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A REMARK ON ASSOCIATED PRIMES IN THE COHOMOLOGY ALGEBRA OF A FINITE GROUP

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Introduction

Let G be a finite group and $H^{*}(G,k)$ be the cohomology algebra of G over an algebraically closed field k of characteristic $p>0$. It is well known that $H^{*}(G,k)$ is a commutative noetherian graded k-algeba. In my talk we discussed properties of the associated primes in $H^{*}(G,k)$ and gave a slight generalizaton of a result of Green [7].

Theorem 0.1. *If $\mathfrak{p}$ is an associated prime in $H^{*}(G,k)$ with $\dim H^{*}(G,k)/\mathfrak{p}=s$, then there exists an elementary abelian p-subgroup E of rank s of G such that the depth of $H^{*}(C_{G}(E),k)$ is s and $\mathfrak{p}=\mathrm{Res}_{G,E}^{-1}(\sqrt{0})$.*

Conversely, let E be an elementary abelian p-subgroup E of rank s of G and assume that the depth of $H^{*}(C_{G}(E),k)$ is s. Then $\mathrm{Res}_{G,E}^{-1}(\sqrt{0})$ is an associated prime in $H^{*}(G,k)$.

The first assertion was known to be true using the notion of the Steenrod Algebra [4].We shall give a different proof using arguments by Carlson [3]. And the second assertion is known to be true if $C_{G}(E)=G$ by Green [7].

1. The Rank-Restriction Condition

Let r be the p-rank of G. Let $\mathcal{A}_{s}=\mathcal{A}_{s}(G)$ be the set of elementary abelian p-subgroups of G $(1\leqq s\leqq r)$ and set $\mathcal{H}_{s}=\mathcal{H}_{s}(G)=\{C_{G}(E)\ ;\ E\in \mathcal{A}_{s}\}$. And for $1\leqq s\leqq r$, let $\mathcal{K}_{s}=\mathcal{K}_{s}(G)$ be the family of subgroups K of G such that the Sylow p-subgroups of $C_{G}(K)$ are not conjugate to a subgroup of any of the groups in $\mathcal{H}_{s}$. (If such a subgroup K does not exists, then we set $\mathcal{K}_{s}=\{1\}$.) Notice that any elementary abelian p-subgroup of $K\in \mathcal{K}_{s}$ have rank at most $s-1$. For a nonempty family $\mathcal{P}$ of subgroups of G, set $\mathrm{Im}\,\mathrm{Tr}_{\mathcal{P},G}=\sum_{P\in \mathcal{P}}\mathrm{Im}\,\mathrm{Tr}_{P,G}$ and $\mathrm{Ker}\,\mathrm{Res}_{G,\mathcal{P}}=\cap_{P\in \mathcal{P}}\mathrm{Ker}\,\mathrm{Res}_{G,P}$. Then by a theorem of Benson (Theorem1.1 [1]) $\sqrt{\mathrm{Im}\,\mathrm{Tr}_{\mathcal{H}_{s},G}}=\sqrt{\mathrm{Ker}\,\mathrm{Res}_{G,\mathcal{K}_{s}}}$.

Let $\zeta_{1},\cdots,\zeta_{r}$ be a homogeneous system of parameters in $H^{*}(G,k)$ satisfying the rank-restriction condition. Then for $1\leqq s\leqq r$, $\mathrm{Res}_{G,E}(\zeta_{s})=0$ for any $E\in \mathcal{A}_{s-1}$, we see that $\zeta_{s}\in\sqrt{\mathrm{Ker}\,\mathrm{Res}_{G,\mathcal{K}_{s}}}=\sqrt{\mathrm{Im}\,\mathrm{Tr}_{\mathcal{H}_{s},G}}$. Thus replacing ζ_{s}'s by a suitable p-power, we may assume that $\zeta_{s}\in \mathrm{Im}\,\mathrm{Tr}_{\mathcal{H}_{s},G}$ for each $1\leqq s\leqq r$. Thus we always have a homogeneous system of parameters $\zeta_{1},\cdots,\zeta_{r}$ in $H^{*}(G,k)$ satisfying the following conditions. See Lemma 8.5 [2] for the condition 3 and Lemma 3.4 and Theorem 9.6 [2] for the condition 4.

Condition 1.1.

(1) $\zeta_{1},\cdots,\zeta_{r}$ *satisfy the rank-restriction condition.*

(2) $\zeta_s \in \operatorname{Im}\operatorname{Tr}_{\mathcal{H}_s,G}$
(3) *For each s and* $E \in \mathcal{A}_s$, *the restrictions of* $\zeta_1, \cdots, \zeta_s$ *to* $H^*(E,k)$ *form a homogeneous system of parameters (and* $\zeta_{s+1}, \cdots, \zeta_r$ *restrict to* 0*). In particular, their restrictions to* $H^*(C_G(E),k)$ *form a regular sequence for* $H^*(C_G(E),k)$.
(4) *If* $H^*(G,k)$ *has depth s, then* $\zeta_1, \cdots, \zeta_s$ *form a regular sequence.*
(5) $\dim H^*(G,k)/(\zeta_{s+1}, \cdots, \zeta_r)$ *is s.*

Lemma 1.2 (Carlson [3]). *Let* $0 \neq \eta \in H^*(G,k)$ *be a homogeneous element. Assume* $\dim H^*(G,k)/\operatorname{Ann}_{H^*(G,k)}(\eta) = s$. *Then the following statements hold.*
(1) *If* $t > 0$, *then for any* $H \in \mathcal{H}_t$, $\operatorname{Res}_{G,H}(\eta) = 0$.
(2) *For some* $H \in \mathcal{H}_s$, $\operatorname{Res}_{G,H}(\eta) \neq 0$.

Proof. Set $\mathfrak{a} = \operatorname{Ann}_{H^*(G,k)}(\eta)$.
1. Suppose that $\operatorname{Res}_{G,H}(\eta) \neq 0$ for some $H \in \mathcal{H}_t$ with $t > 0$ and set $\mathfrak{b} = \operatorname{Ann}_{H^*(G,k)}(\operatorname{Res}_{G,H}(\eta))$. Then as $\operatorname{Res}_{G,H}^{-1}(\mathfrak{b}) \subset \mathfrak{a}$,

$$s = \dim H^*(G,k)/\mathfrak{a} \geqq \dim H^*(G,k)/\operatorname{Res}_{G,H}^{-1}(\mathfrak{b}) = \dim H^*(H,k)/\mathfrak{b} \geqq \operatorname{depth} H^*(H,k)$$

H has a central elementary abelian p-subgroup of rank t and $\operatorname{depth} H^*(H,k) \geqq t$ by a theorem of Duflot [5]. Thus we have a contradiction.
2. Suppose that $\operatorname{Res}_{G,H}(\eta) = 0$ for all $H \in \mathcal{H}_s$. Then by the statement 1, $\operatorname{Res}_{G,H}(\eta) = 0$ for all $H \in \mathcal{H}_t$ with $t \geqq s$. Thus $Ann_{H^*(G,k)}(\eta)$ contains $\zeta_s, \cdots, \zeta_r$ and

$$s = \dim H^*(G,k)/\operatorname{Ann}_{H^*(G,k)}(\eta) \leqq \dim H^*(G,k)/(\zeta_s, \cdots, \zeta_r) \leqq s - 1$$

This is a contradiction. □

2. Results of D.J. Green

In this section we shall prove some results on cohomology algebra of a finite group having a normal elementary abelian p-subgroup. A discussion here depends heavily on investigations by D.J.Green [7].

2.1. In this section let C be a finite group with a central elementary abelian p-subgroup E of order p^s $(s > 0)$ and assume that a p-rank of C is larger than s. Set $\overline{C} = C/E$ and for a subgroup $E \subset H \subset C$, set $\overline{H} = H/E \subset \overline{C}$. Set $\mathcal{A} = \mathcal{A}_1(E)$ be the set of elementary abelian p-subgroups F of C containing E with $[F : E] = p$ and $\mathcal{H} = \{ C_C(F); F \in \mathcal{A} \}$. And let $\mathcal{K}$ be the family of subgroups $K \subset C$ such that the Sylow p-subgroups of $C_C(K)$ are not conjugate to a subgroup of any of the groups in $\mathcal{H}$ (If $\mathcal{H}$ has a subgroup H with p'-index in C, then we set $\mathcal{K} = \{ 1 \}$ as before). Notice that if $K \in \mathcal{K}$, then any elementary abelian p-subgroup of K is contained in E.

Let $F_1, \cdots, F_k$ be a complete set of representatives of the C-conjugacy classes in $\mathcal{A}$. Then $\overline{F_i} \subset \overline{C}$ $(1 \leqq i \leqq)$ are not $\overline{C}$-conjugate. let $\sigma_i' \in H^2(\overline{F_i}, GF(p)) \subset H^2(\overline{F_i}, k)$ be a nonzero fixed element and set $\sigma_i = (\sigma_i')^{p-1}$. Then using the Evens' norm map, there can be constructed a homogeneous element $\lambda \in H^*(\overline{C}, k)$ such that $\operatorname{Res}_{\overline{C},\overline{F_i}}(\lambda)$ is a p-power of σ_i for each i (see a discussion in Section 7.1 [7]). Now set $\kappa = \operatorname{Inf}_{\overline{C},C}(\lambda) \in H^*(C,k)$. κ is a primitive element. We know that $\operatorname{Res}_{C,K}(\kappa) = 0$ for any $K \in \mathcal{K}$ and therefore, replacing κ by some power, we may assume that $\kappa \in \operatorname{Im}\operatorname{Tr}_{\mathcal{H},C}$ (see a proof of Lemma 2.6 [7]). Among the result of D.J.Green [7], the following fact will be used in this note.

Proposition 2.1. *The homogeneous element $\kappa \in H^*(C,k)$ given above satisfies the following properties.*

(1) $\kappa \in \operatorname{Im}\operatorname{Tr}_{\mathcal{H},C}$.
(2) $\operatorname{Res}_{C,H}(\kappa) \in H^*(H,k)$ *is a regular element for every* $H \in \mathcal{H}$.
(3) $\operatorname{Ann}_{H^*(C,k)}(\kappa) = \operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$
(4) *Suppose that* $\xi_1, \cdots, \xi_t$ *is a sequence of homogeneous elements of* $H^*(C,k)$ *whose restrictions form a regular sequence in* $H^*(E,k)$. *Then*
  (a) $\xi_1, \cdots, \xi_t$ *is a regular sequence for* $H^*(C,k)/H^*(C,k)\cdot\kappa$. *In particular, as a* $k[\xi_1, \cdots, \xi_t]$*-module,* $H^*(C,k)/H^*(C,k)\cdot\kappa$ *is free.*
  (b) $\operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$ *is a free* $k[\xi_1, \cdots, \xi_t]$*-module*
(5) *If the depth of* $H^*(C,k)$ *is* s, *then* κ *is a zero divisor in* $H^*(C,k)$. *In particular,* $\operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}} \neq 0$.

Proof. All the statements in the proposition except the assertion (b) in the statement 4 are included in [7]. We shall prove the stetement 4, (b). Set $R = k[\xi_1, \cdots, \xi_t]$. Then $H^*(C,k)$ is a free R-module by a result of Duflot [5]. We have the following exact sequence of R-modules

$$0 \to \operatorname{Ann}_{H^*(C,k)}(\kappa) \to H^*(C,k) \xrightarrow{\cdot\kappa} H^*(C,k) \to H^*(C,k)/H^*(C,k)\kappa \to 0$$

and $\operatorname{Ann}_{H^*(C,k)}(\kappa) = \operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$. Thus by the assertion $(a), 4$, it follows that $\operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$ is a projective R-module. So the result follows because a projective R-module is a free R-module (see, for example, Theorem 2.5 [8]). □

2.2. In this section let N be a finite group with normal elementary abelian subgroup E of rank s and set $C = C_N(E)$. We use notations in the previous section for C. We shall prove the following proposition.

Proposition 2.2. *Assume that the depth of* $H^*(C,k)$ *is* s. *Then*

(1) *As a* kN/C*-module* $\operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$ *contains a regular module* kN/C.
(2) $\operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$ *contains a nonzero* N*-invariant element.*

Set $E = \langle a_1, \cdots, a_s\rangle$, $\alpha_i \in H^1(E,k) = \operatorname{Hom}(E,k)$ be the element dual to a_i and $\beta_i = \beta(\alpha_i)$, where β is the Bockstein map. We have a polynomial subalgebra $k[\beta_1, \cdots, \beta_s]$ in $H^*(E,k)$. Using Evens' norm map, we obtain homogeneous elements $\xi_1, \cdots \xi_s \in H^*(C,k)$ such that $\operatorname{Res}_{C,E}(\xi_i) = \beta_i^{p^a}$ for some p-power p^a. The subalgebra $k[\beta_1, \cdots, \beta_s] \subset H^*(E,k)$ is N-stable , but the subalgebra $k[\xi_1, \cdots, \xi_s] \subset H^*(C,k)$ may not be N-stable. For $g \in N$, let $\beta_i^g = \sum_{j=1}^s \lambda_{ij}\beta_j$, $\lambda_{ij} \in GF(p)$ and consider the element $\xi = \xi_i^g - (\sum_{j=1}^s \lambda_{ij}\xi_j)$. Then $\operatorname{Res}_{C,E}(\xi) = 0$ and therefore $\operatorname{Res}_{C,K}(\xi)$ is nilpotent for any $K \in \mathcal{K}$ because elementary abelian p-subgroups of such K are contained in E. Hence by a theorem of Benson (Theorem 1.1 [1]) some p-power of ξ is contained in $\operatorname{Im}\operatorname{Tr}_{\mathcal{H},C}$. Replacing the $\xi_i's$ by suitable p-powers, we have proved the following.

Lemma 2.3. *There exist homogeneous elements* ξ_i $(1 \leqq i \leqq s)$ *and a* p*-power* p^b *satisfying the following.*

(1) $\operatorname{Res}_{C,E}(\xi) = \beta_i^{p^b}$ *for each* i $(1 \leqq i \leqq s)$.

(2) $k[\xi_1, \cdots, \xi_s]+\operatorname{Im}\operatorname{Tr}_{\mathcal{H},C}$ *is* N*-stable and* $k[\xi_1, \cdots, \xi_s]+\operatorname{Im}\operatorname{Tr}_{\mathcal{H},C}/\operatorname{Im}\operatorname{Tr}_{\mathcal{H},C} \cong k[\beta_1^{p^b}, \cdots, \beta_1^{p^b}]$ *as* N*-modules.*

Proof of **Proposition 2.2**
Now we shall prove the proposition. First we prove the statement 1. Set $R_0 = k[\beta_1^{p^b}, \cdots, \beta_s^{p^b}]$ and $R = k[\xi_1, \cdots, \xi_s]$. Then it is well known that the sequence $\xi_1, \cdots, \xi_s$ is a regular sequence for $H^*(C,k)$ and $H^*(C,k)$ is a free R-module (see [5]).

By Proposition 2.1, $\operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}} \neq 0$. Let n be the least integer such that $V = \operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}} \cap H^n(C,k) \neq 0$. Then V is N-stable and any k-basis of V can be extended to a set of R-free generators of $\operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$. Then by Lemma 1.2 , Proposition 2.1 and the fact that $\operatorname{Im}\operatorname{Tr}_{\mathcal{H},C}$ annihilates $\operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$, it follows that $V \cdot R$ is N-stable and $V \cdot R \cong V \otimes_k R_0$ as N-modules. It is known that R_0 contains kN/C as a kN/C-module (see [9]). Therefore the statement (1) follows.

The statement 2 is an easy consequence of the statement 1.

3. Proof of Theorem 0.1

First we shall show that the first statement of the theorem hold. Let $\mathfrak{p}$ be an associated prime in $H^*(G,k)$ with $\dim H^*(G,k)/\mathfrak{p} = s$ and take a homogeneous element $0 \neq \eta \in H^*(G,k)$ such that $\operatorname{Ann}_{H^*(G,k)}(\eta) = \mathfrak{a}$. Then by Lemma 1.2, there exists $E \in \mathcal{A}_s$ such that $\operatorname{Res}_{G,C_G(E)}(\eta) \neq 0$. Set $C = C_G(E)$, $\eta_0 = \operatorname{Res}_{G,C}(\eta)$ and we shall use the notations in Section 2. Again by Lemma 1.2, $\eta_0 \in \operatorname{Ker}\operatorname{Res}_{C,\mathcal{H}}$.

Let $\alpha \in H^*(C,k)$ such that $\operatorname{Res}_{C,E}(\alpha) = 0$. Then $\alpha \in \sqrt{\operatorname{Ker}\operatorname{Res}_{C,\mathcal{K}}}$ by an argument in Section 1 and therefore $\alpha \in \sqrt{\operatorname{Im}\operatorname{Tr}_{\mathcal{H},C}}$. Thus $\alpha \in \sqrt{\operatorname{Ann}_{H^*(C,k)}(\eta_0)}$.

By a result of Benson in [1] there exists a homogeneous element $\tau \in H^*(C,k)$ such that $\operatorname{Res}_{N,C}\operatorname{Tr}_{C,N}(\tau)$ is a regular element in $H^*(C,k)$ and restricts to zero on every subgroup A of C with $A \not\supset E$, where $N = N_G(E)$. Set $\sigma = \operatorname{Tr}_{C,G}(\tau)$ and consider the element $\alpha = \eta\sigma$. We shall show the following equality hold.

$$\operatorname{Res}_{G,C}(\alpha) = \eta_0 \operatorname{Res}_{N,C}\operatorname{Tr}_{C,N}(\tau)$$

As $\alpha = \eta\operatorname{Tr}_{C,G}(\tau) = \operatorname{Tr}_{C,G}(\operatorname{Res}_{G,C}(\eta)\tau) = \operatorname{Tr}_{C,G}(\eta_0\tau)$, the Mackey double coset formula shows that

$$\operatorname{Res}_{G,C}(\alpha) = \sum_{g \in C\backslash G/C} \operatorname{Tr}_{C^g \cap C, C}\operatorname{Res}_{C^g, C^g \cap C}((\eta_0\tau)^g) = \sum_{g \in C\backslash G/C} \operatorname{Tr}_{C^g \cap C, C}\operatorname{Res}_{C^g, C^g \cap C}(\eta_0\tau^g)$$

Suppose that $C^{g^{-1}} \not\supset E$. Then as $\operatorname{Res}_{C,C\cap C^{g^{-1}}}(\tau) = 0$, it follows that $\operatorname{Res}_{C,C\cap C^{g^{-1}}}(\eta_0\tau) = 0$ and therefore $\operatorname{Res}_{C^g,C^g\cap C}(\eta\tau^g) = 0$. If $E \subset C^{g^{-1}}$ and $E \neq E^{g^{-1}}$, then $F = EE^{g^{-1}}$ is elementary abelian, $F \supsetneq E$ and $C \cap C^{g^{-1}} = C_G(F)$. So $\operatorname{Res}_{C,C\cap C^{g^{-1}}}(\eta_0) = 0$ and therefore $\operatorname{Res}_{C^g,C^g\cap C}(\eta\tau^g) = 0$. Thus we have the desired equality. $\alpha \neq 0$ and therefore $\mathfrak{p} = \operatorname{Ann}_{H^*(G,k)}(\eta) = \operatorname{Ann}_{H^*(G,k)}(\alpha)$. In these notations, we shall show that for $\rho \in H^*(G,k)$, $\rho\alpha = 0$ if and only if $\operatorname{Res}_{G,C}(\rho)\eta_0 = 0$.

Assume that $\operatorname{Res}_{G,C}(\rho)\eta_0 = 0$, then $\operatorname{Res}_{G,C}(\rho)\eta_0\tau = 0$ and $\rho\alpha = \operatorname{Tr}_{C,G}(\operatorname{Res}_{G,C}(\rho)\eta_0\tau) = 0$. Thus we have $\mathfrak{p} = \operatorname{Res}_{G,C}^{-1}(\operatorname{Ann}_{H^*(C,k)}(\eta_0))$.

As $\mathrm{Res}_{C,E}^{-1}(\sqrt{0}) \subset \sqrt{\mathrm{Ann}_{H^*(C,k)}(\eta_0)}$, we have

$$\mathrm{Res}_{G,E}^{-1}(\sqrt{0}) \subset \mathrm{Res}_{G,C}^{-1}(\mathrm{Ann}_{H^*(C,k)}(\eta_0)) = \mathfrak{p}$$

As $\mathrm{Res}_{G,E}^{-1}(\sqrt{0})$ is a prime ideal and $\dim H^*(G,k)/\mathrm{Res}_{G,E}^{-1}(\sqrt{0}) = s$, we can conclude that $\mathrm{Res}_{G,E}^{-1}(\sqrt{0}) = \mathfrak{p}$ and the first statement in the theorem follows.

We next shall prove the second statement in the theorem. Let $E \in \mathcal{A}_s$ and assume that the depth of $H^*(C_G(E),k)$ is s. Set $C = C_G(E)$, $N = N_G(E)$ and we shall use the notations Section 2. As before let τ be a homogeneous element in $H^*(C,k)$ such that $\mathrm{Res}_{N,C}\mathrm{Tr}_{C,N}(\tau)$ is a regular element in $H^*(C,k)$ and restricts to zero on every subgroup A of C with $A \not\supset E$,

By Proposition 2.2, there exists a non zero homogeneous element $\eta_0 \in \mathrm{Ker}\,\mathrm{Res}_{C,\mathcal{H}}$ which is N-invariant. Notice that $\mathrm{Res}_{C,E}^{-1}(\sqrt{0}) \subset \sqrt{\mathrm{Ann}_{H^*(C,k)}(\eta_0)}$. Consider the element $\gamma = \eta_0\tau$. Then an entirely same argument as above, we have

$$\mathrm{Res}_{G,C}\mathrm{Tr}_{C,G}(\gamma) = \mathrm{Res}_{N,C}\mathrm{Tr}_{C,N}(\gamma) = \eta_0\,\mathrm{Res}_{N,C}\mathrm{Tr}_{C,N}(\tau)$$

Set $\alpha = \mathrm{Tr}_{C,G}(\gamma) \in H^*(G,k)$. We shall show that for $\rho \in H^*(G,k)$, $\rho\alpha = 0$ if and only if $\mathrm{Res}_{G,C}(\rho)\eta_0 = 0$.

Assume that $\rho\alpha = 0$. Then $\mathrm{Tr}_{C,G}(\mathrm{Res}_{G,C}(\rho)\gamma) = 0$. Hence $\mathrm{Res}_{G,C}\mathrm{Tr}_{C,G}(\mathrm{Res}_{G,C}(\rho)\gamma) = 0$. By the similar argument in the above we have that

$$\mathrm{Res}_{G,C}\mathrm{Tr}_{C,G}(\mathrm{Res}_{G,C}(\rho)\gamma) = \mathrm{Res}_{N,C}\mathrm{Tr}_{C,N}(\mathrm{Res}_{G,C}(\rho)\gamma) = (\mathrm{Res}_{G,C}(\rho)\eta_0)\,\mathrm{Res}_{N,C}\mathrm{Tr}_{C,N}(\tau)$$

As $\mathrm{Res}_{N,C}\mathrm{Tr}_{C,N}(\tau)$ is a regular element in $H^*(C,k)$, it follows that $\mathrm{Res}_{G,C}(\rho)\eta_0 = 0$.

Conversely, if $\mathrm{Res}_{G,C}(\rho)\eta_0 = 0$, then $\mathrm{Res}_{G,C}(\rho)\gamma = 0$ and $\rho\alpha = \mathrm{Tr}_{C,G}(\mathrm{Res}_{G,C}(\rho)\gamma) = 0$.

A standard argument in commutative noetherian rings says that there exists $\delta \in H^*(G,k)$ such that $\eta = \alpha\delta \neq 0$ and $\mathfrak{p} = \mathrm{Ann}_{H^*(G,k)}(\eta)$ is a prime ideal. Then $\eta_1 = \mathrm{Res}_{G,C}(\delta)\eta_0 \neq 0$ and by an entirely same argument as before, we have

$$\mathrm{Ann}_{H^*(G,k)}(\eta) = \mathrm{Res}_{G,C}^{-1}(\mathrm{Ann}_{H^*(C,k)}(\eta_1))$$

As $\mathrm{Res}_{C,E}^{-1}(\sqrt{0}) \subset \sqrt{\mathrm{Ann}_{H^*(C,k)}(\eta_0)} \subset \sqrt{\mathrm{Ann}_{H^*(C,k)}(\eta_1)}$, $\mathrm{Res}_{G,E}^{-1}(\sqrt{0}) \subset \sqrt{\mathrm{Ann}_{H^*(G,k)}(\eta)} = \mathfrak{p}$ and $\dim H^*(G,k)/\mathfrak{p} = \dim H^*(H,k)/\mathrm{Ann}_{H^*(C,k)}(\eta_1) = s$. Thus $\mathrm{Res}_{G,E}^{-1}(\sqrt{0}) = \mathfrak{p}$ and the second statement in the theorem follows.

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