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A METRIC METHOD FOR THE ANALYSIS OF STATIONARY ERGODIC HAMILTON–JACOBI EQUATIONS

ANTONIO SICONOLFI

1. Overview

The scope of this contribution is to explain how the so–called metric method, which has revealed to be a powerful tool for the analysis of deterministic Hamilton–Jacobi equations, see [4], can be used in the stationary ergodic setting. The material is taken from [1], [2], [3], and to these papers we refer for a more formal and complete treatment of the subject. Other papers of interest are [6] and [7].

We focus on two basic issues, namely the role of random closed stationary sets and the asymptotic analysis of the intrinsic distances leading to the notion of stable norm. These items are of crucial relevance. In a sense the stationary ergodic structure of the Hamiltonian induces a stochastic geometry in the state variable space $\mathbb{R}^N$, where the fundamental entities are indeed the closed random stationary sets which, somehow, play the same role as the points in the deterministic case, see [5] for a general treatment of of random sets theory. Secondly, the ergodicity can be viewed as an extremely weak form of compactness, mostly thanks to some powerful asymptotic results, like Birkhoff and Kingman subadditive theorem, and especially the latter is a fundamental tool for proving the existence of asymptotic norms.

In Section 2 we start by recalling the basic points of the metric method in the deterministic case, then in Section 3 we discuss the notion(s) of critical value.

2. Deterministic case

The basic idea of the metric methodology is very simple: we consider an Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and we assume three conditions, which will be kept throughout the paper, on it:

$$H \text{ is continuous in both arguments;} \tag{1}$$

$$H \text{ is convex in the momentum variable;} \tag{2}$$

$$\lim_{p\to+\infty} H(x,p) = +\infty \quad \text{uniformly in } x. \tag{3}$$

Then, given an associate Hamilton–Jacobi equations in $\mathbb{R}^N$ of the form

$$H(x, Du) = a, \tag{4}$$

for some $a \in \mathbb{R}$, we consider the a–sublevels of the Hamiltonian, defined, for any $x \in R^N$, by

$$Z_a(x) = \{p \mid H(x,p) \le a\}.$$

It can be easily checked that, under the previous assumptions on H, the multifunction Z_a is compact convex–valued (with possibly empty values) and continuous, with respect to the classical Hausdorff metric, at any x_0 where int $Z_a(x_0) \neq \emptyset$, upper semicontinuous at any x_0 around which Z_a is locally nonempty.

We proceed defining the support function of $Z_a(x)$

$$\sigma_a(x,q) = \sup\{p \cdot q \mid p \in Z_a(x)\},$$

where the symbol $\cdot$ indicates the scalar product in $\mathbb{R}^N$, here we are identifying $\mathbb{R}^N$ and its dual, and we adopt the usual convention that $\sigma_a(x,\cdot) \equiv -\infty$, whenever $Z_a(x) = \emptyset$. The function σ_a is convex positively homogeneous in the second variable and inherits the same continuity properties of Z_a with respect to x.

Starting from σ_a, we give a notion of intrinsic length for any (Lipschitz–continuous) curve ξ defined in the interval $[0,1]$ setting

$$\ell_a(\xi) = \int_0^1 \sigma_a(\xi, \dot{\xi})\, dt.$$

Notice that the above integral is invariant for orientation–preserving change of parameter, thanks to positive homogeneity of the integrand, as an intrinsic length should be. It is therefore not restrictive to assume all the curves under consideration to be defined in $[0,1]$. At this level of generality, it is clear that the intrinsic length can be $-\infty$ for some curve.

The final step in this construction is to take the path metric associated to ℓ_a, which is given, for any ordered pair of points of $\mathbb{R}^N$, by the infimum of the intrinsic length of the curves joining the first to the second point. We denote it by S_a. An important property linking S_a to the equation (4) is the following

Proposition 2.1. *The equation* (4) *admits locally Lipschitz–continuous a.e. subsolutions if and only if* $S_a \not\equiv -\infty$.

It is clear that $S_a \not\equiv -\infty$ is in turn equivalent to $S_a(x,y)$ finite for every x, y in $\mathbb{R}^N$. In this case S_a can be analogously defined as the functional metric associated to (4), i.e.

$$S_a(x,y) = \sup\{u(y) - u(x) \mid u \text{ is an a.e. subsolution to (4)}\} \qquad (5)$$

Obviously, a functional distance can be defined, in principle, for any partial differential equation. The peculiar situation here is that it is path distance, in the sense that it comes from the minimization of an integral functional. This relevant property is strictly related to the convex character of the Hamiltonian.

We call S_a, with a slight terminology abuse, intrinsic distance associated to (4). Properly speaking, in fact, it is not a distance, since it lacks the sign and symmetry property, but the crucial point is that it enjoys the triangle inequality. We derive from (5)

Proposition 2.2. S_a *is finite if and only if the intrinsic length of any closed curve is nonnegative.*

3. Critical values

We will be interested in the separation element

$$c_0 = \sup\{a \mid S_a \equiv -\infty\} = \inf\{a \mid S_a \text{ is finite}\}$$

which is called critical value of H. By straightforward stability results on subsolutions, we have that the critical equation

$$H(x, Du) = c_0 \qquad (6)$$

admits a.e. subsolutions and so S_{c_0} is finite. We moreover recall that, at least if the underlying space is noncompact, as we are presently assuming, any supercritical equation (i.e. with $a \geq c_0$) also admits (viscosity) solutions, which, due to the properties of the Hamiltonian can be simply characterized as the continuous functions u such that $H(x_0, D\psi(x_0)) = a$ for any x_0, any ψ of class C^1 locally around x_0, for which x_0 is a local minimizer of $u - \psi$.

We need, for later use, a refinement of the Proposition 2.2. We will indicate by $\ell(\cdot)$ the Euclidean length of a curve.

Proposition 3.1. *Given $a > c_0$ and a compact set $K \subset \mathbb{R}^N$, there is a positive constant α such that any closed curve ξ contained in K satisfies*

$$\ell_a(\xi) \geq \alpha\, \ell(\xi).$$

The setup is different if the ground space of the Hamilton–Jacobi equation is instead compact. The relevant example is the flat torus $\mathbb{T}^N$. If the Hamiltonian H is in fact defined in $\mathbb{T}^N \times \mathbb{R}^N$, identified with the cotangent space of $\mathbb{T}^N$, then the critical equation is unique among the equations (4) for which a solution does exist.

This is also related to a metric phenomenon. The critical distance S_{c_0}, in contrast to what happens for S_a with $a > c_0$, is not locally equivalent to the Euclidean distance.

One can be more precise: a metric degeneration takes place around points through which a sequence of cycles, say ξ_n, pass with

$$\inf_n \ell_{c_0}(\xi_n) = 0 \quad \text{and} \quad \inf_n \ell(\xi_n) > 0.$$

Look at the Proposition 3.1 to better understand the meaning of this condition. These points play an important role in the analysis of critical equations. They made up a set named after Aubry. We stress that if the underlying space is noncompact the critical distance S_{c_0} can be still locally equivalent to the Euclidean distance and, accordingly, the Aubry set can be empty.

Intrinsic distance furthermore plays a crucial role in the representation formulae for (sub)solutions of (4). In the supercritical case, in fact, the functions $x \mapsto S_a(y, x)$ provide a class of fundamental subsolutions to (4), for any fixed $y \in \mathbb{R}^N$. They are also solutions in $\mathbb{R}^n \setminus \{y\}$. More generally, for any C closed subset of the ground space, any function g defined on C and 1– Lipschitz–continuous with respect to S_a the Lax formula

$$\sup\{g(y) + S_a(y, \cdot) \mid y \in C\} \tag{7}$$

gives a subsolution to (4) attaining the value g on C. Such a function is moreover solution in all the space except the *source set* C. This helps understanding the difference about existence of solutions between the compact and noncompact setting. If in fact the ground pace is noncompact, the source set in (7) can be swept away sending it to infinity, obtaining through passage at the limit a solution whenever $a \geq c_0$.

This procedure cannot be applied in the compact case, and the unique possibility to get a solution through Lax formula (7) is that a is equal to the critical value and C is a subset of the Aubry set. In this way, we actually obtain all the critical solutions, and we also characterize the point y of the Aubry set through the property that $S_{c_0}(y, \cdot)$ is a global solution of the critical equation.

As it is well known, in the case the underlying space is the flat torus, such critical solutions play the role of correctors in the periodic homogenization procedure. In the limit equations it appears the so–called effective Hamiltonian $\bar{H}(p_0)$ which is defined, for any $p_0 \in \mathbb{R}^N$ as the critical value of the Hamiltonian $(x, p) \mapsto H(x, p + p_0)$.

4. Periodic Hamiltonians

As we will explain with some more detail later, the periodic case is the simplest example of ergodic stationary setting. However, even in this easy setting are present some difficulties in the application of the metric method arising in more complicated environments..

Dealing with a $\mathbb{Z}^N$–periodic Hamiltonian, we have the basic options of directly working on the quotient space $\mathbb{R}^n/\mathbb{Z}^N = \mathbb{T}^N$ or to keep $\mathbb{R}^N$ as ground space and exploit on it the periodicity condition. The first choice is more simple from the viewpoint of the analysis of critical equations, effective Hamiltonian and so on, since we directly use the compactness of the torus, as previously illustrated. On the other side, this sweeps under carpet, in a sense, the real difficulties in the analysis. Moreover such a choice is confined to the periodic case, in other ergodic stationary ergodic settings, even in the quasi–periodic and almost–periodic case, there is no the possibility of adapting the ground space and we are forced to work in $\mathbb{R}^N$.. For explanatory purposes, let us take, in the periodic case, the difficult road of keeping $\mathbb{R}^N$ as ground space.

Since we are only interested on periodic solutions, we a priori have two distinguished critical values. The first one is the previously defined c_0, which can be equivalently given by

$$c_0 = \min\{a \mid \text{there are subsolution in } \mathbb{R}^N \text{ of } (4)\}.$$

The other relevant value is

$$c = \min\{a \mid \text{there are } \textit{periodic} \text{ subsolution in } \mathbb{R}^N \text{ of } (4)\}.$$

We will call it periodic critical value. It is clear that $c \geq c_0$, but these two values can be very different. To see this, it is enough to consider the family of Hamiltonians appearing in the definition of effective Hamiltonian $\overline{H}$, namely $H(x, p + p_0)$, with p_0 varying in $\mathbb{R}^N$. The presence of the extra additive term does not affect c_0, since if u is a subsolution to (4) in $\mathbb{R}^N$, the same property holds true for $u(x) - p_0 \cdot x$ with respect to the modified equation $H(x, Du + p_0) = a$. When we look instead to periodic solutions, the situation changes, because, even if u is periodic, $u(x) + p_0 \cdot x$ does not inherit this property. We moreover know that under assumptions (1), (2), (3) the effective Hamiltonian is coercive, namely that the periodic critical value associated to $H(x, p + p_0)$ goes to infinity as $|p_0| \to +\infty$.

What is disappointing, at a first sight, regarding the metric method in this case, is that the distances S_a defined on $\mathbb{R}^N$ does not directly give information on the critical periodic value c and, similarly Lax formula does not provide *periodic* (sub)solutions.

When we consider the same periodic Hamiltonian H on the torus $\mathbb{T}^N$, or on $\mathbb{R}^N$ at some level $a \geq c_0$, then the intrinsic length of curves do not change because the a–sublevels of the Hamiltonian are the same in the two cases. But the intrinsic distances are different. The distance between two equivalence classes on $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$, say the classes containing the elements x and y of $\mathbb{R}^N$, respectively, are given by the formula

$$\inf\{S_a(x + z, y + r) \mid z, r \in \mathbb{Z}^N\}. \tag{8}$$

Now, the periodicity of the Hamiltonian allows to simplify it, since

$$S_a(x + z, y + r) = S_a(x + z - r, y),$$

we can equivalently write (8) in the form

$$\inf\{S_a(x+z,y) \mid z \in \mathbb{Z}^N\}.$$

We retain from it an information which will be developed in what follows, namely that even if the distance S_a is not *per se* interesting for the analysis of the periodic case, it can be however useful to consider the distance of points of $\mathbb{R}^N$ from sets enjoying suitable compatibility properties with the periodic structure. This will lead us to the notion of random stationary closed set.

Another remark is about the asymptotic behavior of S_a. When $c > c_0$, in correspondence with equivalence classes on the torus belonging to the Aubry set, which is nonempty, we see in $\mathbb{R}^N$ curves connecting points of the type y, $y+z$, with $y \in \mathbb{R}^N$, $z \in \mathbb{Z}^N \setminus \{0\}$ possessing infinitesimal positive intrinsic length ℓ_c, by juxtaposition of such curves, we get, loosely speaking, connection of some point y to ∞ through curves of infinitesimal intrinsic length. More formally:

Proposition 4.1. *Assume $c > c_0$, there is a point $y \in \mathbb{R}^n$ such that for any positive ε we can find a sequence $z_n \in \mathbb{Z}^N$, with $|z_n|$ positively diverging, such that*

$$0 \le \lim_n S_c(y, y+z_n) \le \varepsilon.$$

The homogenization suggests the right way of performing an asymptotic analysis of the intrinsic distances, we consider a family of Hamiltonians with highly oscillating variables of the form

$$H_\varepsilon(x,p) = H(x/\varepsilon, p).$$

We fix a level a and set for x, q in $\mathbb{R}^N$, $Z_a^\varepsilon(x) = Z_a(x/\varepsilon)$, $\sigma_a^\varepsilon(x,q) = \sigma_a(x/\varepsilon, q)$. We find for the intrinsic distance S_a^ε related to H_ε:

$$\begin{aligned}
S_a^\varepsilon(x,y) &= \inf\left\{\int_0^1 \sigma_a^\varepsilon(\xi,\dot\xi)\,dt \mid \xi(0)=x,\ \xi(1)=y\right\} \\
&= \inf\left\{\int_0^1 \sigma_a(\xi/\varepsilon,\dot\xi)\,dt \mid \xi(0)=x,\ \xi(1)=y\right\} \\
&= \inf\left\{\int_0^1 \varepsilon\,\sigma_a^\varepsilon(\xi/\varepsilon,\dot\xi/\varepsilon)\,dt \mid \xi(0)=x,\ \xi(1)=y\right\} \\
&= \varepsilon\ \inf\left\{\int_0^1 \sigma_a^(\gamma,\dot\gamma)\,dt \mid \gamma(0)=x/\varepsilon,\ \gamma(1)=y/\varepsilon\right\} \\
&= \varepsilon\, S(x/\varepsilon, y/\varepsilon),
\end{aligned}$$

for any x, y. We have therefore proved:

Proposition 4.2. *The metric S_a^ε related to the Hamiltonian H_ε at some level a satisfies*

$$S_a^\varepsilon(x,y) = \varepsilon S(x/\varepsilon, y/\varepsilon).$$

The idea is to pass to the limit of S_a^ε for $\varepsilon \to 0$. In this point we crucially exploit the periodic character of the Hamiltonian, as desired, as well as the validity of triangle inequality for S_a. This is done using the following baby version of the subadditive principle:

Lemma 4.3. *Let z_n a sequence of numbers satisfying the subadditive property*

$$z_n + z_m \le z_n + z_m \quad \textit{for any } n,\ m,$$

and assume that $\inf_n \frac{z_n}{n}$ is finite, then $\frac{z_n}{n}$ converges and $\lim_n \frac{z_n}{n} = \inf_n \frac{z_n}{n}$.

The diction *baby* that we have employed is relative to the fact that to treat the general ergodic stationary case we will need a more sophisticated version of this principle holding for sequences of random variables satisfying suitable conditions. This is named after Kingman.

From Lemma 4.3 we derive:

Theorem 4.4. *The family* $\varepsilon S_a(x/\varepsilon, y/\varepsilon)$ *locally uniformly converges in* $\mathbb{R}^N \times \mathbb{R}^N$ *to* $\phi_a(y-x)$, *where* ϕ_a *is is positively homogeneous and sublinear, and consequently convex.*

The function ϕ_a is a norm of Minkowski type, called stable norm associated to the distance S_a . In general it can be degenerate, i.e. vanishing for some nonzero vectors, and even negative. The important point however is that the periodic critical value c can be characterized in terms of properties of the stable norms. Taking also into account Proposition 4.1 we in fact have

Theorem 4.5. $c = \inf\{a \geq c_0 \mid \phi_a$ *is nondegenerate*$\}$. *If* $c > c_0$ *then* ϕ_c *is degenerate but nonnegative.*

5. Stationary ergodic setting

In this section we pass describe the general stationary setting. As usual, we sacrifice precision in favour of ease and simplicity.

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which the action of $\mathbb{R}^N$ gives rise to an N- dimensional ergodic dynamical system. In other terms it is defined a family of mappings $\tau_x : \Omega \to \Omega$, for $x \in \mathbb{R}^N$, which satisfy the following properties:

(1) the *group property:* $\tau_0 = id, \quad \tau_{x+y} = \tau_x \circ \tau_y$;
(2) the mappings $\tau_x : \Omega \to \Omega$ are measurable and measure preserving, i.e. $\mathbb{P}(\tau_x E) = \mathbb{P}(E)$ for every $E \in \mathcal{F}$;
(3) the map $(x, \omega) \mapsto \tau_x \omega$ from $\mathbb{R}^N \times \Omega$ to Ω is jointly measurable.

The ergodicity condition on $(\tau_x)_{x \in \mathbb{R}^N}$ can be expressed in the following equivalent ways:

(i) every measurable function f defined on Ω such that, for every $x \in \mathbb{R}^N$, $f(\tau_x \omega) = f(\omega)$ a.s. in Ω, is almost surely constant;
(ii) every set $A \in \mathcal{F}$ such that $\mathbb{P}(\tau_x A \,\Delta\, A) = 0$ for every $x \in \mathbb{R}^N$ has probability either 0 or 1, where Δ stands for the symmetric difference.

We moreover consider an Hamiltonian $H(x, p, \omega)$

$$H : \mathbb{R}^N \times \mathbb{R}^N \times \Omega \to \mathbb{R}$$

which still satisfies the conditions (1), (2), (3) in (x, p) for every ω, is measurable in ω and enjoys the following compatibility property, called stationarity, with the previously described dynamical system

$$H(\cdot + z, \cdot, \omega) = H(\cdot, \cdot, \tau_z \omega) \quad \text{for every } (z, \omega) \in \mathbb{R}^N \times \Omega.$$

Any given periodic $H_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ can be seen as a specific realization of a suitably defined stationary ergodic Hamiltonian. We take as Ω the set $[0, 1[^N$, as $\mathbb{P}$ the N–dimensional Lebesgue measure. and as $\mathcal{F}$ the σ–algebra of Borel subsets of Ω. The action of $\mathbb{R}^N$ on Ω is given by

$$\tau_x(\omega) = x + \omega \ (\text{mod } \mathbb{Z}^N),$$

and it is clearly ergodic. A stationary Hamiltonian is then obtained by setting

$$H(x,p,\omega) = H_0(x+\omega,p).$$

We proceed considering the family of stochastic Hamilton–Jacobi equations

$$H(x,Du,\omega) = a \quad a \in \mathbb{R} \tag{9}$$

and look for admissible subsolutions of it. By this we mean Lipschitz random functions $u(x,\omega)$ (i.e. u Lipschitz–continuous in x a.s. in ω and jointly measurable in (x,ω)) which are almost surely in ω a.e. subsolution to (9), and in addition satisfy the following stationarity condition:

for every $z \in \mathbb{R}^N$, there exists a set Ω_z with probability 1 such that for every $\omega \in \Omega_z$

$$u(\cdot + z,\omega) = u(\cdot,\tau_z\omega) \quad \text{on } \mathbb{R}^N.$$

Beside stability, we also give a weaker notion of admissibility. We say u admissible if it has stationary increments, i.e. for every $z \in \mathbb{R}^N$, there exists a set Ω_z of probability 1 such that

$$v(x+z,\omega) - v(y+z,\omega) = v(x,\tau_z\omega) - v(y,\tau_z\omega) \quad \text{for all } x,y \in \mathbb{R}^N$$

for every $\omega \in \Omega_z$, and, in addition it is almost surely sublinear at infinity, i.e.

$$\lim_{|x|\to+\infty} \frac{u(x,\omega)}{|x|} = 0 \quad \text{a.s. in } \omega.$$

It can be proved that any stationary e function is also admissible. In the same way, with obvious adaptations, it is given the notions of stationary and admissible (viscosity) solutions.

We can now define, as we did in the periodic case, two different critical values.

$$\begin{aligned} c &= \inf\{a \in \mathbb{R} \mid (9) \text{ admits admisssible subsolutions }\}, \\ c_0(\omega) &= \inf\{a \in \mathbb{R} \mid (9) \text{ has a subsolution}\} \end{aligned}$$

Note that c_0 is in principle a random variable, but it can proved, thanks to the ergodicity assumption, that it is indeed a.s. constant. The stationary critical value c can be equivalently defined replacing admissible with stationary subsolutions. However the class of admissible subsolutions is preferable since it enjoys stronger stability property. In particular it can be proved, by means of an Ascoli–type theorem adjusted to the random environment, that the critical equation

$$H(x,Du,\omega) = c \tag{10}$$

have an admissible subsolution but not necessarily a stationary one. Note that this phenomenon is new with respect to the periodic case where a periodic (i.e. stationary) critical subsolution always exists. Therefore the infimum in the definition of c can be replaced by a maximum.

As in the compact deterministic case, we have:

Proposition 5.1. *The critical equation* (10) *is the unique in the family* (9) *for which an admissible solution may exist.*

We finally, straightforwardly adapting the procedure used in the deterministic case, the intrinsic distances related to the family of equations (9), at least for $a \geq c_0$. We obtain a family of random distances $S_a(\cdot, \cdot, \omega)$, but in their definition ω plays the role of parameter. Therefore the same remarks of the periodic case apply here. To repeat: the intrinsic random distances S_a are not directly useful in our analysis. Some other steps should be accomplished.

6. CLOSED RANDOM STATIONARY SETS

Here we follow the first track indicated in Section 4 to adapt the intrinsic metrics to the needs of our analysis. Namely, we consider the distance of points of $\mathbb{R}^N$ from special sets compatible with the stationary ergodic structure we are working with. These set are first of all random closed sets. That is to say random variables taking values in the family of closed subsets of $\mathbb{R}^N$, where the notion of measurability must be understood in the sense of Effros. Namely, we require that a closed random stationary set $X(\omega)$ is a closed subset of $\mathbb{R}^N$ for any ω and

$$\{\omega \mid X(\omega \cap K \neq \emptyset\} \in \mathcal{F}$$

when K varies among the compact subset of $\mathbb{R}^N$. Moreover we require X to be stationary, This means that for every $z \in \mathbb{R}^N$ there exists a set Ω_z of probability 1 such that

$$X(\tau_z \omega) = X(\omega) - z \qquad \text{for every } \omega \in \Omega_z.$$

Note that, as a consequence, the set $\{\omega \, : \, X(\omega) \neq \emptyset\}$, which is measurable by the Effros measurability of X, is invariant with respect to the group of translations $(\tau_x)_{x \in \mathbb{R}^N}$ by stationarity, so it has probability either 0 or 1 by the ergodicity assumption.

A relevant property of the random closed stationary is about their asymptotic structure, which yields in particular that they are spread with some uniformity in the space.

Proposition 6.1. *Let X be an almost surely nonempty closed stationary set in $\mathbb{R}^N$. Then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that*

$$\lim_{r \to +\infty} \frac{|(X(\omega) + B_R) \cap B_r|}{|B_r|} \geq 1 - \varepsilon \qquad a.s. \text{ in } \Omega,$$

whenever $R \geq R_\varepsilon$.

We exploit such random sets to give a stochastic version of Lax formula. Let $C(\omega)$ be an almost surely nonempty stationary closed random set in $\mathbb{R}^N$. Take a Lipschitz random function g and set, for $a \geq c_0$,

$$u(x, \omega) := \inf\{g(y, \omega) + S_a(y, x, \omega) \, : \, y \in C(\omega)\} \quad x \in \mathbb{R}^N, \tag{11}$$

where we agree that $u(\cdot, \omega) \equiv 0$ when either $C(\omega) = \emptyset$ or the infimum above is $-\infty$. The following holds:

Proposition 6.2. *Let g be a stationary Lipschitz random function and $C(\omega)$, u as above. Let us assume that, for some $a \geq c_0$, the infimum in (11) is finite a.s. in ω. Then u is a stationary random subsolution to (9) and satisfies $u(\cdot, \omega) \leq g(\cdot, \omega)$ on $C(\omega)$ a.s. in ω. Moreover, u is a solution of (9) in $\mathbb{R}^N \setminus C(\omega)$ a.s. in ω.*

When g is itself an admissible subsolution of (9), we can state a stronger version of the previous result.

Proposition 6.3. *Let g be an admissible random subsolution of* (9) *and $C(\omega)$, u as above. Then u is an admissible random subsolution of* (9). *In addition, it is a viscosity solution of* (9) *in $\mathbb{R}^N \setminus C(\omega)$, and takes the value $g(\cdot,\omega)$ on $C(\omega)$ a.s. in ω.*

7. Stable norms

In this section, generalizing the results of periodic case, we show the existence of asymptotic norm–type functions associated with S_a, whenever $a \geq c_0$. Given $\varepsilon > 0$, we define

$$S_a^\varepsilon(x,y,\omega) = \ eps \ S_a(x/\varepsilon, y/\varepsilon, \omega)$$

for every $x, y \in \mathbb{R}^N$ and $\omega \in \Omega$,.

Theorem 7.1. *Let $a \geq c_f$. There exists a convex and positively 1-homogeneous function $\phi_a : \mathbb{R}^N \to \mathbb{R}$ such that*

$$S_a^\varepsilon(x,y,\omega) \underset{\varepsilon \to 0}{\rightrightarrows} \phi_a(y-x), \qquad x,y \in \mathbb{R}^N. \tag{12}$$

for any ω in a set Ω_a of probability 1. In addition, ϕ_a is nonnegative for $a = c$, and nondegenerate, i.e. satisfying $\phi_a(\cdot) \geq \delta_a |\cdot|$ for some $\delta_a > 0$, when $a > c$.

This result is based on the following fundamental subadditive theorem which takes the plase of the baby version employed in the periodic case.

Theorem 7.2 (Kingman's Subadditive Ergodic Theorem). *Let $\{f_{m,n} : 0 \leq m \leq n\}$ be random variables which satisfy the following properties:*

(a) $f_{0,0} = 0$ and $f_{m,n} \leq f_{m,k} + f_{k,m}$ for every $m \leq k \leq n$;

(b) $\{f_{m,m+k} : m \geq 0,\ k \geq 0\}$ have the same distribution law than $\{f_{m+1,m+k+1} : m \geq 0,\ k \geq 0\}$, i.e. for every $0 \leq m_1 < \cdots < m_n$, $0 \leq k_1 < \cdots < k_n$, $n \in \mathbb{N}$

$$\mathbb{P}\left(\cap_{i=1}^n f^{-1}_{m_1,m_1+k_1}(A_i)\right) = \mathbb{P}\left(\cap_{i=1}^n f^{-1}_{m_1+1,m_1+k_1+1}(A_i)\right)$$

for any open subset A_i of $\mathbb{R}$;

(c) $\int_\Omega (f_{0,1}(\omega))^+ \, d\mathbb{P}(\omega) < +\infty$.

Then the following holds:

(i) $\mu := \lim_{n \to \infty} \frac{1}{n} \int_\Omega f_{0,n}(\omega)\, d\mathbb{P}(\omega) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_\Omega f_{0,n}(\omega)\, d\mathbb{P}(\omega) \in [-\infty, +\infty)$;

(ii) $f_\infty(\omega) := \lim_{n \to \infty} \frac{f_{0,n}(\omega)}{n}$ *exists for $\mathbb{P}$-almost every $\omega \in \Omega$;*

(iii) $\int_\Omega f_\infty(\omega)\, d\mathbb{P}(\omega) = \mu$ *and, if $\mu > -\infty$, then*

$$\frac{f_{0,n}}{n} \to f_\infty \quad in\ L^1(\Omega).$$

Theorem 7.3. *For every $a \geq c_f$, the stable norm ϕ_a is the support function of the a-sublevel of the effective Hamiltonian $\overline{H}(p_0)$ which associates to every $p_0 \in \mathbb{R}^N$ the stationary critical value of the Hamiltonian $H(x, p + p_0.\omega)$.*

We list some consequences of the previous results in the next statement, compare with Proposition 4.5

Theorem 7.4. *The following properties hold true:*

i. $\min_{\mathbb{R}^N} \overline{H} = c_0$;

ii. *let* $a \geq c_0$ *such that the corresponding stable norm is nondegenerate. Then equation* (9) *admits stationary subsolutions;*

iii. $c = \inf\{a \geq c_0 \mid \phi_a$ *is nondegenerate*$\}$. *If* $c > c_0$ *then* ϕ_c *is degenerate but nonnegative.*

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Dip. di Matematica, Università di Roma "La Sapienza", P.le Aldo Moro 2, 00185 Roma, Italy

E-mail address: `siconolf@mat.uniroma1.it`