

Lagrangian dynamics on an infinite-dimensional torus

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Abstract

The space $L^2(0, 1)$ has a natural Riemannian structure on the basis of which we introduce an $L^2(0, 1)$ -infinite dimensional torus \mathbb{T} . We consider the group \mathcal{G} of bijections $G : [0, 1] \rightarrow [0, 1]$ which preserve Lebesgue measure. We also consider a class of Hamiltonians defined on the cotangent bundle of \mathbb{T} , invariant under the action of \mathcal{G} . We establish existence of a viscosity solution for a *cell problem* on \mathbb{T} , that are invariant under the action of \mathcal{G} . We apply this to the study of one-dimensional nonlinear Vlasov system with periodic potential. (This is a joint work with A. Tudorascu [12]).

1 A Weak KAM theorem on a Hilbert space

Existence of solution of the so-called cell problem in the Hamilton-Jacobi theory remains an unsolved problem in Hilbert spaces such as $L^2(0, 1)$. Motivated by applications in kinetic theory, we consider a special class of Hamiltonian for which we are able to solve the cell problem. Let us fix a \mathbb{Z} -periodic function $W \in C^2(\mathbb{R})$: $W(z + 1) = W(z)$ for all $z \in \mathbb{R}$. We consider the Hamiltonian and the Lagrangian $H, L : L^2(0, 1) \times L^2(0, 1) \rightarrow \mathbb{R}$ are given by

$$H(M, N) = \frac{1}{2} \|N\|_{\nu_0}^2 + \frac{1}{2} \mathcal{W}(M), \quad L(M, N) = \frac{1}{2} \|N\|_{\nu_0}^2 - \frac{1}{2} \mathcal{W}(M). \quad (1)$$

Here, ν_0 is the restriction of \mathcal{L}^1 to $(0, 1)$ where \mathcal{L}^1 , the one-dimensional Lebesgue measure. We have set

$$\mathcal{W}(M) := \int_{(0,1)^2} W(Mz - My) dz dy. \quad (2)$$

We assume that

$$W(z) = W(-z) \leq W(0) = 0. \quad (3)$$

The requirement that W is symmetric is not restrictive since we may substitute W by its symmetric part in (2) without altering the definition of \mathcal{W} . The fact that $W(0) = 0$ is not restrictive either since we may substitute W by $W - W(0)$ without affecting the analysis below.

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Only the fact that W attains its maximum at 0 is a restriction which we have imposed to be able to draw some explicit conclusions.

In this talk we start with a problem which at a first glance looks purely abstract. It is the following cell problem: fix $c \in \mathbb{R}$. Find $\mathcal{U}(\cdot; c) : L^2(0, 1) \rightarrow \mathbb{R}$ viscosity solution of

$$H(M, \nabla_{L^2} \mathcal{U}(M; c) + c) = \frac{c^2}{2} \quad (4)$$

such that $\mathcal{U}(\cdot; c)$ is Lipschitz and periodic in the sense that

$$\mathcal{U}(M + Z; c) = \mathcal{U}(M; c)$$

for all $Z \in L^2_{\mathbb{Z}}(0, 1)$. Here, $L^2_{\mathbb{Z}}(0, 1)$ is the set of $M \in L^2(0, 1)$ whose ranges are contained in \mathbb{Z} . We define the $L^2_{\mathbb{Z}}(0, 1)$ -torus by

$$\mathbb{T} := L^2(0, 1) / L^2_{\mathbb{Z}}(0, 1). \quad (5)$$

We also impose that $\mathcal{U}(\cdot; c)$ be rearrangement invariant in the sense that

$$\mathcal{U}(M_1; c) = \mathcal{U}(M_2; c)$$

whenever $M_1, M_2 \in L^2(0, 1)$ have the same distribution. What we mean is that

$$\nu_0[M_1^{-1}(B)] = \nu_0[M_2^{-1}(B)]$$

for every interval $B \subset \mathbb{R}$. We recall that in measure theory, if E and F are two topological spaces and ν is a Borel measure on E , $M_{\#}\nu$ is the measure on F called the push-forward to ν by M and defined by

$$M_{\#}\nu(A) = \nu(M^{-1}(A))$$

for all Borel sets $A \subset F$. So, $M_1, M_2 \in L^2(0, 1)$ have the same distribution means that $M_{1\#}\nu_0 = M_{2\#}\nu_0$.

The set of Lipschitz functions $U : L^2(0, 1) \rightarrow \mathbb{R}$ that are rearrangement invariant has been completely characterized in [12] as those satisfying $U(M \circ G) = U(M)$ for all $M \in L^2(0, 1)$ and all $G \in \mathcal{G}$. Here, \mathcal{G} be the set of bijections $G : [0, 1] \rightarrow [0, 1]$ such that G, G^{-1} are Borel and preserve \mathcal{L}^1 .

In some sense our problem consists in proving existence of viscosity solution of (4) on \mathbb{T}/\mathcal{G} . This is an extension of the finite dimensional weak KAM theory [9] [15] [16] [17] to a Hilbert space.

It is shown [12] that \mathbb{T}/\mathcal{G} is a compact set for the strong topology inherited from $L^2(0, 1)$. Hence, standard methods of Hamilton-Jacobi theory can be applied to find a solution $\mathcal{U}(\cdot; c)$ of (4) on \mathbb{T}/\mathcal{G} . Indeed, for each $\epsilon > 0$ one considers the unique solution [5] of

$$\epsilon V_{\epsilon} + H(M, \nabla_{L^2} V_{\epsilon}(M; c) + c) = 0. \quad (6)$$

Since H satisfies the invariance properties

$$H(M \circ G, N \circ G) = H(M, N) = H(M + Z, N)$$

for all $M, N \in L^2(0, 1)$, $G \in \mathcal{G}$ and $Z \in L^2_{\mathbb{Z}}(0, 1)$, uniqueness of solution in (6) ensures that

$$V_\epsilon(M + Z; c) = V_\epsilon(M; c) = V_\epsilon(M \circ G). \quad (7)$$

In fact the invariance property of V_ϵ can also be established as a simple consequence of the representation formula:

$$V_\epsilon(M) := \inf_{\sigma} \{ \mathcal{A}_\epsilon(\sigma) : \sigma \in AC^2_{loc}(0, \infty; L^2(0, 1)), \sigma_0 = M \}. \quad (8)$$

Here,

$$L_c(M, N) = L(M, N) - c \int_0^1 N(z) dz, \quad \text{and} \quad \mathcal{A}_\epsilon(\sigma) = \int_0^\infty e^{-\epsilon t} L_c(\sigma_t, -\dot{\sigma}_t) dt.$$

Remark 1.1. *We have the semigroup property: for each $T > 0$*

$$V_\epsilon(M) = \inf_{M^*} \{ e^{-\epsilon T} V_\epsilon(M^*) + W_T(M, M^*) : M \in L^2(0, 1) \},$$

where,

$$W_T(M, M^*) := \inf_{\sigma} \left\{ \int_0^T L_c(\sigma_t, -\dot{\sigma}_t) dt : \sigma_0 = M, \sigma_T = M^*, \sigma \in AC^2(0, T; L^2(0, 1)) \right\}. \quad (9)$$

Exploiting (1.1), one proves that V_ϵ is Lipschitz, with a Lipschitz constant bounded by κ_c , independent of $\epsilon \in (0, 1)$. Hence, it attains its minimum on the compact set \mathbb{T}/\mathcal{G} . Set $U_\epsilon := V_\epsilon - \min V_\epsilon$. Then $\{U_\epsilon\}_{\epsilon \in (0, 1)} \subset C(\mathbb{T}/\mathcal{G})$ is equicontinuous. Since \mathbb{T}/\mathcal{G} is a compact set, up to a subsequence, $\{U_\epsilon\}_{\epsilon \in (0, 1)}$ uniformly to a function $\mathcal{U}(\cdot; c)$ of (4) on \mathbb{T}/\mathcal{G} . Also, $\{\epsilon V_\epsilon\}_{\epsilon \in (0, 1)}$ is bounded uniformly bounded. Since the Lipschitz constant of ϵV_ϵ is bounded by $\epsilon \kappa_c$ and $\{\epsilon V_\epsilon\}_{\epsilon \in (0, 1)}$ is equicontinuous on \mathbb{T}/\mathcal{G} we conclude that it converges uniformly to a constant q . It is easy to check that $\{\epsilon V_\epsilon(0)\}_{\epsilon \in (0, 1)}$ converges to $c^2/2$ to conclude that $q = c^2/2$. By the invariance property (7) we conclude that $\{\epsilon V_\epsilon\}_{\epsilon \in (0, 1)}$ converges uniformly to $c^2/2$ on $L^2(0, 1)$. We exploit the semigroup property (1.1) to obtain

$$\mathcal{U}(\sigma_0; c) \leq \int_0^T L_c(\sigma_t, -\dot{\sigma}_t) dt + \mathcal{U}(\sigma_T; c) + \frac{1}{2} c^2 T \quad (10)$$

for all $T > 0$ and all $\sigma \in H^1(0, T; L^2(0, 1))$.

Next, we make a delicate statement and refer the reader to [12] for a detailed proof: for each $M \in L^2(0, 1)$ **monotone** nondecreasing there exists a so-called calibrated curve σ^c originating at M , associated to $\mathcal{U}(\cdot; c)$ in the sense that $\sigma^c \in H^2(0, \infty; L^2(0, 1))$, $\sigma^c_0 = M$ and whenever $T > 0$,

$$\mathcal{U}(M; c) = \int_0^T L_c(\sigma^c_t, -\dot{\sigma}^c_t) dt + \mathcal{U}(\sigma^c_T; c) + \frac{1}{2} c^2 T. \quad (11)$$

By (11) and (10), for each $T > 0$, σ^c is a minimizer of

$$\sigma \rightarrow \mathcal{A}_T(\sigma) := \int_0^T L_c(\sigma_t, -\dot{\sigma}_t) dt$$

over the set of path $\sigma \in H^1(0, T; L^2(0, 1))$ such that $\mathcal{U}(\sigma_0) = \mathcal{U}(\sigma_0^c)$ and $\mathcal{U}(\sigma_T) = \mathcal{U}(\sigma_T^c)$. In particular, σ^c is a minimizer of \mathcal{A}_T over the set of path $\sigma \in H^1(0, T; L^2(0, 1))$ such that $\sigma_0 = \sigma_0^c$ and $\sigma_T = \sigma_T^c$. Thus, σ^c satisfies the Euler-Lagrange equation

$$\ddot{\sigma}_t^c z = - \int_I W'(\sigma_t^c z - \sigma_t^c y) dy. \quad (12)$$

Remark 1.2. *Assuming that M is monotone nondecreasing is essential for obtaining a calibrated curve originating at M . The detail of the argument can be found in [12].*

For each $t \geq 0$ we define a probability measure $f_t := (\sigma_t \times \dot{\sigma}_t)_{\#} \nu_0$ on \mathbb{R}^2 and a probability measure $\varrho_t := (\sigma_t)_{\#} \nu_0$ on \mathbb{R} . What we mean is

$$\varrho_t(A) = \nu_0(\{z \in [0, 1] \mid \sigma_t z \in A\}), \quad f_t(B) = \nu_0(\{z \in [0, 1] \mid (\sigma_t z, \dot{\sigma}_t z) \in B\}) \quad (13)$$

for $A \subset \mathbb{R}$ and $B \subset \mathbb{R}^2$ Borel sets. If we denote the first projection of \mathbb{R}^2 onto \mathbb{R} by $\pi_1(x, v) = x$ then $\pi_{1\#} f_t = \varrho_t$. We say that ϱ_t is the first marginal of f_t .

Thanks to (12) f satisfies the Vlasov system

$$\begin{cases} \partial_t f_t + v \partial_x f_t = \partial_x P_t \partial_v f_t \\ \varrho_t(x) = \pi_{1\#} f_t \\ P_t(x) = \int_{\mathbb{R}} V(x - \bar{x}) d\rho_t(x). \end{cases} \quad (14)$$

The first equation in (14) has to be understood in the sense of distribution. Note that (14) is an infinite dimensional Hamiltonian system on the Wasserstein space [1] [10]. The Hamiltonian is given by

$$\mathcal{H}(f) := \int_{\mathbb{R}^2} \left[\frac{v^2}{2} + \frac{1}{2} \int_{\mathbb{R}^2} W(x - y) df(y, w) \right] df(x, v).$$

There is a Hamiltonian vector field $X_{\mathcal{H}}$ [10] such that (14) is equivalent to $\dot{f} = X_{\mathcal{H}}(f)$.

We can now state the main theorem of this talk.

Theorem 1.3. *Let $c \in \mathbb{R}$. Then for each $M \in L^2(0, 1)$ monotone nondecreasing there exists $N \in L^2(0, 1)$ such that for $\bar{f} := (M \times N)_{\#} \nu_0$ (14) admits a solution f satisfying $f_0 = \bar{f}$,*

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^2} |v + c|^2 df_t(x, v) = 0, \quad \sup_{t > 0} t \int_{\mathbb{R}^2} \left| \frac{x}{t} + c \right|^2 df_t(x, v) < \infty.$$

Proof: We use (11) and the fact that $\mathcal{W} \leq 0$ to obtain that for each $T > 0$,

$$2\mathcal{U}(M; c) - 2\mathcal{U}(\sigma_T^c; c) = \int_0^T \left[\int_{\mathbb{R}^2} |v + c|^2 df_t(x, v) - \mathcal{W}(\sigma_t) \right] dt \geq \int_0^T \int_{\mathbb{R}^2} |v + c|^2 df_t(x, v) dt \quad (15)$$

Set

$$u(t) := \int_{\mathbb{R}^2} |v + c|^2 df_t(x, v).$$

As a Lipschitz function on \mathbb{T}/\mathcal{G} , \mathcal{U} is continuous there. Since \mathbb{T}/\mathcal{G} is a compact set, we obtain that \mathcal{U} is bounded and so, by (15), $u \in L^1(0, \infty)$. But,

$$u'(t) = \int_0^1 \ddot{\sigma}_t^c(z) (\dot{\sigma}_t^c(z) + c) dz$$

We use (12) and the fact that W' is bounded to conclude that

$$\sup_{t>0} \|\ddot{\sigma}_t^c\|_{L^\infty(0,1)} < \infty. \quad (16)$$

Conservation of the Hamiltonian H along the path σ^c yields

$$\|\dot{\sigma}_t^c\|_{L^2(0,1)}^2 + \mathcal{W}(\sigma_t^c) = \|N\|_{L^2(0,1)}^2 + \mathcal{W}(M).$$

Since \mathcal{W} is bounded, we conclude that

$$\bar{e} := \sup_{t>0} \|\dot{\sigma}_t^c\|_{L^2(0,1)} < \infty. \quad (17)$$

We use (16) and (17) to conclude that $u' \in L^\infty(\mathbb{R})$. This, together with the fact that $u \in L^1(0, \infty)$ yields $\lim_{t \rightarrow \infty} u(t) = 0$, which is the first assertion of the theorem.

Next,

$$\|\sigma_t^c - \sigma_0^c + ct\|_{L^2(0,1)} = \left\| \int_0^t (\dot{\sigma}_s^c + c) ds \right\|_{L^2(0,1)} \leq \int_0^t \|\dot{\sigma}_s^c + c\|_{L^2(0,1)} ds \leq \sqrt{t} \sqrt{\int_0^t \|\dot{\sigma}_s^c + c\|_{L^2(0,1)}^2 ds}$$

and so,

$$\|\sigma_t^c - \sigma_0^c + ct\|_{L^2(0,1)} \leq \sqrt{t} \sqrt{\int_0^t u(s) ds} \leq \sqrt{t} \|u\|_{L^1(0,\infty)}^{1/2}$$

This, together with the fact that

$$\sqrt{\int_{\mathbb{R}^2} |x + ct|^2 df_t(x, v)} = \|\sigma_t^c + ct\|_{L^2(0,1)} \leq \|\sigma_t^c - \sigma_0^c + ct\|_{L^2(0,1)} + \|\sigma_0^c\|_{L^2(0,1)},$$

yields the proof. QED.

Remark 1.4. *In the jargon of the weak KAM theory, we have proven that for each $c \in \mathbb{R}$ and each $M \in L^2(0, 1)$ monotone nondecreasing, there exist $N \in L^2(0, 1)$ and a path $t \rightarrow f_t$ starting at $(M \times N)_{\#} \nu_0$, satisfying $\dot{f} = X_{\mathcal{H}}(f)$ and of rotation number c .*

We have chosen the Vlasov system as a simple model to illustrate the use of the weak KAM theory for understanding qualitative behavior of PDEs, for several reasons. Firstly, they provide a simple link between finite and infinite dimensional systems. Secondly, they are one of the most frequently used kinetic models in statistical mechanics. Existence and uniqueness of global solutions for the initial value problem are well understood [3] [8]. In this paper we have searched for special solutions which allow for a connection with a more conventional way of regarding (14) as Hamiltonian. We assume the initial data to be of the form $f_0 = (M, N)_{\#} \nu_0$ where $M, N \in L^2(X)$ so that the unique solution of (14) retains the same structure.

2 Appendix; Basic facts

Definition 2.1. Let V be a real valued proper functional defined on $L^2(0, 1)$ with values in $\mathbb{R} \cup \{\pm\infty\}$. Let $M_0 \in L^2(0, 1)$ and $\xi \in L^2(0, 1)$. We say that ξ belongs to the (Fréchet) subdifferential of V at M_0 and we write $\xi \in \partial V(M_0)$ if

$$V(M) - V(M_0) \geq \langle \xi, M - M_0 \rangle + o(\|M - M_0\|)$$

for all $M \in L^2(0, 1)$.

We say that ξ belongs to the superdifferential of V at M_0 and we write $\xi \in \partial V(M_0)$ if $-\xi \in \partial(-V)(M_0)$.

Remark 2.2. As expected, when the sets $\partial V(M_0)$ and $\partial V(M_0)$ are both nonempty, then they coincide and consist of a single element. That element is the L^2 -gradient of V at M_0 , denoted by $\nabla_{L^2} V(M_0)$.

We can now define [5] the notion of viscosity solution for a general Hamilton-Jacobi equation of the type

$$F(M, \nabla_{L^2} U(M)) = 0. \tag{HJ}$$

Definition 2.3. Let $V : L^2(0, 1) \rightarrow \mathbb{R}$ be continuous.

(i) We say that V is a viscosity subsolution for (HJ) if

$$F(M, \zeta) \leq 0 \text{ for all } M \in L^2(0, 1) \text{ and all } \zeta \in \partial V(M). \tag{18}$$

(ii) We say that V is a viscosity supersolution for (HJ) if

$$F(M, \zeta) \geq 0 \text{ for all } M \in L^2(0, 1) \text{ and all } \zeta \in \partial V(M). \tag{19}$$

(iii) We say that V is a viscosity solution for (HJ) if V is both a subsolution and a supersolution for (HJ).

Remark 2.4. If U is a viscosity solution, then, in view of remark 2.2, we deduce that (HJ) is satisfied at all $M \in L^2(0, 1)$ where $\partial U(M) \cap \partial U(M) \neq \emptyset$, which are precisely the points where U is differentiable.

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References

- [1] L. AMBROSIO, W. GANGBO, *Hamiltonian ODEs in the Wasserstein Space of Probability Measures*, Comm. Pure Appl. Math. **LXI** (2008), 18-53.
- [2] L. AMBROSIO, N. GIGLI, G. SAVARÉ, *Gradient flows in metric spaces and the Wasserstein spaces of probability measures*, Lectures in mathematics, E.T.H. Zurich, Birkhäuser, 2005.
- [3] W. BRAUN, K. HEPP, *The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles*, Comm. Math. Phys. **56** (2) (1977), 101-113.
- [4] H. BREZIS, *Analyse fonctionnelle; théorie et applications*, Masson, Paris (1983).
- [5] M. G. CRANDALL, P. L. LIONS. *Hamilton-Jacobi equations in infinite dimensions I. Uniqueness of viscosity solutions*, J. Funct. Anal. **62** (1985), 379-396.
- [6] M. CRANDALL, P.L. LIONS. *Hamilton-Jacobi equations in infinite dimensions II. Existence of viscosity solutions*, J. Funct. Anal. **65** (1986), 368-405.
- [7] M. CRANDALL, P.L. LIONS. *Hamilton-Jacobi equations in infinite dimensions III*, J. Funct. Anal. **68** (1986), 214-247.
- [8] R.L. DOBRUSHIN. *Vlasov equations*, Funct. Anal. Appl. **13** (1979), 115-123.
- [9] A. FATHI. *Weak KAM theory in Lagrangian dynamics*, preliminary version, Lecture notes (2003).
- [10] W. GANGBO, H.K. KIM AND T. PACINI. *Differential forms on Wasserstein space and infinite-dimensional Hamiltonian systems*. To appear in Memoirs of AMS.
- [11] W. GANGBO, T. NGUYEN, A. TUDORASCU. *Euler-Poisson systems as action minimizing paths in the Wasserstein space*, Arch. Rat. Mech. Anal. (to appear) (2008).
- [12] W. GANGBO, A. TUDORASCU. *Lagrangian Dynamics on an infinite-dimensional torus; a Weak KAM theorem*. Preprint.
- [13] W. GANGBO, A. TUDORASCU. *Homogenization for Hamilton-Jacobi equations in probability spaces*, (in progress).
- [14] P.L. LIONS, G. PAPANICOLAOU, S.R.S. VARADHAN. *Homogenization of Hamilton-Jacobi equations*, unpublished, cca 1988.
- [15] J. MATHER. *Minimal measures*. Comment. Math. Helv. **64**, no. 3, 375–394, 1989.
- [16] J. MATHER. *Action minimizing invariant measures for positive definite Lagrangian systems*. Math. Z. **207**, no. 2, 169–207, 1991.
- [17] J. MATHER. *Variational construction of connecting orbits*. Ann. Inst. Fourier (Grenoble) **43**, no. 5, 1349–1386, 1993.
- [18] C. VILLANI. *Topics in optimal transportation*. Graduate Studies in Mathematics **58**, American Mathematical Society (2003).