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Variational Inequalities with Gradient Constraint and Applications to Optimal Dividend Payments

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1 Variational inequalities arisen from dividend payments

We consider the variational inequality of the form:

$$\begin{aligned}
&(a) \qquad w'(x) \geq 1, \quad x>0, \quad w'(0+)>1,\\
&(b) \qquad -\alpha w+\frac{1}{2}\sigma^{2}w''+\mu w' \leq 0, \quad x>0,\\
&(c) \qquad (-\alpha w+\frac{1}{2}\sigma^{2}w''+\mu w')(w'-1)^{+}=0, \quad x>0,\\
&(d) \qquad w(0)=0, \qquad \mu, \sigma>0 : \text{constants}.
\end{aligned}$$

Define

$$w(x)=\begin{cases} w_{0}(x), & x\leq m,\\ x-m+w_{0}(m), & x>m,\end{cases}$$

where w_{0} is the solution of

$$\mathcal{A}w_{0}:=-\alpha w_{0}+\frac{1}{2}w_{0}''+\mu w_{0}=0, \quad x\leq m,$$

and $m>0$ is chosen as $w_{0}'(m)=1$.

Theorem 1.1 $w\in C^{2}(0,\infty)\cap C[0,\infty)$ *is a concave solution of the variational inequality (a)-(d).*

The variational inequality (a)-(d) is closely related to optimal dividend payments. The reserve R_{t} of an insurance company at time $t\geq 0$ is assumed to be governed by

$$R_{t}=\mu t+\sigma B_{t}-L_{t}, \quad R_{0}=x-L_{0}\geq 0,$$

where B_t is a standard Brownian motion, $\mu, \sigma > 0$ constants, $x \geq 0$ the initial position of reserve and L_t the rate of dividend payment at time t (0 acts absorbing barrier for R_t). Note that $R_0 = x - L_0$ means that if there is a pay-out of dividends at time 0, then R_t instantaneously decreases from x to $x - L_0$. The dividend process $\{L_t\}$ is called admissible if

$$L_t : \mathcal{F}_t := \sigma(B_s, s \leq t)\text{-measurable}, \quad x - L_0 \geq 0,$$

$$L_t \text{ is nonnegative, nondecreasing, continuous,}$$

and we denote by $\mathcal{L}$ the class of all admissible dividend processes $\{L_t\}$.

The objective is to find an optimal dividend payment $\{L_t^*\} \in \mathcal{L}$ so as to maximize the expected total pay-out of dividend

$$J_x(L) = E[\int_0^\tau e^{-\alpha t} dL_t], \quad L \in \mathcal{L},$$

where $\alpha > 0$ is the discount rate and τ the absorption time, $\tau = \inf\{t \geq 0 : R_t = 0\}$.

Theorem 1.2 *We have*

$$J_x(L) \leq w(x).$$

Define

$$R_t^* = x + \mu t + \sigma B_t - L_t^*, \quad R_0^* = x - L_0^* \geq 0,$$

$$L_t^* = \max_{s \leq t}(x + \mu s + \sigma B_s - m)^+.$$

Theorem 1.3 *We assume that the initial position $x \leq m$. Then $\{L_t^*\}$ is optimal.*

Remark 1.4 *Instead of the variational inequality, we consider the Black-Scholes Model:*

$$(a) \qquad w'(x) \geq 1, \quad x > 0, \quad w'(0+) > 1,$$

$$(b) \qquad -\alpha w + \frac{1}{2}\sigma^2 x^2 w'' + \mu x w' \leq 0, \quad x > 0,$$

$$(c) \qquad (-\alpha w + \frac{1}{2}\sigma^2 x^2 w'' + \mu x w')(w' - 1)^+ = 0, \quad x > 0,$$

$$(d) \qquad w(0) = 0,$$

where $\mu, \sigma > 0$ *constants. Then* $w(x) = x$ *and (a) fails if* $\alpha > \mu$.

Remark 1.5 *Consider the following variational inequality:*

$$
\begin{aligned}
&(a) \qquad w'(x) \geq 1, \quad x > 0, \quad w'(0+) > 1, \\
&(b) \qquad -\alpha w + \frac{1}{2}\sigma^2 x^2 w'' + \mu w' \leq 0, \quad x > 0, \\
&(c) \qquad (-\alpha w + \frac{1}{2}\sigma^2 x^2 w'' + \mu w')(w' - 1)^+ = 0, \quad x > 0, \\
&(d) \qquad w(0) = 0.
\end{aligned}
$$

Then this variational inequality seems to have no solution.

2 Variational inequalities in the Stochastic Ramsey problem

From now on, we consider the variational inequality associated with optimal dividends for the stochastic Ramsey model. We define the following quantities:

K_t = capital stock of a firm at time t,

K^γ = the Cobb-Douglas function for the amount of capital stock K, $\qquad 0 < \gamma < 1$,

B_t = 1-dim. Brownain motion,

$\mathcal{F}_t = \sigma(B_s, s \leq t)$,

σ = diffusion constant, $\sigma > 0$

x = initial position, $x > 0$.

Dividends are paid from the profit of the firm for shareholders and the remainder accumulates in capital stock. We assume that the flow of dividend payments at time t can be written as $K_t dD_t$, where dD_t denotes the per capital stock dividend payments. Let $\mathcal{A}$ be the class of all nonnegative, nondecreasing, continuous, $\{\mathcal{F}_t\}$-adapted stochastic processes $D = \{D_t\}$ such that $x_D := x - D_0 > 0$. Given a policy $D \in \mathcal{A}$, the capital stock process $\{K_t\}$ evolves according to

$$dK_t = K_t^\gamma dt + \sigma K_t dB_t - K_t dD_t, \qquad K_0 = x - D_0 > 0.$$

Our objective is to find an optimal policy $D^* = \{D_t^*\}$ so as to maximize the expected total pay-out functional with discount factor $\alpha > 0$:

$$J(D) = E[\int_0^\infty e^{-\alpha t} K_t dD_t], \quad \forall D \in \mathcal{A}.$$

The associated variational inequality is given by

$$(VI) \qquad \begin{aligned} &\bullet \quad v'(x) \geq 1, \quad x > 0, \quad v'(0+) > 1, \\ &\bullet \quad -\alpha v + \frac{1}{2}\sigma^2 x^2 v'' + x^\gamma v' \leq 0, \quad x > 0, \\ &\bullet \quad (-\alpha v + \frac{1}{2}\sigma^2 x^2 v'' + x^\gamma v')(v' - 1)^+ = 0, \quad x > 0. \end{aligned}$$

For the existence of K_t, we have the following.

Proposition 2.1 *For each $D \in \mathcal{A}$, there exists uniquely a positive solution $\{K_t\}$ of*

$$dK_t = K_t^\gamma dt + \sigma K_t dB_t - K_t dD_t, \qquad K_0 = x_D = x - D_0 > 0.$$

such that

$$E[K_t] \leq 2^\beta (x_D + t^\beta),$$

$$E[K_t^2] \leq 2^{2\beta} e^{\sigma^2 t}(x_D^2 + t^{2\gamma\beta}/\sigma^2),$$

where $\beta = 1/(1-\gamma)$.

Outline of the proof. We set $k_t = K_t^{1-\gamma}$. Then, by Ito's formula

$$\begin{aligned} dk_t &= (1-\gamma)K_t^{-\gamma} dK_t + \frac{\sigma^2}{2} K_t^2 (1-\gamma)(-\gamma) K_t^{-\gamma-1} dt \\ &= (1-\gamma)dt + \sigma K_t^{1-\gamma} dB_t - K_t^{1-\gamma} dD_t \\ &\quad + \frac{\sigma^2}{2}(1-\gamma)(-\gamma)K_t^{1-\gamma} dt \\ &= (1-\gamma)\{(1 - \frac{\sigma^2}{2}\gamma k_t)dt + \sigma k_t dB_t - k_t dD_t\}, \\ &\quad k_0 = x_D^{1-\gamma}. \end{aligned}$$

By linearity, there exists a unique positive solution $\{k_t\}$.

Proposition 2.2 *Assume* $\sigma = 0$. *Then there exists a concave solution* $v_0 \in C^2(0,\infty)$ *of (VI).*

Outline of the proof. We solve the equation $-\alpha h + x^\gamma h' = 0$ to have

$$h(x) = Q\exp\{\alpha x^{1-\gamma}/(1-\gamma)\}.$$

Define

$$v_0(x) = \begin{cases} h(x) & \text{if } \; x \le x_*, \\ x - x_* + h(x_*) & \text{if } \; x_* < x, \end{cases}$$

Choose $x_* = (\gamma/\alpha)^{1/(1-\gamma)}, Q > 0$ such that $h'(x_*) = 1$. Then we have

$$h''(x_*) = 0,$$

and

$$-\alpha v_0 + x^\gamma v_0' = -\alpha\{x - x_* + h(x_*)\} + x^\gamma \le 0 \;\text{ for }\; x > x_*.$$

3 Probabilistic solution of the penalty equation

We consider the penalty equation

$$(p) \qquad -\alpha u + \frac{1}{2}\sigma^2 x^2 u'' + x^\gamma u' + \frac{x}{\varepsilon}(u'-1)^- = 0, \quad x > 0,$$

which can be rewritten as

$$-\alpha u + \frac{1}{2}\sigma^2 x^2 u'' + x^\gamma u' + \frac{x}{\varepsilon}\max_{0\le c\le 1}(1-u')c = 0, \quad x > 0.$$

Let $\mathcal{C}$ be the class of all $\{\mathcal{F}_t\}$-progressively measurable processes $c = \{c_t\}$ such that $0 \le c_t \le 1$, *a.s.* for all $t \ge 0$. For any $c \in \mathcal{C}$, let $\{X_t\}$ be the solution of

$$dX_t = X_t^\gamma dt + \sigma X_t dB_t - \frac{1}{\varepsilon}c_t X_t dt, \qquad X_0 = x > 0.$$

Define

$$u(x) = \sup_{c\in\mathcal{C}} E[\int_0^\infty e^{-\alpha t}\frac{1}{\varepsilon}c_t X_t dt],$$

where the supremum is taken over all systems $(\Omega, \mathcal{F}, P, \{c_t\}, \{B_t\})$. Then we observe that the penalty equation (p) is a Hamilton-Jacobi-Bellman equation.

Theorem 3.1 *We have*

$$0 \le u(x) \le v_0(x) \le C(1+x), \quad x > 0,$$

for some constant $C > 0$.

Theorem 3.2 *For any* $\rho > 0$, *there exists* $C_{\rho,\varepsilon} > 0$ *such that*

$$|u(x) - u(y)| \le C_{\rho,\varepsilon}|x-y| + \rho(1+x+y), \quad x, y > 0.$$

Theorem 3.3 u *is concave on* $(0, \infty)$.

4 Solution of the penalty equation

In this section, we show that the probabilistic solution u is a classical solution of the penalty equation (p).

Definition 4.1 *Let* $w \in C(0,\infty)$. *Then* w *is called a viscosity solution of (p) if*

(a) w *is a viscosity subsolution of (p), that is, for any* $\phi \in C^2(0,\infty)$ *and any local maximum point* $z > 0$ *of* $w - \phi$,

$$-\alpha w + \frac{1}{2}\sigma^2 x^2 \phi'' + x^\gamma \phi' + \frac{x}{\varepsilon}(\phi' - 1)^-\Big|_{x=z} \ge 0,$$

and (b) w *is a viscosity supersolution of (p), that is, for any* $\phi \in C^2(0,\infty)$ *and any local minimum point* $\bar{z} > 0$ *of* $w - \phi$,

$$-\alpha w + \frac{1}{2}\sigma^2 x^2 \phi'' + x^\gamma \phi' + \frac{x}{\varepsilon}(\phi' - 1)^-\Big|_{x=\bar{z}} \le 0.$$

By Theorems 3.1 and 3.2, we can show that the dymanic programming principle holds for u, i.e.,

$$u(x) = \sup_{c \in \mathcal{C}} E[\int_0^s e^{-\alpha t}\frac{1}{\varepsilon}c_t X_t dt + e^{-\alpha s}u(X_s)]$$

for any $s \ge 0$. By the theory of viscosity solutions, taking into account Proposition 2.1, we have the viscosity property of u. For details, we refer to [9].

Theorem 4.2 *u is a viscosity solution of (p).*

Theorem 4.3 *We have*

$$u \in C^2(0,\infty).$$

5 Solution of the variational inequality

In this section, we study the convergence of $u = u_\varepsilon$ to a viscosity solution v of the variational inequality (VI) as $\varepsilon \to 0$.

5.1 Limit of the penalized problem

Definition 5.1 *Let $w \in C(0,\infty)$. Then w is called a viscosity solution of (VI), if the following assertions are satisfied:*

(a) *For any $\phi \in C^2$ and any local minimum point $\bar{z} > 0$ of $w - \phi$,*

$$\phi'(\bar{z}) \geq 1, \quad -\alpha w + \frac{1}{2}\sigma^2 x^2 \phi'' + x^\gamma \phi' \Big|_{x=\bar{z}} \leq 0,$$

(b) *For any $\phi \in C^2$ and any local maximum point $z > 0$ of $w - \phi$,*

$$(-\alpha w + \frac{1}{2}\sigma^2 x^2 \phi'' + x^\gamma \phi')(\phi' - 1)^+ \Big|_{x=z} \geq 0.$$

By concavity and Theorem 3.1, we get

$$0 \leq u_\varepsilon'(x) x \leq u_\varepsilon(x) - u_\varepsilon(0) \leq v_0(x), \quad x > 0.$$

Hence, for any $0 < a < b$,

$$\sup_\varepsilon \|u_\varepsilon'\|_{C[a,b]} < \infty.$$

By the Ascoli-Arzelà theorem and Theorem 4.2, we have the following.

Theorem 5.2 *There exists a subsequence $\{u_{\varepsilon_n}\}$ such that*

$$u_{\varepsilon_n} \to v \in C(0,\infty) \quad \textit{locally uniformly in } (0,\infty) \textit{ as } \varepsilon_n \to 0.$$

Furthermore, v is a viscosity solution of (VI).

5.2 Regularity

In this subsection, we study the regularity of the viscosity solution v of (VI). By concavity, we can show that

$$u'_{\varepsilon_n} \geq 1 \quad \text{on} \quad [a,b].$$

We rewrite the penalty equation as

$$-u''_\varepsilon = \frac{2}{\sigma^2 x^2}\{-\alpha u_\varepsilon + x^\gamma u'_\varepsilon + \frac{x}{\varepsilon}(u'_\varepsilon - 1)^-\}.$$

Thus we have:

Theorem 5.3 *For any $0 < a < b$, we have*

$$\sup_{n\geq 1} \|u''_{\varepsilon_n}\|_{C[a,b]} < \infty.$$

By Theorem 5.3, extracting a subsequence, we have

$$u'_{\varepsilon_n} \quad \to \quad v' \quad \text{locally uniformly in} \quad (0,\infty) \quad \textit{as} \quad n \to \infty,$$

and v' is locally Lipschitz on $(0,\infty)$.

Theorem 5.4 *We have*

$$v \in C^{1,1}_{loc}(0,\infty), \quad \textit{piecewise } C^2, \quad v' \geq 1 \quad \textit{on} \quad (0,\infty).$$

Furthermore, by using Proposition 2.2, we can state the following.

Theorem 5.5 *We have*

$$v'(0+) > 1,$$

and there exists $x^* > 0$ such that

$$x^* = \inf\{x > 0 : \; v'(x) = 1\}.$$

6 Optimal dividend payments

In this section, we give a synthesis of the optimal policy $D^* \in \mathcal{A}$ of the maximization problem.

Consider the SDE with reflecting barrier conditions:

$$\begin{aligned} &(a) \qquad dK_t^* = (K_t^*)^\gamma dt + \sigma K_t^* dB_t - K_t^* dD_t^*, \quad K_0^* = x - D_0^* > 0, \\ &(b) \qquad D_t^* = (x - x^*)^+ + \int_0^t 1_{\{K_s^* = x^*\}} dD_s^*, \\ &(c) \qquad D_t^* \text{ is continous a.s.}, \\ &(d) \qquad K_t^* \in \mathcal{R}, \quad \forall t \geq 0, \quad \text{a.s.}, \\ &(e) \qquad \int_0^t 1_{\{K_s^* = x^*\}} ds = 0, \quad \forall t \geq 0, \quad \text{a.s.}, \end{aligned}$$

where $\mathcal{R} := (0, x^*]$ for $x^* = \inf\{x > 0 : v'(x) = 1\}$.

Theorem 6.1 *We assume that the initial position $x \leq x^*$, (by making $D_0 = x - x^*$ if $x > x^*$). Then the optimal policy $D^* = \{D_t^*\}$ is given by (a) - (e).*

Lemma 6.2 *There exists a unique solution $(\{K_t^*\}, \{D_t^*\})$ of (a) - (e).*

Proof. There exists a unique solution $\{(M_t, \Delta_t)\}$ of the SDE with reflecting barrier conditions:

- $dM_t = (1 - \gamma)(dt - \frac{\sigma^2 \gamma}{2} M_t dt + \sigma M_t dB_t) - d\Delta_t, \quad M_0 = x^{1-\gamma} - \Delta_0 > 0,$
- $\Delta_t = (x^{1-\gamma} - (x^*)^{1-\gamma})^+ + \int_0^t 1_{\{M_s \in \partial\mathcal{S}\}} d\Delta_s,$
- Δ_t is continous a.s.,
- $M_t \in \mathcal{S}, \quad \forall t \geq 0, \quad$ a.s.,
- $\int_0^t 1_{\{M_s \in \partial\mathcal{S}\}} ds = 0, \quad \forall t \geq 0, \quad$ a.s.,

where $\mathcal{S} = [0, (x^*)^{1-\gamma}]$ and $\{\Delta_t\}$ is a bounded variation process. Define

$$K_t^* = M_t^\beta, \quad D_t^* = \Delta_0^\beta + \int_0^t \beta M_s^{-1} 1_{\{M_s > 0\}} d\Delta_s, \quad \beta := 1/(1 - \gamma).$$

Then, Ito's formula completes the proof.

Proof of Theorem 6.1. Let $D \in \mathcal{A}$ be arbitrary. By the variational inequality and the continuity of $\{D_t\}$, we can apply the generalized Ito formula to $\{K_t\}$ for convex functions (cf. [5]). Then

$$\begin{aligned} e^{-\alpha s}v(K_s) - v(x_D) &= \int_0^s e^{-\alpha t}\{-\alpha v + \frac{1}{2}\sigma^2 x^2 v'' + x^\gamma v'\}\Big|_{x=K_t} dt \\ &+ \int_0^s e^{-\alpha t}v'(K_t)\sigma K_t dB_t - \int_0^s e^{-\alpha t}v'(K_t)K_t dD_t \\ &\leq \int_0^s e^{-\alpha t}v'(K_t)\sigma K_t dB_t - \int_0^s e^{-\alpha t}v'(K_t)K_t dD_t, \quad a.s. \quad s \geq 0. \end{aligned}$$

Hence

$$E[\int_0^{\tau_R} e^{-\alpha t}K_t dD_t] \leq v(x_D) \leq v(x).$$

where $\tau_R := R \wedge \inf\{t \geq 0 : K_t \geq R \text{ or } K_t \leq 1/R\}$ for $R > 0$. Letting $R \to \infty$,

$$J(D) = E[\int_0^\infty e^{-\alpha t}K_t dD_t] \leq v(x).$$

By the same argument as above, we get

$$v(x) = E[\int_0^\infty e^{-\alpha t}v'(K_t^*)K_t^* dD_t^*].$$

Since D_t^* increases only when $K_t^* = x^*$ and $v'(x^*) = 1$,

$$v(x) = E[\int_0^\infty e^{-\alpha t}v'(K_t^*)1_{\{K_t^*=x^*\}}K_t^* dD_t^*] = E[\int_0^\infty e^{-\alpha t}K_t^* dD_t^*] = J(D^*),$$

which completes the proof.

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