(PGL(2,q)) = a_{C_{q+1}} + a_{C_{q-1}} - 2$ if $q \neq 3,5,7,9,17$.*

Theorem 4.3 gives $CSm(PGL(2,q)) = LO(PGL(2,q))$. Furthermore, we obtain the following.

Theorem 5.3. *$Sm(PGL(2,q)) = LO(PGL(2,q))$.*

6. Small Groups

In this section we discuss by viewing from the order of a Sylow 2-subgroup of an Oliver group. If G is an Oliver group of odd order then G is a gap group and $LO(G)$ is a subset of $CSm(G)$.

Theorem 6.1. *If G is an Oliver group whose order is divisible by 2 not by 4 then $LO(G)$ is a subset of $CSm(G)$.*

Example 6.2. *Let K be a finite abelian group of odd order whose rank is greater than 2. Let h be an automorphism on K which sends $k \in K$ to it's inverse k^{-1}. Put $G = \langle h, K \rangle$. Then G is an Oliver nongap group satisfying $CSm(G) = LO(G)$.*

Theorem 6.3. *Let N be a normal subgroup of G. Suppose that $a_G \leq a_{G,N}(N) + 1$. The induction mapping $\operatorname{Ind}_N^G \colon LO(N) \otimes \mathbb{Q} \to LO(G) \otimes \mathbb{Q}$ is surjective.*

From now on, we suppose that G is a finite Oliver group, $[G : G^{\mathrm{nil}}] = 2$ and $a_G \geq 2$. Note that $a_{G,G^{\mathrm{nil}}}(G \smallsetminus G^{\mathrm{nil}}) = a_G - a_{G,G^{\mathrm{nil}}}(G^{\mathrm{nil}})$. The above theorem yields the following.

Theorem 6.4. *If $a_G \leq a_{G,G^{\mathrm{nil}}}(G^{\mathrm{nil}}) + 1$ then $LO(G) = WLO(G) = CSm(G)$.*

So, we are interesting in the case when $a_{G,G^{\mathrm{nil}}}(G \smallsetminus G^{\mathrm{nil}}) = a_G - a_{G,G^{\mathrm{nil}}}(G^{\mathrm{nil}}) \geq 2$.

Let $\mathcal{F}$ be the set of isomorphism classes of finite Oliver nongap groups K such that $4 \mid |K|$, $[K : K^{\mathrm{nil}}] = 2$, and $a_{K,K^{\mathrm{nil}}}(K \smallsetminus K^{\mathrm{nil}}) \geq 2$. Note that $|G|$ is divisible by 8 if $|G|$ is divisible by 4 and less than or equal to 2000. The set of all representatives of elements in $\mathcal{F}$ consists of 5 groups

$$G_{648}, PGL(2,9), G_{1296}, G_{1944a}, G_{1944b}.$$

Here they are given as follows.

$$\begin{tikzcd} G_{108} \arrow[d] & C_3^3 \arrow[l, "\triangleleft"'] \\ C_2^2 & \end{tikzcd} \qquad \begin{tikzcd} G_{648} \arrow[d] & G_{324} \arrow[l, "\triangleleft"'] \arrow[d] & G_{108} \arrow[l, "\triangleleft"'] \\ C_2 & C_3 & \end{tikzcd}$$

$$\begin{tikzcd}
G_{1944a} \arrow[d] & G_{972a} \arrow[l, "\triangleleft"'] \arrow[d] & G_{108} \arrow[l, "\triangleleft"'] & G_{1944b} \arrow[d] & G_{972b} \arrow[l, "\triangleleft"'] \arrow[d] & G_{108} \arrow[l, "\triangleleft"'] \\
C_2 & C_9 & & C_2 & C_3^2 &
\end{tikzcd}$$

$$\begin{tikzcd}
G_{1296} \arrow[d] & G_{648} \arrow[l, "\triangleleft"'] \arrow[d] & G_{216} \arrow[l, "\triangleleft"'] \arrow[d] & C_3^3 \arrow[l, "\triangleleft"'] \\
C_2 & C_3 & Q_8 &
\end{tikzcd}$$

G_{648} gives the isomorphism class of the smallest group in $\mathcal{F}$. G_{1296} has center C_2 and the quotient group by it's center is isomorphic to G_{704}. For these groups G, it holds that $CSm(G) = Sm(G)$. $a_G = 4, 2, 10, 6, 6$ and $a_{G,G^{\text{nil}}}(G \smallsetminus G^{\text{nil}}) = 3, 2, 4, 3, 3$ respectively. There are only five groups up to order 2000. However we have the following.

Proposition 6.5. *There are infinitely many finite groups G such that $[G : G^{\text{nil}}] = 2$ and $a_{G,G^{\text{nil}}}(G \smallsetminus G^{\text{nil}}) \geq 2$.*

Problem 6.6. *Is there a finite nongap group G and involutions x and y of $G \smallsetminus O^2(G)$ such that $[G : G^{\text{nil}}] = 2$, x and y are not conjugate in G, and $C_G(x)$ and $C_G(y)$ are both not 2-groups.*

There is no such a group if the order is less than or equal to 2000.

Proposition 6.7. *Suppose that there is a finite nongap group satisfying the property in the above problem. Then there are infinitely many finite nongap groups satisfying the same property.*

7. Direct product gap groups

In this section, we consider about when a direct product group is a gap group. First we remark that

Proposition 7.1 ([6, 12])**.** *Let K be a finite group with $\mathcal{P}(K) \cap \mathcal{L}(K) = \emptyset$ and H be a 2-group. $K \times H$ is a gap group if and only if so is K.*

We call a finite group G is a *generalized dihedral group* if $[G : O^2(G)] = 2$ and there is an involution $h \in G \smallsetminus O^2(G)$ such that $hgh = g^{-1}$ for any $g \in O^2(G)$. A generalized dihedral group is a subgroup of certain direct product group of dihedral groups.

Proposition 7.2 ([13, Lemma 7.2])**.** *Suppose $[K : K^{\text{nil}}] = 2$ and $\mathcal{P}(K) \cap \mathcal{L}(K) = \emptyset$. For an odd prime p and a nontrivial p-group H, $K \times H$ is a gap group if and only if K is not a generalized dihedral group.*

Moreover we have the following.

Proposition 7.3. *Suppose that $[K : K^{\text{nil}}] = 2$ and $\mathcal{P}(K) \cap \mathcal{L}(K) = \emptyset$. If $|\pi(H/[H,H])| \geq 2$, or $|\pi(H/[H,H])| = 1$ and K is not a generalized dihedral group then $K \times H$ is a gap group, where $[H,H]$ is a commutator subgroup of H.*

If K or H is a gap group then so is $K \times H$. We put

$$\kappa(K) = \bigcup_{x \in K \smallsetminus O^2(K)} \pi(\langle x \rangle).$$

$\kappa(K)$ is a subset of $\pi(K)$ and if $K \neq O^2(K)$ then it contains 2.

Theorem 7.4. *Suppose that K and H are nongap groups with $[K : K^{\mathrm{nil}}] = [H : H^{\mathrm{nil}}] = 2$. Let L be a unique subgroup of $K \times H$ with index 2 which is neither K nor H. Further suppose that $\mathcal{P}(L) \cap \mathcal{L}(L) = \emptyset$. The following claims are equivalent.*

(1) *L is a gap group.*
(2) (i) *$a_{K,O^2(K)}(K \smallsetminus O^2(K)) \geq 1$ and there is a 2-element x of $H \smallsetminus O^2(H)$ with $|x| \geq 4$, or*
(ii) *$a_{H,O^2(H)}(H \smallsetminus O^2(H)) \geq 1$ and there is a 2-element y of $K \smallsetminus O^2(K)$ with $|y| \geq 4$, or*
(iii) *$a_{K,O^2(K)}(K \smallsetminus O^2(K)) \geq 1$, $a_{H,O^2(H)}(K \smallsetminus O^2(H)) \geq 1$ and $|\kappa(K) \cup \kappa(H)| \geq 3$.*

Corollary 7.5. *Let K, H, and L be groups as in Theorem 7.4. If*

(1) *$a_{K,O^2(K)}(K \smallsetminus O^2(K)) = a_{H,O^2(H)}(K \smallsetminus O^2(H)) = 0$, or*
(2) *$a_{K,O^2(K)}(K \smallsetminus O^2(K)) \geq 1$ and H is not a generalized dihedral group, or*
(3) *$a_{H,O^2(H)}(H \smallsetminus O^2(H)) \geq 1$ and K is not a generalized dihedral group,*

then $K \times H$ is a nongap group. Furthermore, the converse is also true if $\mathcal{P}(O^2(K)) \cap \mathcal{L}(O^2(K)) = \emptyset$ and $\mathcal{P}(O^2(H)) \cap \mathcal{L}(O^2(H)) = \emptyset$.

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