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Nonlocal minimal surfaces

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Abstract

This paper describes a joint work with L. Caffarelli and O. Savin. We introduce a new notion of minimal surfaces, replacing the BV norm by the H^α one, with $\alpha < \frac{1}{2}$. The resulting sets are called α-minimal sets. The main result is a de Giorgi type theorem: if an α-minimal set is flat enough, then it is smooth.

1 Introduction

In this work with L. Caffarelli and O. Savin [1], our goal is to understand the regularity properties of sets whose indicator function is a *local* minimiser of the H^α norm, $\alpha < \frac{1}{2}$.

To make sense of this, let us do a brief review of the de Giorgi theory of minimal surfaces. We say that Σ is a *minimal surface* in B_1 if any perturbation of Σ within B_1 increases its area. The question that arises first is the regularity, and a large effort was devoted to it in the 1930's and the 1960's (Bernstein, Radò, Almgren, Federer...) but the definite blow came from de Giorgi, and we explain his result now. In the de Giorgi theory [4] we see the surface Σ as the boundary of a set E. Ask that $\mathbf{1}_E$ be in $BV(B_1)$.

Definition. *E minimal in B_1 iff for all F such that*

- *$\mathbf{1}_F$ is in $BV(B_1)$*
- *$F = E$ on ∂B_1 we have*

$$|\int_{B_1} |D\mathbf{1}_E| \leq \int_{B_1} |D\mathbf{1}_F|.$$

This coincides with classical definition if ∂E is smooth.

Theorem. (de Giorgi) *[i]. E minimal in B_1, then if $B_1 \cap \partial E$ is flat enough (i.e. can be trapped in a very flat box) then $B_{1/2} \cap \partial E$ is a $C^{1,\gamma}$ (hence analytic) graph. [ii]. The dimension of the singular set is $\leq N - 8$.*

Let us then see what happens when we replace BV by H^α, $\alpha > 0$. We consider sets E such that $\mathbf{1}_E$ is in $H^\alpha(\mathbb{R}^N)$ (forget problems at infinity for the moment), we have

$$\begin{aligned} \|\mathbf{1}_E\|^2_{H^\alpha(\mathbb{R}^N)} &= \int \frac{(\mathbf{1}_E(x) - \mathbf{1}_E(y))^2}{|x-y|^{N+2\alpha}}\, dxdy \\ &= |2 \int_E \int_{\mathbb{R}^N\setminus E} \frac{dxdy}{|x-y|^{N+2\alpha}} \\ := L(E, \mathbb{R}^N\setminus E) \end{aligned}$$

Note 1. We MUST have $\alpha < \frac{1}{2}$. Indeed, NO indicator function is in H^α, $\alpha \geq \frac{1}{2}$.
Note 2. The quantity $\|\mathbf{1}_E\|_{H^\alpha(\mathbb{R}^N)}$ makes sense if E is smooth and bounded.

Let us then proceed to the definition of *α-minimal sets.*

Definition 1. *E is α-minimal in B_1 iff for all F such that*
- $\mathbf{1}_F \in H^\alpha(\mathbb{R}^N)$,
- $F = E$ *outside* B_1 *we have*

$$|\|\mathbf{1}_E\|_{H^\alpha} \leq \|\mathbf{1}_F\|_{H^\alpha}.$$

This looks like the definition of a local minimum, but still does not treat unbounded sets. To take into account and unbounded E: remove the nonconvergent part $\|\mathbf{1}_E\|_{H^\alpha}$, i.e.

Definition 2. *E is α-minimal in B_1 iff for all $A \subset B_1$ such that*

$$\mathbf{1}_{E\cup A},\ \mathbf{1}_{(\mathbb{R}^N\setminus E)\cup A} \in H^\alpha_{loc}(\mathbb{R}^N)$$

we have

$$\begin{array}{ll} \textit{if } A \subset E, \textit{ then} & L(A, \mathbb{R}^N\setminus E) \leq L(A, E\setminus A) \\ \textit{if } A \subset \mathbb{R}^N\setminus E, \textit{ then} & L(A, E) \leq L(A, \mathbb{R}^N\setminus(E \cup A)) \end{array}$$

Interpretation. We may remember this barbarian looking condition by saying that the interaction of A with the rest of the world through ∂E is less than that through ∂A.

2 Motivation and main result

Definitions 1 or 2 should certainly imply some sort of Euler-Lagrange equation, and this is our next task. What follows, although philosophically correct, has no mathematical rigour of any kind.

Let A be a 'small' set containing the point $x \in \partial E$, where we are going to write the optimality condition. Linearise

$$|\|\mathbf{1}_E\|^2_{H^\alpha} \leq \|\mathbf{1}_F\|^2_{H^\alpha}$$

with

$$\int_{E\cup A}\int_{\mathbb{R}^N\setminus(E\cup A)} = \int_E\int_{\mathbb{R}^N\setminus E} + \int_A(\int_{\mathbb{R}^N\setminus E} - \int_E) - \int_A\int_A.$$

This is, as already said technically wrong, because the kernel $|x-y|^{N+2\alpha}$ does not leave the integrals of interest any chance to converge. Let us however proceed, and the result is

$$\begin{aligned}\kappa_\alpha(x) &:= \int \frac{\mathbf{1}_{\mathbb{R}^N\setminus E}(y)-\mathbf{1}_E(y)}{|x-y|^{N+2\alpha}}\,dy\\ &= 0\\ &= \int_0^{+\infty}\frac{dr}{r^{1+2\alpha}}\frac{\text{area}(S_r^{N-1}(x)\cap(\mathbb{R}^N\setminus E))-\text{area}(S_r^{N-1}(x)\cap E)}{r^{N-1}}\end{aligned}$$

The last amount may be seen as the (algebraic) excess area of ∂E over its tangent plane at x (provided it exists). Moreover we see that some regularity is needed for the last integral to exist: it diverges if, for instance, ∂E has a corner at x.

Remark 1. We have $\lim_{\alpha\to 1/2}(\frac{1}{2}-\alpha)\kappa_\alpha(x)=\kappa(x)$ (the mean curvature at x) if ∂E is C^2.

Remark 2. We have $\lim_{\alpha\to 1/2}(\frac{1}{2}-\alpha)\|\mathbf{1}_E\|_{H^\alpha}=\|\mathbf{1}_E\|_{BV}$ if ∂E is bounded and C^2.

A further justification of our minimisation problem is the study of Allen-Cahn functionals. Consider indeed the classical Allen-Cahn energy:

$$J_\varepsilon(u)=\int(\frac{1}{2}|Du|^2+\frac{1}{\varepsilon}G(u))\,dx,\quad G\text{: standard double-well potential.}$$

As is well-known (Modica-Mortola), a converging sequence of minimisers converges, as $\varepsilon\to 0$, to the indicator of a minimal set. Consider the nonlocal Allen-Cahn energy:

$$J_\varepsilon(u)=\frac{1}{2}\int\frac{(u(x)-u(y))^2}{|x-y|^{N+2\alpha}}+\frac{1}{\varepsilon}\int G(u)$$

A converging sequence of minimisers will converge, as $\varepsilon\to 0$, to the indicator of an α-minimal set.

PROOF. An H^α indicator is an admissible test function. This bounds the ε^{-2} terms, and a classical semiconinuity argument concludes. •

Note that, for $\alpha>1/2$, we have (and this is much less trivial) convergence to classical minimal sets (Gonzalez [3]).

Our main result reads as follows.

Theorem. ([1]) *[i]. If E minimal in B_1; if $\partial E\cap B_1$ is flat enough then $\partial E\cap B_{1/2}$ is a $C^{1,\gamma}$ graph.*

[ii]. The dimension of the singular set is $\le N-2$.

Definition. 1. *The flatness of the cylinder $\Sigma=\{x'\in B,\ |x_N|\le h\}$ is*

$$\text{flatness}(\Sigma)=\frac{h}{\text{largest diameter of a ball}\subset B}.$$

2. *E is δ-flat at x if, in a system of coordinates (say, (x',x_N)) we have*

$$\{x_N\le 0\}\cap B_1(x)\subset E\cap B_1(x)\subset\{x_N\le\delta\}.$$

3 Proof of regularity: the main steps

To prove the theorem we do an improvement of flatness. More precisely, if we prove a statement of the type
Theorem. E α-minimal in B_1, and

$$\{|x'| \leq 1,\ x_N \leq -\varepsilon\} \subset E \cap B_1 \subset \{|x'| \leq 1,\ x_N \leq \varepsilon\}.$$

There are $\varepsilon_0 > 0$, $\delta_0 \in (2\varepsilon_0, 1)$ and $\gamma_0 \in (0,1)$ universal such that, if $\varepsilon \leq \varepsilon_0$:

$$\{|\tilde{x}'| \leq 1,\ \tilde{x}_N \leq -\delta_0\gamma_0\varepsilon\} \subset E \cap B_{\delta_0} \subset \{|\tilde{x}'| \leq 1,\ \tilde{x}_N \leq \delta_0\gamma_0\varepsilon\}.$$

in a possibly different system of coordinates $(\tilde{x}', \tilde{x}_N)$.
then we are done. An alternative formulation is: if ∂E is ε-flat in B_1, then ∂E is $\gamma_0\varepsilon$-flat in B_{δ_0}. This classically implies (see [4]) $C^{1,\gamma}$ regularity.

The strategy is by contradiction (Savin [5]). We assume the existence of a sequence of α-minimal sets (E_ε) such that
- we have $\{|x'| \leq 1, \quad x_N \leq -\varepsilon\} \subset E_\varepsilon \cap B_1 \subset \{|x'| \leq 1, \quad x_N \leq \varepsilon\}$
- and improvement of flatness does not hold.

Our goal is the following: consider the dilations $\partial E'_\varepsilon = \{(x', \frac{x_N}{\varepsilon}), \quad (x', x_N) \in \partial E\}$. We wish to prove:
- $(\partial E'_\varepsilon)_\varepsilon$ converges to a graph $\{x', \phi(x')\}$ in $B_{1/2}$
- and ϕ satisfies a nice equation (e.g. $(-\Delta_{x'})^{\frac{1+2\alpha}{2}}\phi = 0$).

This will imply the contradiction. We proceed in two steps.

Step 1: Convergence to a graph

The tool here is a Harnack type inequality which says that, roughly, that if ∂E is well localised in B_1, it is even better localised in a smaller ball.
Theorem. *Consider E α-minimal, and $0 \in \partial E$, such that we have*

$$\{|x'| \leq 1, \quad x_N \leq -\varepsilon\} \subset E \cap B_1 \subset \{|x'| \leq 1, \quad x_N \leq \varepsilon\}$$

Then there is $\varepsilon_0 > 0$, $\delta_0 \in (0,1)$ universal such that, if $\varepsilon \leq \varepsilon_0$:

$$\{|x'| \leq \frac{1}{2}, \quad x_N \leq -\delta_0\varepsilon\} \subset E \cap B_{1/2} \subset \{|x'| \leq \frac{1}{2}, \quad x_N \leq \delta_0\varepsilon\}$$

Corollary. *$(\partial E'_\varepsilon)_\varepsilon$ converges to a graph $\{x', \phi(x')\}$ in $B_{1/2}$. Moreover, ϕ is Hölder with exponent* $\dfrac{|\mathrm{Log}\delta_0|}{\mathrm{Log}2}$
PROOF. Apply the Harnack k times such that $(2\delta_0)^k\varepsilon \leq \varepsilon_0$.

Step 2: A viscosity inequality

Similarly to the theorem of Caffarelli-Cordoba (a minimal surfaces satisfies the curvature equation in the viscosity sense) we prove the following

Theorem. *E α-minimal, and $x \in \partial E$. Assume a ball touches ∂E from below at x. Then*

$$\kappa_\alpha(x) := \int \frac{\mathbf{1}_{\mathbb{R}^N \setminus E}(y) - \mathbf{1}_E(y)}{|x-y|^{N+2\alpha}}\, dy \geq 0$$

Using the fact that we have a graph at the scale ε, we plug this into the viscosity relation... and to our profound discontent we obtain

$$\varepsilon(-\Delta)^{\frac{1+2\alpha}{2}}\phi = \text{Lipschitz} + \text{h.o.t.}$$

The nonlocality of the problem has struck!

To remedy this, we take an intermediate scale ensuring that

- the zero-order part disappears,
- at the limit, ϕ does not grow too fast so that $(-\Delta)^{\frac{1+2\alpha}{2}}\phi$ is well-defined.

To put this programme to work, we replace the initial improvement of flatness statement by the more sophisticated one:

Theorem. *E α-minimal, $0 \in \partial E$. Pick $\sigma < 2\alpha$. There exists k_0 integer such that: if there is a sequence $(\Sigma_k)_{0 \leq k \leq k_0}$ such that, for $k \leq k_0$:*

$$\partial E \cap B_{2^{-k}} \text{ trapped in } \Sigma_k,$$

with Σ_k cylinder such that

$$\frac{\text{flatness}(\Sigma_{k+1})}{\text{flatness}(\Sigma_k)} = \frac{1}{2^\sigma}.$$

Then $\partial E \cap B_{2^{-(k_0+1)}}$ is trapped in a cylinder of flatness $2^{-(k_0+1)\sigma}$.

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