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ON THE DISTRIBUTION OF PISOT AND CNS POLYNOMIALS

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1. Introduction

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Let $d \geq 1$ be an integer and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$. Consider the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$: for $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ let

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor),$$

where $\mathbf{ra} = r_1 a_1 + \cdots + r_d a_d$ denotes the inner product. We call $\tau_{\mathbf{r}}$ a *shift radix system* (*SRS* for short) if for all $\mathbf{a} \in \mathbb{Z}^d$ we can find some $k > 0$ with $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$. This concept was introduced by Akiyama et al. [1]. We proved that it is a common generalization of canonical number systems in residue class rings of polynomial rings (see [8, 10, 12]) as well as of β-expansions of real numbers, [13]. For the investigation of properties of SRS it turned out convenient to introduce some sets.

For $d \in \mathbb{N}$, $d \geq 1$ let

$$\begin{aligned} \mathcal{D}_d &:= \left\{ \mathbf{r} \in \mathbb{R}^d : \forall \mathbf{a} \in \mathbb{Z}^d \ (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic} \right\}, \\ \mathcal{D}_d^0 &:= \left\{ \mathbf{r} \in \mathbb{R}^d : \forall \mathbf{a} \in \mathbb{Z}^d \, \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \right\}. \end{aligned}$$

It is clear that $\mathcal{D}_d^0 \subset \mathcal{D}_d$ and $\mathbf{r}$ is SRS iff $\mathbf{r} \in \mathcal{D}_d^0$. In [1] we proved among others that $\mathcal{D}_d, \mathcal{D}_d^0$ are Lebesgue measurable and $\mathcal{D}_d^0$ admits some convexity property. On the other hand the results of [2] shoved that the boundary already of $\mathcal{D}_2^0$ is very complicated. Further we proved in [1] that we can embed the discrete sets of Pisot, Salem and CSN polynomials in these continues sets. In [3] and [4] we studied the distribution of Pisot, Salem and CNS polynomials. In the present paper we give a survey about the last mentioned results. Further we present the sketch of the proof one of the main results.

2. Pisot and Salem polynomials

Let $P(X) = X^d - b_1 X^{d-1} - \cdots - b_d \in \mathbb{Z}[X]$.

- If all but one root of P is located in the open unit disc then P is called a *Pisot polynomial.* Its dominant root is called *Pisot number.*
- If all but one root of P is located in the closed unit disc and at least one of them has modulus 1 then P is called a *Salem polynomial.* Its dominant root is called *Salem number.*

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If P is a Pisot or Salem polynomial, we will denote its dominating root by β.

Let $\mathrm{Fin}(\beta)$ be the set of positive real numbers having finite greedy expansion with respect to β. We say that $\beta > 1$ has property (F) if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty).$$

It was shown by Frougny and Solomyak [7] that (F) can hold only for Pisot numbers β. Analogously to $\mathcal{D}_d$ and $\mathcal{D}_d^0$ define for each $d \in \mathbb{N}$, $d \geq 1$ the sets

$$\mathcal{B}_d = \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : P(X) \text{ is a Pisot or Salem polynomial}\}$$

and

$$\mathcal{B}_d^0 = \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : P(X) \text{ is a Pisot polynomial with property (F)}\},$$

where $P(X) = X^d - b_1 X^{d-1} - \cdots - b_d$. We obviously have $\mathcal{B}_d^0 \subseteq \mathcal{B}_d$.

If $(b_1, \ldots, b_d) \in \mathcal{B}_d$ then let β be the dominating root of

$$P(X) = X^d - b_1 X^{d-1} - \cdots - b_d.$$

Consider the map $\psi : \mathcal{B}_d \to \mathbb{R}^{d-1}$:

$$\psi(b_1, \ldots, b_d) = (r_d, \ldots, r_2),$$

where $r_2, \ldots, r_d$ are such that

$$X^d - b_1 X^{d-1} - \cdots - b_d = (X - \beta)(X^{d-1} + r_2 X^{d-2} + \cdots + r_d).$$

As $(b_1, \ldots, b_d) \in \mathcal{B}_d$, the polynomial $X^{d-1} + r_2 X^{d-2} + \cdots + r_d$ has all its roots in the closed unit circle. Thus

$$\psi(\mathcal{B}_d) \subseteq \overline{\mathcal{D}_{d-1}}.$$

In [1] we proved:

$$\psi(\mathcal{B}_d^0) \subseteq \mathcal{D}_{d-1}^0.$$

This means we can embed the discrete sets $\mathcal{B}_d$ and $\mathcal{B}_d^0$ in the continues sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$ respectively, i.e, SRS can be considered as a generalization of the β-representations.

The sets $\mathcal{B}_d, \mathcal{B}_d^0$ are obviously discrete and infinite. To study their distribution we fix the first coordinate. More precisely, for $M \in \mathbb{N}_{>0}$ we set

$$\mathcal{B}_d(M) := \left\{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} : (M, b_2, \ldots, b_d) \in \mathcal{B}_d\right\}$$

and

$$\mathcal{B}_d^0(M) := \left\{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} : (M, b_2, \ldots, b_d) \in \mathcal{B}_d^0\right\}.$$

It is clear that $\mathcal{B}_d^0(M) \subseteq \mathcal{B}_d(M)$, moreover $\mathcal{B}_d(M)$ is finite. Indeed, as $M = \beta +$ other roots of $X^d - MX^{d-1} - b_2 X^{d-2} - b_d$ and the roots of $X^d - MX^{d-1} - b_2 X^{d-2} - b_d$ except of β are lying in the unit disc, thus $|\beta| \leq M + d - 1$. Hence there are easily computable constants $c_i(M, d)$ such that $|b_i| \leq c_i(M, d)$, which ensures the finiteness of $\mathcal{B}_d(M)$. With these notations we proved in [4] the following theorem.

Theorem 1. *We have*

$$\left| \frac{|\mathcal{B}_d(M)|}{M^{d-1}} - \lambda_{d-1}(\mathcal{D}_{d-1}) \right| = O(M^{-d+1+1/d}), \tag{1}$$

and

$$\lim_{M \to \infty} \frac{|\mathcal{B}_d^0(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_{d-1}^0), \tag{2}$$

where λ_{d-1} denotes the $d-1$-dimensional Lebesgue measure and $|A|$ the cardinality of the finite set A.

Notice that (2) is weaker than (1). As the boundary of $\mathcal{D}_{d-1}$ is smooth, we were able to estimate accurately the number of images under ψ lying near to the boundary. This was not possible for $\mathcal{D}_{d-1}^0$, because its boundary is quite complicated.

In Theorem 1 and later in Theorem 2 the volume or Lebesgue measure of $\mathcal{D}_d$ appears in the main term. This was calculated by Fam [6]. Using the Barnes G-function we have

$$\lambda_d(\mathcal{D}_d) = \begin{cases} \frac{2^{2n^2+n}\Gamma(n+1)G(n+1)^4}{G(2n+2)} & (d=2n), \\ \frac{2^{2n^2+3n+1}G(n+2)^4}{\Gamma(n+1)G(2n+3)} & (d=2n+1). \end{cases}$$

Note that for positive integers the Barnes G-function equals the superfactorials: $G(n+2) = \prod_{j=1}^n j!$ for $n \in \mathbb{N}$. Moreover, observe that by [6, Formula (2.13)] we have $\lim_{d\to\infty} \lambda_d(\mathcal{D}_d) = 0$. On the other hand the diameter of $\mathcal{D}_d$ tends to infinity with d. Indeed, the vector of the coefficients of the k-th cyclotomic polynomial Φ_k belongs to the boundary of $\mathcal{D}_{\varphi(k)}$ and by a result of Emma Lehmer [11] the maximum of the absolute value of the coefficients of Φ_k is not bounded, see also [9].

3. CNS POLYNOMIALS

Assume $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0$ with $p_0 \geq 2$ and set $\mathcal{N} = \{0, 1, \ldots, p_0 - 1\}$. Denote by x the image of X under the canonical epimorphism from $\mathbb{Z}[X]$ to $R := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$. Each coset of R has a unique element of degree at most $d-1$, say

$$A(X) = A_{d-1}X^{d-1} + \cdots + A_1X + A_0 \quad (A_0, \ldots, A_{d-1} \in \mathbb{Z}). \tag{3}$$

Let $\mathcal{G} := \{A(X) \in \mathbb{Z}[X] \,:\, \deg A < d\}$ and

$$T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1})X^i,$$

where $A_d = 0$ and $q = \lfloor A_0/p_0 \rfloor$. Then $T_P \,:\, \mathcal{G} \to \mathcal{G}$ and

$$A(X) = (A_0 - qp_0) + XT_P(A), \text{ where } A_0 - qp_0 \in \mathcal{N}.$$

If for each $A \in \mathcal{G}$ there is a $k \in \mathbb{N}$ such that $T_P^k(A) = 0$ we call P a *canonical number system polynomial* (*CNS polynomial*). Let $P(X)$ be a monic irreducible CNS polynomial and denote α one of its roots. Then $\mathcal{G}$ is isomorphic to $\mathbb{Z}[\alpha]$ and α is the bases of a canonical number system in $\mathbb{Z}[\alpha]$. Canonical number systems were introduced for quadratic number fields by Kátai and Kovács [8] and for number rings by Kovács and Pethő [10]. You find this general definition in [12, 1].

Similarly to Pisot polynomials, associated to CNS polynomials we define for each $d \in \mathbb{N}$, $d \geq 1$ the sets

$$\mathcal{C}_d := \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d \,:\, |p_0| \geq 2 \text{ and } T_P \text{ has only finite orbits}\}$$

and

$$\mathcal{C}_d^0 := \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d : |p_0| \geq 2 \text{ and } \forall A \in \mathcal{G}\ \exists \ell \in \mathbb{N} : T_P^\ell(A) = 0\},$$

where $P = X^d + p_{d-1}X^{d-1} + \cdots + p_0$. In [1] we proved that

$$(p_0, p_1, \ldots, p_{d-1}) \in \mathcal{C}_d \text{ (resp. } \mathcal{C}_d^0)$$

if and only if

$$\left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}\right) \in \mathcal{D}_d \text{ (resp. } \mathcal{D}_d^0).$$

With other words SRS is a generalization of CNS. Again $\mathcal{C}_d$ and $\mathcal{C}_d^0$ are infinite discrete sets. To obtain finite portions of them it is enough to fix one coordinate.

For $M \in \mathbb{N}_{>0}$ we set

$$\mathcal{C}_d(M) := \left\{(p_1, \dots, p_{d-1}) \in \mathbb{Z}^{d-1} \, : \, (M, p_1, \dots, p_{d-1}) \in \mathcal{C}_d\right\}$$

and

$$\mathcal{C}_d^0(M) := \left\{(p_1, \dots, p_{d-1}) \in \mathbb{Z}^{d-1} \, : \, (M, p_1, \dots, p_{d-1}) \in \mathcal{C}_d^0\right\}.$$

It is clear that $\mathcal{C}_d^0(M) \subseteq \mathcal{C}_d(M)$. Moreover $\mathcal{C}_d(M)$ is finite. Indeed, it is easy to see (c.f. [1]) that if the coefficients of a polynomial belong to $\mathcal{C}_d$ then all roots are lying outside the unit circle. As their product is equal to M, their modulus are bounded by M, thus $|p_i|, i = 1, \dots, d-1$ is bounded to.

With the above notations we proved in [3]

Theorem 2. *We have*

$$\lim_{M \to \infty} \frac{|\mathcal{C}_d(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_{d-1}),$$

and similarly

$$\lim_{M \to \infty} \frac{|\mathcal{C}_d^0(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_{d-1}^0).$$

Notice that in Theorem 2 in contrast to Theorem 1 we were able to establish only the main term in the distribution function. This is natural for $\mathcal{C}_d^0(M)$ by the same reason, described after Theorem 1.

4. Sketch of the proof of Theorem 1

In this section we present the main steps of the proof of Theorem 1. You may found the details in [4].

4.1. **Properties of two auxiliary mappings.** For $M \in \mathbb{Z}$ let the mapping $\chi_M \, : \, \mathbb{R}^{d-1} \mapsto \mathbb{Z}^d$ be such that if $\mathbf{r} = (r_d, \dots, r_2)$ then $\chi_M(\mathbf{r}) = \mathbf{b} = (b_1, \dots, b_d)$, where

$$\begin{aligned} b_1 &= M, b_d = \left\lfloor r_d(M + r_2) + \frac{1}{2} \right\rfloor \quad \text{and} \\ b_i &= \left\lfloor r_i(M + r_2) - r_{i+1} + \frac{1}{2} \right\rfloor, i = 2, \dots, d-1. \end{aligned}$$

If $\mathbf{b} = (b_1, \dots, d_d) \in \mathcal{B}_d$, then $\chi_{b_1}(\psi(\mathbf{b})) = \mathbf{b}$, i.e., χ_{b_1} is a left invers of ψ.

To prove Theorem 1 we need some properties of the sets

$$\mathcal{S}_d(M) = \chi_M(\overline{\mathcal{D}_{d-1}}) \quad \text{and} \quad \mathcal{S}_d^0(M) = \chi_M(\overline{\mathcal{D}_{d-1}^0})$$

and

$$\mathcal{S}_d = \cup_{M \in \mathbb{Z}} \mathcal{S}_d(M) \quad \text{and} \quad \mathcal{S}_d^0 = \cup_{M \in \mathbb{Z}} \mathcal{S}_d^0(M).$$

Our first Lemma shows that if $|M|$ is large enough then the polynomials associated to the elements of $\mathcal{S}_d(M)$ behaves in some sense similar as Pisot or Salem polynomials.

Lemma 3. *Let $M \in \mathbb{Z}$, $(M, b_2, \dots, b_d) = (b_1, \dots, b_d) \in \mathcal{S}_d(M)$ and $P(X) = X^d - b_1X^{d-1} - \dots - b_d$. There exist constants $c_1 = c_1(d), c_2 = c_2(d)$ such that if $|M|$ is large enough than $P(X)$ has a real root β for which the inequalities*

$$|\beta - b_1| < c_1 \tag{4}$$

$$\left|\beta - b_1 - \frac{b_2}{b_1}\right| < \frac{c_2}{|b_1|} + O\left(\frac{1}{b_1^2}\right), \tag{5}$$

hold.

Now we are in the position to extend the definition of ψ from the set $\mathcal{B}_d$ to $\mathcal{S}_d$. If $(b_1, \dots, b_d) \in \mathcal{S}_d$ and $|b_1|$ is large enough, then let β be the dominating root of the polynomial

$$P(X) = X^d - b_1X^{d-1} - \dots - b_d,$$

which exists by Lemma 3. Then let

$$\psi(b_1, \dots, b_d) = (r_d, \dots, r_2),$$

where the real numbers $r_2, \dots, r_d$ are defined in a way that they satisfy the relation

$$X^d - b_1X^{d-1} - \dots - b_d = (X - \beta)(X^{d-1} + r_2X^{d-2} + \dots + r_d).$$

We also introduce an other mapping $\tilde{\psi} : \mathbb{Z}^d \mapsto \mathbb{Q}^{d-1}$ by

$$\tilde{\psi}(b_1, \dots, b_d) = \left(\frac{b_d}{b_1 + \frac{b_2}{b_1}}, \frac{b_{d-1}}{b_1 + \frac{b_2}{b_1}} + \frac{b_d}{b_1^2}, \dots, \frac{b_2}{b_1 + \frac{b_2}{b_1}} + \frac{b_3}{b_1^2}\right).$$

The next lemma shows that if $(b_1, \dots, b_d) \in \mathcal{S}_d$ then $\tilde{\psi}(b_1, \dots, b_d)$ is a good approximation of $\psi(b_1, \dots, b_d)$. We actually prove

Lemma 4. *Let $(b_1, \dots, b_d) \in \mathcal{S}_d$ and assume that $|b_1|$ is large enough. Then*

$$\left|\tilde{\psi}(b_1, \dots, b_d) - \psi(b_1, \dots, b_d)\right|_\infty < \frac{c_3}{b_1^2} + O\left(\frac{1}{|b_1|^3}\right),$$

where c_3 is depending only on d.

$\mathcal{B}_d$ and $\mathcal{B}_d(M)$ are subsets of a lattice. This nice property does not remain valid after the application of ψ. However, the next lemma shows that the set $\tilde{\psi}(\mathcal{S}_d)$ is lattice like. More precisely we have

Lemma 5. *Let $\mathbf{b} = (b_1, \dots, b_d), \mathbf{b}' = (b'_1, \dots, b'_d) \in \mathcal{S}_d$ such that there exists a $1 \le j \le d$ with $b_i = b'_i, i \ne j$ and $b'_j = b_j + 1$. Then*

$$|\tilde{\psi}(\mathbf{b})_k - \tilde{\psi}(\mathbf{b}')_k| = \begin{cases} 0, & \text{if } j > 2 \text{ and } k \ne d-j+1, d-j+2 \\ \frac{1}{|b_1|} + O(b_1^{-2}), & \text{if } j > 2 \text{ and } k = d-j+1 \\ & \quad \text{or } j = 2, k = d-1 \\ O(b_1^{-2}), & \text{if } j > 2 \text{ and } k = d-j+2 \\ & \quad \text{or } j = 2, k < d-1 \\ |b_{d-k+1}| \ \frac{1}{b_1^2} + O(|b_1|^{-3} \ , & \text{if } j = 1. \end{cases}$$

4.2. **A lemma on the roots of polynomials.** It is well known that the roots of real polynomials are continues functions of the coefficients. The next lemma is a quantitative version of this fact.

Lemma 6. *Let $d \in \mathbb{N}$ and $\rho, \varepsilon \in \mathbb{R}_{>0}$. Then there exists a constant $c_4 > 0$ depending only on d and ρ with the following property: if all roots $\alpha \in \mathbb{C}$ of the polynomial $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0 \in \mathbb{R}[X]$ satisfy $|\alpha| < \rho$ and $Q(X) = X^d + q_{d-1}X^{d-1} + \cdots + q_0 \in \mathbb{R}[X]$ is chosen such that $|p_i - q_i| < \varepsilon, i = 0, \dots, d-1$ then for each root β of $Q(X)$ there exists a root α of $P(X)$ satisfying*

$$|\beta - \alpha| < c_4 \varepsilon^{1/d}. \tag{6}$$

In particular, all roots β of $Q(X)$ satisfy $|\beta| < \rho + c_4\varepsilon^{1/d}$.

Let

$$\mathcal{E}_d(r) := \{(r_1, \dots, r_d) \in \mathbb{R}^d \,:\, X^d + r_d X^{d-1} + \cdots + r_1 \text{ has only roots } y \in \mathbb{C} \text{ with } |y| < r\}.$$

The next lemma gives a precise estimate for the volume of the strip near to the boundary of $\mathcal{D}_d$. It is very important to prove the first part of Theorem 1.

Lemma 7. *Let $0 < \eta < 1$. Then we have*

$$\lambda_d\left(\mathcal{E}_d\left(1+\eta\right) \setminus \mathcal{D}_d\right) \le 2^{d(d+1)/2} \lambda_d(\mathcal{E}_d(1))\eta$$

and

$$\lambda_d\left(\mathcal{D}_d \setminus \mathcal{E}_d\left(1-\eta\right)\right) \le 2^{d(d+1)/2} \lambda_d(\mathcal{E}_d(1))\eta.$$

4.3. **Proof of Theorem 1 for $\mathcal{D}_d$.** Now we are in the position to finish the first assertion of Theorem 1. Let $M > 0$ and put

$$W(\mathbf{x}, s) = \{\mathbf{y} \in \mathbb{R}^d \,:\, |\mathbf{x} - \mathbf{y}|_\infty \le s/2\} \; (\mathbf{x} \in \mathbb{R}^d, s \in \mathbb{R})$$

and

$$\mathcal{W}_{d-1}(M) = \cup_{\mathbf{x} \in \mathcal{B}_d(M)} W(\psi(\mathbf{x}), M^{-1}).$$

Then we claim

$$\lambda_{d-1}(\mathcal{W}_{d-1}(M)) = \frac{|\mathcal{B}_d(M)|}{M^{d-1}} \left(1 + O\left(\frac{1}{M}\right)\right). \tag{7}$$

Indeed, let $\mathbf{x}, \mathbf{y} \in \mathcal{B}_d(M)$ such that $\mathbf{x} - \mathbf{y} = \mathbf{e}_j$ for some $j \in \{2, \dots, d\}$. Then by Lemmata 4 and 5

$$\begin{aligned} |\psi(\mathbf{x})_k - \psi(\mathbf{y})_k| &\le |\psi(\mathbf{x})_k - \tilde{\psi}(\mathbf{x})_k + \tilde{\psi}(\mathbf{x})_k - \tilde{\psi}(\mathbf{y})_k + \tilde{\psi}(\mathbf{y})_k - \psi(\mathbf{y})_k| \\ &\le \begin{cases} \frac{1}{M} + O\ \frac{1}{M^2}\ , & \text{if } (j,k) = (2, d-1), \text{ or } j > 2, k = d-j+1 \\ \frac{1}{M^2}\ , & \text{otherwise.} \end{cases} \end{aligned}$$

Thus

$$\lambda_{d-1}(W(\psi(\mathbf{x}), M^{-1}) \cap W(\psi(\mathbf{y}), M^{-1})) = O\left(\frac{1}{M^d}\right). \tag{8}$$

As $\mathbf{x}$ has at most 2^d neighbors we get

$$\lambda_{d-1}\left(\bigcup_{\substack{\mathbf{x},\mathbf{y} \in \mathcal{B}_d(M) \\ \mathbf{x} \ne \mathbf{y}}} (W(\psi(\mathbf{x}), M^{-1}) \cap W(\psi(\mathbf{y}), M^{-1}))\right) = O\left(\frac{|\mathcal{B}_d(M)|}{M^d}\right)$$

and the claim is proved.

Hence in the sequel it is enough to consider $\mathbf{x} \in \mathcal{B}_d(M)$.

Lower estimate for $\lambda_{d-1}(\mathcal{D}_{d-1})$.

Put $\eta = c_4(2M)^{-1/(d-1)}$. Let $\mathbf{x} \in \mathcal{B}_d(M)$ such that $\psi(\mathbf{x}) \in \mathcal{E}_{d-1}(\eta) \subseteq \mathcal{D}_{d-1}$. Let $\mathbf{y} \in W(\psi(\mathbf{x}), M^{-1})$. Then $\rho(\psi(\mathbf{x})) < 1-\eta$ and as $|\psi(\mathbf{x}) - \mathbf{y}|_\infty \leq \frac{1}{2M}$ we get $\rho(\mathbf{y}) < 1$. Thus

$$\bigcup_{\substack{\mathbf{x}\in\mathcal{B}_d(M)\\ \rho(\psi(\mathbf{x}))<1-\eta}} W(\psi(\mathbf{x}), M^{-1}) \subseteq \mathcal{D}_{d-1}. \tag{9}$$

By Lemma 7 the measure of the set

$$\mathcal{D}_{d-1} \setminus \mathcal{E}_{d-1}(1-\eta)$$

is bounded by $O(M^{-1/(d-1)})$. Moreover this set satisfies the conditions of a Theorem of H. Davenport [5]. Thus the number of $\mathbf{x} \in \mathcal{B}_d(M)$ such that $1-\eta \leq \rho(\psi(\mathbf{x})) \leq 1$ is at most $O(M^{d-1-1/(d-1)})$. Combining this with (8) and (9) we obtain

$$\lambda_{d-1}(\mathcal{D}_{d-1}) \geq \frac{|\mathcal{B}_d(M)|}{M^{d-1}}\left(1 - c_7 M^{-1/(d-1)}\right).$$

Upper estimate for $\lambda_{d-1}(\mathcal{D}_{d-1})$.

We construct for every $\mathbf{r} = (r_d, \dots, r_2) \in \mathcal{D}_{d-1}$ and M large enough, an integer vector $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{Z}^d$ such that $\psi(\mathbf{b})$ is located near enough to $\mathbf{r}$.

Consider

$$\tilde{\psi}(\mathbf{b}) = \left(\frac{b_d}{b_1 + \frac{b_2}{b_1}}, \frac{b_{d-1}}{b_1 + \frac{b_2}{b_1}} + \frac{b_d}{b_1^2}, \dots, \frac{b_2}{b_1 + \frac{b_2}{b_1}} + \frac{b_3}{b_1^2}\right).$$

Set $\eta = 2c_4(2M)^{-1/(d-1)}$. Thus by Lemma 6 we get

$$\rho(\psi(\mathbf{b})) \leq \rho(\mathbf{r}) + \eta \leq 1 + \eta.$$

This means that if M is large enough then all but one root of $X^d - b_1 X^{d-1} - \dots - b_d$ have absolute value at most $1+\eta$ and one root is close to M.

We have further

$$\begin{aligned}\mathcal{D}_{d-1} &\subseteq \bigcup_{\substack{\mathbf{x}\in\mathbb{Z}^d\\ \psi(\mathbf{x})\in\mathcal{E}_{d-1}(1+\eta)}} W(\psi(\mathbf{x}), M^{-1}) \\ &= \bigcup_{\mathbf{x}\in\mathcal{B}_d(M)} W(\psi(\mathbf{x}), M^{-1}) \cup \bigcup_{\substack{\mathbf{x}\in\mathbb{Z}^d\\ \psi(\mathbf{x})\in\mathcal{E}_{d-1}(1+\eta)\setminus\mathcal{E}_{d-1}(1)}} W(\psi(\mathbf{x}), M^{-1}).\end{aligned}$$

We conclude that the volume of the set $\mathcal{E}_{d-1}(1+\eta)\setminus\mathcal{D}_{d-1}$ is at most $O(M^{-1/(d-1)})$.

As the conditions of the above mentioned Theorem of Davenport [5] hold again we get that the number of $\mathbf{x} \in \mathbb{Z}^d$ such that $\psi(\mathbf{x})$ lies in $\mathcal{E}_{d-1}(1+\eta) \setminus \mathcal{D}_{d-1}$ is at most $O(M^{d-1-1/(d-1)})$. Thus there is a constant $c_8 > 0$ such that

$$\lambda_{d-1}(\mathcal{D}_{d-1}) \leq \frac{|\mathcal{B}_d(M)|}{M^{d-1}}\left(1 + c_8 M^{-1/(d-1)}\right).$$

Combining the lower and upper estimates for $\lambda_{d-1}(\mathcal{D}_{d-1})$ we finish the proof of the first part of Theorem 1.

5. Problem

To fix a coefficient is an unusual way to measure a set of polynomials. Unfortunately, we were not able to prove a to Theorem 1 analogous result for Pisot polynomials with bounded height, i.e, if the maximum modulus of the coefficients is bounded. Therefore we propose the following problem:

For $M \in \mathbb{N}_{>0}$ set

$$\mathcal{B}'_d(M) := \left\{(b_1, b_2, \dots, b_d) \in \mathbb{Z}^d \cap \mathcal{B}_d \,:\, \max\{|b_1|, \dots, |b_d|\} = M\right\}$$

and

$$\mathcal{B}'^0_d(M) := \left\{(b_1, b_2, \dots, b_d) \in \mathbb{Z}^d \cap \mathcal{B}^0_d \,:\, \max\{|b_1|, \dots, |b_d|\} = M\right\}.$$

Do

$$\lim_{M\to\infty} \frac{|\mathcal{B}'_d(M)|}{M^{d-1}} \qquad \text{and/or} \qquad \lim_{M\to\infty} \frac{|\mathcal{B}'^0_d(M)|}{M^{d-1}}$$

exist?

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