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Higher depth regularized products and zeta functions of Milnor type*

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1 Introduction

For a complex sequence $\boldsymbol{a}=\{a_n\}_{n\in I}$, the (zeta) regularized product of $\boldsymbol{a}$ is defined by

$$\prod_{n\in I} a_n := \exp\Big(-\frac{d}{ds}\zeta_{\boldsymbol{a}}(s)\Big|_{s=0}\Big),$$

where $\zeta_{\boldsymbol{a}}(s) := \sum_{n\in I} a_n^{-s}$ is the zeta function attached to $\boldsymbol{a}$. Here, we assume that $\zeta_{\boldsymbol{a}}(s)$ converges absolutely in some right half plane, admits a meromorphic continuation to some region containing the origin and is holomorphic at the origin. This gives a kind of generalization of the usual product. In fact, if $\boldsymbol{a}$ is a finite sequence, then one can see that $\prod_{n\in I} a_n = \prod_{n\in I} a_n$. The most important and fundamental example of the regularized product is the following Lerch formula;

$$\prod_{n\geq 0}(n+z) = \exp\Big(-\frac{d}{ds}\zeta(s,z)\Big|_{s=0}\Big) = \frac{\sqrt{2\pi}}{\Gamma(z)}, \tag{1.1}$$

where $\Gamma(z)$ is the gamma function and $\zeta(s,z) := \sum_{n\geq 0}(n+z)^{-s}$ is the Hurwitz zeta function. In particular, letting $z=1$, we have $\prod_{n\geq 1} n \,(=\infty!) = \sqrt{2\pi}$. Notice that, if $\prod_{n\in I}(a_n+z)$ exists, then, as a function of z, it defines an entire function whose zeros are located at $z=-a_n$ for $n\in I$.

Let $\zeta(s) := \sum_{n\geq 1} n^{-s}$ be the Riemann zeta function and $\mathcal{R}$ the set of all non-trivial zeros of $\zeta(s)$. The following formula was obtained by Deninger [D, Theorem 3.3] (see also [SS, V]);

$$\Xi(z) := \prod_{\rho\in\mathcal{R}}\Big(\frac{z-\rho}{2\pi}\Big) = 2^{-\frac{1}{2}}(2\pi)^{-2}\pi^{-\frac{z}{2}}\Gamma\Big(\frac{z}{2}\Big)\zeta(z)z(z-1) = \frac{1}{2^{\frac{3}{2}}\pi^2}\Lambda(z), \tag{1.2}$$

where $\Lambda(z) := \frac{1}{2}z(z-1)\Gamma(\frac{z}{2})\zeta(z)$ is the complete Riemann zeta function. The aim of this note is to give "higher depth" generalizations of the formula (1.2) for Hecke L-functions. Namely, we explicitly calculate "higher depth regularized products" for the zeros of Hecke L-functions.

We here explain the higher depth regularized products above. In [Mi], from the viewpoint of the Kubert identity which plays an important role in the study of Iwasawa theory, Milnor introduced a "higher depth gamma function" $\boldsymbol{\Gamma}_r(z)$ defined by

$$\boldsymbol{\Gamma}_r(z) := \exp\Big(\frac{d}{ds}\zeta(s,z)\Big|_{s=1-r}\Big) \tag{1.3}$$

and studied, for examples, special values, a Stirling formula (that is, an asymptotic formula as $z\to+\infty$) and functional relations among them (see also [KOW]). Notice that, by the Lerch

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formula (1.1), we have $\mathbf{\Gamma}_1(z) = \frac{\Gamma(z)}{\sqrt{2\pi}}$, whence $\mathbf{\Gamma}_r(z)$ indeed gives a generalization of $\Gamma(z)$. Based on the study of Milnor, we define a *higher depth (or depth r) regulalized product* of the sequence $\boldsymbol{a}$ by

$$\prod_{n\in I}^{[r]} a_n := \exp\Big(-\frac{d}{ds}\zeta_{\boldsymbol{a}}(s)\Big|_{s=1-r}\Big),$$

where we further assume that $\zeta_{\boldsymbol{a}}(s)$ admits a meromorphic continuation to some region containing $s = 1-r$ and is holomorphic at the point. It is clear that the case $r=1$ reproduces the usual regularized product; $\prod_{n\in I}^{[1]} a_n = \prod_{n\in I} a_n$. Note that it can be written as $\mathbf{\Gamma}_r(z)^{-1} = \prod_{n\geq 0}^{[r]}(n+z)^1$.

To state our main result, let us recall Hecke L-functions. Let K be an algebraic number field of degree n and of discriminant d_K, $\mathcal{O}_K$ the ring of integers of K, and r_1 and r_2 the number of real and complex places of K, respectively. Let χ be a Hecke grössencharacter with conductor $\mathfrak{f}$ and

$$L_K(s;\chi) := \prod_{\mathfrak{p}}\Big(1-\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-1} = \sum_{\mathfrak{a}}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} \qquad (\mathrm{Re}\,(s)>1)$$

the Hecke L-function associate with χ. Here, $\mathfrak{p}$ runs over all prime ideals of $\mathcal{O}_K$ and $\mathfrak{a}$ over all integral ideals of $\mathcal{O}_K$ (we understand that $\chi(\mathfrak{p}) = 0$ if $\mathfrak{p}$ and $\mathfrak{f}$ are not coprime). It is well known that $L_K(s;\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$ with a possible simple pole at $s=1$ and has a functional equation $\Lambda_K(1-s;\overline{\chi}) = W_K(\chi)\Lambda_K(s;\chi)$ where $W_K(\chi)$ is a constant with $|W_K(\chi)| = 1$ and $\Lambda_K(s;\chi)$ is the entire function defined by

$$(1.4)\qquad \Lambda_K(s;\chi) := \Big(\frac{1}{2}s(s-1)\Big)^{\varepsilon_\chi}\Big(\frac{N(\mathfrak{f})|d_K|}{2^{2r_2}\pi^n}\Big)^{\frac{s}{2}} L_K(s;\chi)\prod_{v\in S_\infty(K)}\Gamma\Big(\frac{N_v(s+i\varphi_v)+|m_v|}{2}\Big).$$

Here, $S_\infty(K)$ is the set of all archimedean places of K, $\varepsilon_\chi = 1$ if χ is principal and 0 otherwise. Moreover, for $v\in S_\infty(K)$, $N_v = 1$ if v is real and 2 otherwise, and $\varphi_v = \varphi(\chi)\in\mathbb{R}$ with $\sum_{v\in S_\infty(K)} N_v\varphi_v = 0$ and $m_v = m(\chi)\in\mathbb{Z}$ are uniquely determined by

$$\chi\big((\alpha)\big) = \prod_{v\in S_\infty(K)} |\alpha_v|^{-iN_v\varphi_v}\Big(\frac{\alpha_v}{|\alpha_v|}\Big)^{m_v} \qquad (\alpha\in O_K \text{ with } \alpha\equiv 1 \bmod^\times \mathfrak{f}),$$

where $\bmod^\times$ indicates the multiplicative congruence and α_v is the image of α with respect to the embedding $K\hookrightarrow K_v$ with $K_v = \mathbb{R}$ or $\mathbb{C}$. We remark that, if $\varphi_v = m_v = 0$ for all $v\in S_\infty(K)$, then χ is called a class character.

Let $\mathcal{R}_K(\chi)$ be the set of all non-trivial zeros of $L_K(s;\chi)$ and $\xi_K(s,z;\chi)$ the zeta function attached to the sequence $\{\frac{z-\rho}{2\pi}\}_{\rho\in\mathcal{R}_K(\chi)}$, that is[2],

$$\xi_K(s,z;\chi) := \sum_{\rho\in\mathcal{R}_K(\chi)}\Big(\frac{z-\rho}{2\pi}\Big)^{-s} \qquad (\mathrm{Re}\,(s)>1,\ \mathrm{Re}\,(z)>1).$$

Moreover, let

$$\Xi_{K,r}(z;\chi) := \prod_{\rho\in\mathcal{R}_K(\chi)}^{[r]}\Big(\frac{z-\rho}{2\pi}\Big) = \exp\Big(-\frac{d}{ds}\xi_K(s,z;\chi)\Big|_{s=1-r}\Big).$$

Remark that, when $\mathrm{Re}\,(z)>1$, the function $\Xi_{K,r}(z;\chi)$ can be defined because it will be shown that $\xi_K(s,z;\chi)$ admits a meromorphic continuation to the whole plane $\mathbb{C}$ as a function of s and, in particular, is holomorphic at $s=1-r$ for any $r\in\mathbb{N}$ (Proposition 2.2). Now our main result is given as follows.

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[1]For $r\geq 2$, if $\prod_{n\in I}^{[r]}(a_n+z)$ exists, then it defines in general a multivalued function with branch points at $z=-a_n$ for $n\in I$. See [KWY] for more precise discussions. In particular, $\mathbf{\Gamma}_r(z)$ is a multivalued function with branch points at $z=-n$ for $n\geq 0$ or defines a holomorphic function in $\mathbb{C}\setminus(-\infty,0]$.

[2]From now on, the sum $\sum_{\rho\in\mathcal{R}_K(\chi)}$ means $\lim_{T\to\infty}\sum_{\rho\in\mathcal{R}_K(T;\chi)}$ where $\mathcal{R}_K(T;\chi) := \{\rho\in\mathcal{R}_K(\chi)\,|\,|\mathrm{Im}\,(\rho)|<T\}$.

Theorem 1.1. *For* $\mathrm{Re}(z) > 1$, *it holds that*

$$\Xi_{K,r}(z;\chi) = \Big(\frac{z}{2\pi}\Big)^{\varepsilon_\chi(\frac{z}{2\pi})^{r-1}} \Big(\frac{z-1}{2\pi}\Big)^{\varepsilon_\chi(\frac{z-1}{2\pi})^{r-1}} L_K^{(r)}(z;\chi)^{(-1)^{r-1}(r-1)!(2\pi)^{1-r}} \times \prod_{v\in S_\infty(K)} (N_v\pi)^{-\frac{(N_v\pi)^{1-r}}{r}B_r(\frac{N_v(z+i\varphi_v)+|m_v|}{2})} \boldsymbol{\Gamma}_r\Big(\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Big)^{(N_v\pi)^{1-r}}. \tag{1.5}$$

Here, $B_r(z)$ *is the* r*th Bernoulli polynomial,* $\boldsymbol{\Gamma}_r(z)$ *is the Milnor gamma function defined by* (1.3) *and* $L_K^{(r)}(z;\chi)$ *is a holomorphic function in* $\mathrm{Re}(z) > 1$ *defined by the following Euler product;*

$$L_K^{(r)}(s;\chi) := \prod_{\mathfrak{p}} H_r\Big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-(\log N(\mathfrak{p}))^{1-r}} \qquad (\mathrm{Re}(s) > 1), \tag{1.6}$$

where $H_r(z) := \exp(-Li_r(z))$ *with* $Li_r(z) := \sum_{m=1}^{\infty}\frac{z^m}{m^r}$ *being the polylogarithm of degree* r.

We call $L_K^{(r)}(s;\chi)$ a "poly-Hecke L-function" of degree r. Remark that this is a generalization of $L_K(s;\chi)$. Actually, since $Li_1(z) = -\log(1-z)$ and hence $H_1(z) = 1-z$, we have $L_K^{(1)}(s;\chi) = L_K(s;\chi)$. Some analytic properties of this new "L-" function are given in the last section.

As a corollary of this theorem, letting $r=1$ with noting that $B_1(z) = z - \frac{1}{2}$, $\boldsymbol{\Gamma}_1(z) = \frac{\Gamma(z)}{\sqrt{2\pi}}$ and $L_K^{(1)}(z;\chi) = L_K(z;\chi)$, we obtain the following regularized product expressions of Hecke L-functions.

Corollary 1.2. *It holds that*

$$\prod_{\rho\in\mathcal{R}_K(\chi)}\Big(\frac{z-\rho}{2\pi}\Big) = \frac{(N(\mathfrak{f})|d_K|)^{-\frac{z}{2}}}{2^{\varepsilon_\chi+\frac{1}{2}r_1+i\varphi_{\mathbb{C}}+\frac{1}{2}m_{\mathbb{C}}}\pi^{2\varepsilon_\chi+m}}\Lambda_K(z;\chi),$$

where $\varphi_{\mathbb{C}} := \sum_{v:\,complex}\varphi_v$, $m_{\mathbb{C}} := \sum_{v:\,complex}|m_v|$ *and* $m := \sum_{v\in S_\infty(K)}|m_v|$. *In particular, if* χ *is a class character, that is,* $\varphi_v = m_v = 0$ *for all* $v\in S_\infty(K)$, *then we have*

$$\prod_{\rho\in\mathcal{R}_K(\chi)}\Big(\frac{z-\rho}{2\pi}\Big) = \frac{(N(\mathfrak{f})|d_K|)^{-\frac{z}{2}}}{2^{\varepsilon_\chi+\frac{1}{2}r_1}\pi^{2\varepsilon_\chi}}\Lambda_K(z;\chi). \tag{1.7}$$

□

Furthermore, letting $\chi = \mathbf{1}$ (of course $\mathbf{1}$ is a class character) and writing $\zeta_K(s) := L_K(s;\mathbf{1})$, that is, $\zeta_K(s)$ is the Dedekind zeta function of K, $\mathcal{R}_K := \mathcal{R}_K(\mathbf{1})$ and $\Lambda_K(s) := \Lambda_K(s;\mathbf{1})$ in (1.7), respectively, one obtains the regularized product expression of the Dedekind zeta function.

Corollary 1.3. *It holds that*

$$\prod_{\rho\in\mathcal{R}_K}\Big(\frac{z-\rho}{2\pi}\Big) = \frac{|d_K|^{-\frac{z}{2}}}{2^{\frac{1}{2}r_1+1}\pi^2}\Lambda_K(z). \tag{1.8}$$

□

Now we immediately obtain the equation (1.2) from (1.8) by letting $K = \mathbb{Q}$.

This note is a survey of the paper [WY]. For the readers who are interested in this topic or want to know more precise proofs, please refer the paper above (see also [KWY, Y] where "higher depth determinants" of Laplacians on compact Riemannian manifolds are similarly studied).

2 Sketch of the proof of Theorem 1.1

In this section, we give a brief proof of Theorem 1.1. Remark that the proof is completely based on that of the equation (1.2) due to Deninger [D]. To do that, we first recall the Weil explicit formula refined by Barner [Ba]. For a function F of bounded variation (i.e., $V_{\mathbb{R}}(F) < +\infty$ where $V_{\mathbb{R}}(F)$ is the total variation of F on $\mathbb{R}$), we define the function $\Phi_F(s)$ ($s \in \mathbb{C}$) by

$$\Phi_F(s) := \int_{-\infty}^{\infty} F(x)e^{(s-\frac{1}{2})x}dx.$$

Moreover, for a Hecke character χ and $v \in S_\infty(K)$, we put $F_v(x;\chi) := F(x)e^{-i\varphi_v x}$. Then, the Weil explicit formula is given as follows.

Lemma 2.1 ([Ba, Theorem 1]). *Let χ be a Hecke character and $F : \mathbb{R} \to \mathbb{C}$ a function of bounded variation satisfying the following three conditions*[3]*:*

(a) *There is a positive constant b such that $V_{\mathbb{R}}(F(x)e^{(\frac{1}{2}+b)|x|}) < +\infty$.*

(b) *F is "normalized", that is, $2F(x) = F(x+0) + F(x-0)$ ($x \in \mathbb{R}$).*

(c) *For any $v \in S_\infty(K)$, it holds that $F_v(x;\chi) + F_v(-x;\chi) = 2F(0) + O(|x|)$ as $|x| \to 0$.*

Then, the following equation holds:

$$\begin{aligned}(2.1)\qquad \sum_{\rho\in\mathcal{R}_K(\chi)} \Phi_F(\rho) &= \varepsilon_\chi(\Phi_F(0) + \Phi_F(1)) + F(0)\log\frac{N(\mathfrak{f})|d_K|}{2^{2r_2}\pi^n} \\ &\quad - \sum_{\mathfrak{p}}\sum_{l=1}^{\infty} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{l}{2}}}\left(\chi(\mathfrak{p}^l)F(\log N(\mathfrak{p})^l) + \overline{\chi}(\mathfrak{p}^l)F(-\log N(\mathfrak{p})^l)\right) \\ &\quad + \sum_{v\in S_\infty(K)} W_v(F;\chi),\end{aligned}$$

where

$$W_v(F;\chi) := \int_0^\infty \left(\frac{N_vF(0)}{x} - \left(F_v(x;\chi) + F_v(-x;\chi)\right)\frac{e^{(\frac{2-|m_v|}{N_v}-\frac{1}{2})x}}{1-e^{-\frac{2x}{N_v}}}\right)e^{-\frac{2x}{N_v}}dx.$$

□

For $\mathrm{Re}(z) > 1$ and $\mathrm{Re}(s) > 1$, let

$$F(x) := \begin{cases} x^{s-1}e^{-(z-\frac{1}{2})x} & (x \geq 0), \\ 0 & (x < 0). \end{cases}$$

Then, one can easily check that the function $F(x)$ satisfies the conditions (a), (b) and (c) in Lemma 2.1 and see that $\Phi_F(w) = \frac{\Gamma(s)}{(z-w)^s}$, whence $\Phi_F(0) = \frac{\Gamma(s)}{z^s}$ and $\Phi_F(1) = \frac{\Gamma(s)}{(z-1)^s}$. Therefore, using the explicit formula (2.1) with this F (together with the integral representations of $\zeta(s,z)$ and the gamma function), we obtain the following expression of $\xi_K(s,z;\chi)$.

Proposition 2.2. *For $\mathrm{Re}(z) > 1$, we have*

$$\begin{aligned}(2.2)\qquad \xi_K(s,z;\chi) &= \varepsilon_\chi\left(\left(\frac{2\pi}{z}\right)^s + \left(\frac{2\pi}{z-1}\right)^s\right) + \frac{(2\pi)^s}{2\pi i}\int_{L_-}\frac{L_K'}{L_K}(z-t;\chi)t^{-s}dt \\ &\quad - \sum_{v\in S_\infty(K)} (N_v\pi)^s\zeta\left(s, \frac{N_v(z+i\varphi_v)+|m_v|}{2}\right),\end{aligned}$$

[3]These are called the "Barner conditions".

where L_- is the contour consisting of the lower edge of the cut from $-\infty$ to $-\delta$, the circle $t = \delta e^{i\psi}$ for $-\pi \leq \psi \leq \pi$ and the upper edge of the cut from $-\delta$ to $-\infty$. This gives a meromorphic continuation of $\xi_K(s, z; \chi)$ as a function of s to the whole plane $\mathbb{C}$ with a simple pole at $s = 1$. □

As stated below, the theorem is obtained by directly calculating the derivatives of $\xi_K(s, z; \chi)$ at $s = 1 - r$ from the expression (2.2).

Proof of Theorem 1.1. Write $\xi_K(s, z; \chi) = A_1(s, z) + A_2(s, z) + A_3(s, z)$ where

$$A_1(s, z) := \varepsilon_\chi\Big(\Big(\frac{2\pi}{z}\Big)^s + \Big(\frac{2\pi}{z-1}\Big)^s\Big),$$
$$A_2(s, z) := \frac{(2\pi)^s}{2\pi i}\int_{L_-}\frac{L_K'}{L_K}(z - t; \chi)t^{-s}dt,$$
$$A_3(s, z) := -\sum_{v \in S_\infty(K)}(N_v\pi)^s\zeta\Big(s, \frac{N_v(z + i\varphi_v) + |m_v|}{2}\Big).$$

At first, it is easy to see that

$$-\frac{d}{ds}A_1(s, z)\Big|_{s=1-r} = \varepsilon_\chi\Big(\frac{z}{2\pi}\Big)^{r-1}\log\frac{z}{2\pi} + \varepsilon_\chi\Big(\frac{z-1}{2\pi}\Big)^{r-1}\log\frac{z-1}{2\pi}.$$

The derivative of $A_2(s, z)$ at $s = 1 - r$ is calculated as

$$\begin{aligned}-\frac{d}{ds}A_2(s, z)\Big|_{s=1-r} &= \frac{(2\pi)^{1-r}}{2\pi i}\int_{L_-}\frac{L_K'}{L_K}(z - t; \chi)t^{r-1}\log\frac{t}{2\pi}dt\\ &= (-1)^r(2\pi)^{1-r}\int_0^\infty\frac{L_K'}{L_K}(z + x; \chi)x^{r-1}dx\\ &= (-1)^{r-1}(r-1)!(2\pi)^{1-r}\log L_K^{(r)}(z; \chi).\end{aligned}$$

In the second equality, we have calculated the integral by dividing the contour L_- into three parts; $L_- = (-\infty e^{-\pi i}, -\delta e^{-\pi i}) \sqcup \{\delta e^{i\psi} \mid -\pi \leq \psi \leq \pi\} \sqcup (-\infty e^{\pi i}, -\delta e^{\pi i})$ (and letting $\delta \to 0$) and, in the last equality, we have used the formula

$$\frac{L_K'}{L_K}(z; \chi) = -\sum_{\mathfrak{p}}\sum_{l=1}^{\infty}\log N(\mathfrak{p})\cdot\chi(\mathfrak{p})^l\cdot N(\mathfrak{p})^{-lz} \qquad (\mathrm{Re}(z) > 1)$$

and the Euler product expression (1.6) of the poly-Hecke L-function $L_K^{(r)}(z; \chi)$. Finally, using the well-known formula $\zeta(1 - r, z) = -\frac{B_r(z)}{r}$, we have

$$\begin{aligned}&-\frac{d}{ds}A_3(s, z)\Big|_{s=1-r}\\ &= -\sum_{v \in S_\infty(K)}(N_v\pi)^{1-r}\Big[\frac{\log(N_v\pi)}{r}B_r\Big(\frac{N_v(z + i\varphi_v) + |m_v|}{2}\Big) - \log\Gamma_r\Big(\frac{N_v(z + i\varphi_v) + |m_v|}{2}\Big)\Big].\end{aligned}$$

Combining these three equations, one obtains the desired result. □

3 Poly-Hecke L-functions

The poly-Hecke L-functions, which are naturally appeared in the derivatives of the zeta function $\xi_K(s, z; \chi)$ at non-positive integer points, are mysterious functions at this moment. They are defined by the Euler product (1.6) and, as we have seen before, give generalizations of Hecke L-functions. Therefore one may expect that they satisfy similar properties which so-called L- or zeta functions have, for example, a meromorphic contiuation, a functional equation and a "Riemann hypothesis". In this section, as a closing remark, we give an analytic continuation of $L_K^{(r)}(s;\chi)$ for $r \geq 2$ to (not the whole plane $\mathbb{C}$ but) an infititely many slitted region in $\mathbb{C}$.

Let $\Omega_K(\chi)$ be the set of all complex numbers which are not of the form of $\rho - \lambda$ where ρ is a trivial or a non-trivial zero of $L_K(s;\chi)$ or, if χ is principal, $1-\lambda$ for $\lambda \geq 0$ (we show the region $\Omega_K(\chi)$ in Figure 1 in the case where χ is a principal character). Notice that, from the expression (1.4), all trivial zeros of $L_K(s;\chi)$ are given by $-\frac{|m_v|+2l}{N_v} - i\varphi_v$ where $v \in S_\infty(K)$ and $l \in \mathbb{Z}_{\geq 0}$.

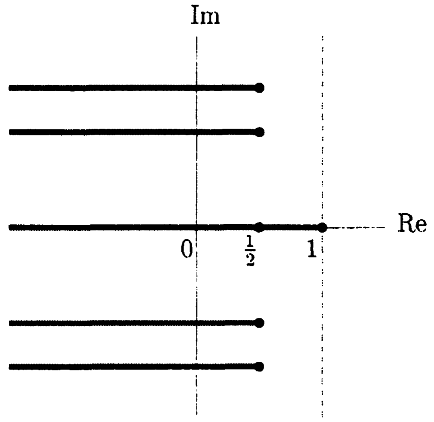

Figure 1: The region $\Omega_K(\chi)$ (if χ is principal)

Now let $r \geq 2$. From the differential equation $\frac{d}{dz}Li_r(z) = \frac{1}{z}Li_{r-1}(z)$ of the polylogarithm, one can see that the poly-Hecke L-function $L_K^{(r)}(s;\chi)$ satisfies the differential equation

$$\frac{d^{r-1}}{ds^{r-1}} \log L_K^{(r)}(s;\chi) = (-1)^{r-1} \log L_K(s;\chi) \qquad (\mathrm{Re}\,(s) > 1).$$

Using this formula, by induction on r, we obtain the following result.

Theorem 3.1. *Let* $\mathrm{Re}\,(a) > 1$. *Then, we have*

$$L_K^{(r)}(s;\chi) = Q_K^{(r)}(s,a) \exp\Bigg(\underbrace{\int_a^s \int_a^{\xi_{r-1}} \cdots \int_a^{\xi_2}}_{r-1} \log L_K(\xi_1;\chi) d\xi_1 \cdots d\xi_{r-1}\Bigg)^{(-1)^{r-1}}.$$

Here $Q_K^{(r)}(s,a) := \prod_{k=0}^{r-2} L_K^{(r-k)}(a;\chi)^{\frac{(-1)^k}{k!}(s-a)^k}$ *and the path for each integral is contained in* $\Omega_K(\chi)$. *The expression gives an analytic continuation of* $L_K^{(r)}(s;\chi)$ *to the region* $\Omega_K(\chi)$. □

It seems to be difficult to continue $L_K^{(r)}(s;\chi)$ to the whole plane $\mathbb{C}$ as a *single-valued* holomorphic (or meromorphic) function. In fact, from an easy observation, one can prove the following

Corollary 3.2. *The extended Riemann hypothesis for* $L_K(s;\chi)$ *is equivalent to say that the function* $(s-1)^{-\varepsilon_\chi(s-1)} L_K^{(2)}(s;\chi)$ *is single-valued and holomorphic in* $\mathrm{Re}\,(s) > \frac{1}{2}$. □

Remark 3.3. Let

$$\widetilde{L}_K^{(r)}(s;\chi) := \prod_{\mathfrak{p}} H_r\Big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-1} \quad (\mathrm{Re}\,(s) > 1)$$

(recall that $L_K^{(r)}(s;\chi) := \prod_{\mathfrak{p}} H_r(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s})^{-(\log N(\mathfrak{p}))^{1-r}}$). Then we have $\widetilde{L}_K^{(1)}(s;\chi) = L_K(s;\chi)$, whence $\widetilde{L}_K^{(r)}(s;\chi)$ also gives a generalization of $L_K(s;\chi)$. It does not, however, seem to have an analytic continuation to the whole plane $\mathbb{C}$. In fact, in [KW], it was shown that $\widetilde{\zeta}^{(r)}(s) := \widetilde{L}_{\mathbb{Q}}^{(r)}(s;\mathbf{1})$ has an analytic continuation to the region $\mathrm{Re}\,(s) > 0$ but has a natural boundary at $\mathrm{Re}\,(s) = 0$.

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