数理解析研究所講究録
第 1714 巻 2010 年 63-79

THE GROTHENDIECK-TEICHMÜLLER GROUP, THE DOUBLE SHUFFLE GROUP AND THE MOTIVIC GALOIS GROUP

HIDEKAZU FURUSHO

CONTENTS

1. The Grothendieck-Teichmüller group

In his celebrated papers on quantum groups [Dr86, Dr90, Dr91] Drinfel'd came to the notion of *quasitriangular quasi-Hopf quantized universal enveloping algebras.* It is a topological algebra which differs from a topological Hopf algebra in the sense that the coassociativity axiom and the cocommutativity axiom is twisted by an associator and an R-matrix satisfying a pentagon axiom and two hexagon axioms. One of the main theorems in [Dr91] is that any quasitriangular quasi-Hopf quantized universal enveloping algebra modulo twists (in other words gauge transformations [Ka]) is obtained as a quantization of a pair (called its classical limit) of a Lie algebra and its symmetric invariant 2-tensor. Quantizations are constructed by 'universal' associators. The associator set M (definition 1.2) is defined to be the set of group-like universal associator.

Let us fix notations and conventions:

Convention 1.1. Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure and $U\mathfrak{F}_2 = k\langle\langle X_0, X_1\rangle\rangle$ a non-commutative formal power series ring with two variables X_0 and X_1. Its element $\varphi = \varphi(X_0, X_1)$ is called *group-like* if it satisfies $\Delta(\varphi) = \varphi \otimes \varphi$ with $\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0$

Date: June 13, 2010.

This article is for the RIMS-kokyuroku of 'Conference on Quantum Groups and Quantum Topology' held on April 19 and 20, 2010 in RIMS, Kyoto.

and $\Delta(X_1)=X_1\otimes 1+1\otimes X_1$ and its constant term is equal to 1. For a monic monomial W, $c_W(\varphi)$ means the coefficient of W in φ. For any k-algebra homomorphism $\iota: U\mathfrak{F}_2\to S$ the image $\iota(\varphi)\in S$ is denoted by $\varphi(\iota(X_0),\iota(X_1))$.

Definition 1.2 ([Dr91]). The *associator set* $\underline{M}$ (resp. M) is the pro-algebraic variety whose set of k-valued points consists of pairs (μ,φ) with $\mu\in k$ (resp. $\mu\in k^\times$) and group-like series $\varphi\in U\mathfrak{F}_2$ satisfying *Drinfel'd's two hexagon equations* in $U\mathfrak{F}_2$:

(1.1)

$$\exp\{\frac{\mu(t_{13}+t_{23})}{2}\}=\varphi(t_{13},t_{12})\exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{13},t_{23})^{-1}\exp\{\frac{\mu t_{23}}{2}\}\varphi(t_{12},t_{23}),$$

(1.2)

$$\exp\{\frac{\mu(t_{12}+t_{13})}{2}\}=\varphi(t_{23},t_{13})^{-1}\exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{12},t_{13})\exp\{\frac{\mu t_{12}}{2}\}\varphi(t_{12},t_{23})^{-1}$$

and *his pentagon equation* in $U\mathfrak{a}_4$:

(1.3)

$$\varphi(t_{12},t_{23}+t_{24})\varphi(t_{13}+t_{23},t_{34})=\varphi(t_{23},t_{34})\varphi(t_{12}+t_{13},t_{24}+t_{34})\varphi(t_{12},t_{23}).$$

Here $U\mathfrak{a}_4$ means the universal enveloping algebra of the *completed pure braid Lie algebra* $\mathfrak{a}_4$ *over* k *with 4 strings*, generated by t_{ij} $(1\leqslant i,j\leqslant 4)$ with defining relations $t_{ii}=0$, $t_{ij}=t_{ji}$, $[t_{ij},t_{ik}+t_{jk}]=0$ (i,j,k: all distinct) and $[t_{ij},t_{kl}]=0$ (i,j,k,l: all distinct).

Remark 1.3. It is proved in [Dr91] (reproved in [Ba]) that $M(\mathbf{Q})$ is non-empty.

The category of representations of a quasitriangular quasi-Hopf quantized universal enveloping algebra [Dr91] forms a quasitensored category, in other words, a *braided tensor category* [JS]; its associativity constraint and its commutativity constraint are subject to one pentagon axiom and two hexagon axioms. The Grothendieck-Teichmüller group GRT_1 is introduced in [Dr91] as a group of deformations of such category which change its associativity constraint keeping all three axioms.

Definition 1.4 ([Dr91]). The (unipotent part of the graded) *Grothendieck-Teichmüller* (pro-algebraic) *group* GRT_1 is defined by $\underline{M}\backslash M$, that is, the pro-algebraic variety whose set of k-valued points consists of group-like series $\varphi\in U\mathfrak{F}_2$ satisfying two hexagon equations (1.1), (1.2) and the pentagon equations (1.3) with $\mu=0$.

In [Dr91] it is shown that GRT_1 is closed by the multiplication [1]

$$(1.4)\qquad \varphi_2 \circ \varphi_1 := \varphi_1(\varphi_2 X_0 \varphi_2^{-1}, X_1) \cdot \varphi_2 = \varphi_2 \cdot \varphi_1(X_0, \varphi_2^{-1} X_1 \varphi_2)$$

for φ_1, $\varphi_2 \in GRT_1(k)$.

Remark 1.5. Let $\underline{F_2}$ be the free pro-unipotent algebraic group with two generators e^{X_0} and e^{X_1} and $\underline{AutF_2}$ be the pro-algebraic group which represents $k \mapsto Aut\underline{F_2}(k)$. By the map sending $X_0 \mapsto X_0$ and $X_1 \mapsto \varphi X_1 \varphi^{-1}$, the group $\underline{GRT_1}$ is regarded as a subgroup of $\underline{AutF_2}$.

Remark 1.6. We note that the group is also closely related to the philosophy of *un jeu de "Lego-Teichmüller"* posed by Grothendieck in *Esquisse d'un programme* [Gr]. And that is why Drinfel'd named it the Grothendieck-Teichmüller group.

Our theorem here is on the defining equations of the associator set M (and hence of the Grothendieck-Teichmüller group GRT_1.)

Theorem 1.7 ([F10]). *Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies Drinfel'd's pentagon equation* (1.3). *Then there exists an element (unique up to signature) $\mu \in \bar{k}$ such that the pair (μ, φ) satisfies his two hexagon equations* (1.1) *and* (1.2). *Actually this μ is equal to $\pm(24 c_{X_0X_1}(\varphi))^{\frac{1}{2}}$.*

It should be noted that we need to use an (actually quadratic) extension of a field k in order to reduce the *GT-relations* (1.1)~(1.3), into one pentagon equation (1.3). Particularly the theorem claims that the pentagon equation is essentially a single defining equation of the Grothendieck-Teichmüller group.

Proof of theorem 1.7. The proof of theorem 1.7 is reduced to the following by standard arguments of Lie algebra.

Proposition 1.8 ([F10]). *Let $\mathfrak{F}_2$ be the set of Lie-like elements φ in $U\mathfrak{F}_2$ (i.e. $\Delta(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi$). Let φ be an element of $\mathfrak{F}_2$ which is commutator Lie-like* [2] *with $c_{X_0X_1}(\varphi) = 0$. Suppose that φ satisfies 5-cycle relation:*

$$\varphi(X_{12}, X_{23}) + \varphi(X_{34}, X_{45}) + \varphi(X_{51}, X_{12}) + \varphi(X_{23}, X_{34}) + \varphi(X_{45}, X_{51}) = 0$$

in $\hat{\mathfrak{P}}_5$. Then it also satisfies 3- and 2-cycle relation:

$$\varphi(X, Y) + \varphi(Y, Z) + \varphi(Z, X) = 0 \textit{ with } X + Y + Z = 0,$$

$$\varphi(X, Y) + \varphi(Y, X) = 0.$$

[1] For our convenience, we change the order of multiplication in the original definition of [Dr91].

[2] We call a series $\varphi = \varphi(X_0, X_1)$ commutator Lie-like if it is Lie-like and $c_{X_0} = c_{X_1} = 0$, in other words $\varphi \in \mathfrak{F}_2' := [\mathfrak{F}_2, \mathfrak{F}_2]$.

Here $\mathfrak{P}_5$ stands for the completion (with respect to the natural grading) of the pure sphere braid Lie algebra with 5 strings; the Lie algebra generated by X_{ij} $(1 \leqslant i, j \leqslant 5)$ with clear relations $X_{ii} = 0$, $X_{ij} = X_{ji}$, $\sum_{j=1}^{5} X_{ij} = 0$ $(1 \leqslant i, j \leqslant 5)$ and $[X_{ij}, X_{kl}] = 0$ if $\{i, j\} \cap \{k, l\} = \emptyset$.

Proof . There is a projection from $\mathfrak{P}_5$ to the completed free Lie algebra $\mathfrak{F}_2$ generated by X and Y by putting $X_{i5} = 0$, $X_{12} = X$ and $X_{23} = Y$. The image of 5-cycle relation gives 2-cycle relation.

For our convenience we denote $\varphi(X_{ij}, X_{jk})$ $(1 \leqslant i, j, k \leqslant 5)$ by φ_{ijk}. Then the 5-cycle relation can be read as

$$\varphi_{123} + \varphi_{345} + \varphi_{512} + \varphi_{234} + \varphi_{451} = 0.$$

We denote LHS by P. Put σ_i $(1 \leqslant i \leqslant 12)$ be elements of $\mathfrak{S}_5$ defined as follows: $\sigma_1(12345) = (12345)$, $\sigma_2(12345) = (54231)$, $\sigma_3(12345) = (13425)$, $\sigma_4(12345) = (43125)$, $\sigma_5(12345) = (53421)$, $\sigma_6(12345) = (23514)$, $\sigma_7(12345) = (23415)$, $\sigma_8(12345) = (35214)$, $\sigma_9(12345) = (53124)$, $\sigma_{10}(12345) = (24135)$, $\sigma_{11}(12345) = (52314)$ and $\sigma_{12}(12345) = (23541)$. Then

$$\begin{aligned}
\sum_{i=1}^{12} \sigma_i(P) = \ & \varphi_{123} + \varphi_{345} + \varphi_{512} + \varphi_{234} + \varphi_{451} \\
& + \varphi_{542} + \varphi_{231} + \varphi_{154} + \varphi_{423} + \varphi_{315} \\
& + \varphi_{134} + \varphi_{425} + \varphi_{513} + \varphi_{342} + \varphi_{251} \\
& + \varphi_{431} + \varphi_{125} + \varphi_{543} + \varphi_{312} + \varphi_{254} \\
& + \varphi_{534} + \varphi_{421} + \varphi_{153} + \varphi_{342} + \varphi_{215} \\
& + \varphi_{235} + \varphi_{514} + \varphi_{423} + \varphi_{351} + \varphi_{142} \\
& + \varphi_{234} + \varphi_{415} + \varphi_{523} + \varphi_{341} + \varphi_{152} \\
& + \varphi_{352} + \varphi_{214} + \varphi_{435} + \varphi_{521} + \varphi_{143} \\
& + \varphi_{531} + \varphi_{124} + \varphi_{453} + \varphi_{312} + \varphi_{245} \\
& + \varphi_{241} + \varphi_{135} + \varphi_{524} + \varphi_{413} + \varphi_{352} \\
& + \varphi_{523} + \varphi_{314} + \varphi_{452} + \varphi_{231} + \varphi_{145} \\
& + \varphi_{235} + \varphi_{541} + \varphi_{123} + \varphi_{354} + \varphi_{412}.
\end{aligned}$$

By the 2-cycle relation, $\varphi_{ijk} = -\varphi_{kji}$ $(1 \leqslant i, j, k \leqslant 5)$. This gives

$$\begin{aligned}\sum_{i=1}^{12}\sigma_i(P) =&\varphi_{123}+\varphi_{234}\\&+\varphi_{231}+\varphi_{423}\\&+\varphi_{342}+\varphi_{312}+\varphi_{342}\\&+\varphi_{235}+\varphi_{423}\\&+\varphi_{234}+\varphi_{523}\\&+\varphi_{352}+\varphi_{312}+\varphi_{352}\\&+\varphi_{523}+\varphi_{231}\\&+\varphi_{235}+\varphi_{123}\\=&2(\varphi_{123}+\varphi_{231}+\varphi_{312})+2(\varphi_{234}+\varphi_{342}+\varphi_{423})\\&+2(\varphi_{235}+\varphi_{352}+\varphi_{523})\\=&2\Big\{\varphi(X_{12},X_{23})+\varphi(X_{23},X_{31})+\varphi(X_{31},X_{12})\Big\}\\&+2\Big\{\varphi(X_{23},X_{34})+\varphi(X_{34},X_{42})+\varphi(X_{42},X_{23})\Big\}\\&+2\Big\{\varphi(X_{23},X_{35})+\varphi(X_{35},X_{52})+\varphi(X_{52},X_{23})\Big\}.\end{aligned}$$

By $[X_{12},X_{12}+X_{31}+X_{32}]=[X_{23},X_{12}+X_{31}+X_{32}]=0$ and $\varphi\in\mathfrak{F}'_2$, $\varphi(X_{12},X_{23})=\varphi(-X_{31}-X_{32},X_{23})=\varphi(X_{34}+X_{35},X_{23})$. By $[X_{31},X_{12}+X_{31}+X_{32}]=[X_{12},X_{12}+X_{31}+X_{32}]=0$ and $\varphi\in\mathfrak{F}'_2$, $\varphi(X_{31},X_{12})=\varphi(X_{31},-X_{31}-X_{32})=\varphi(-X_{23}-X_{34}-X_{35},X_{34}+X_{35})$. By $[X_{34},X_{42}+X_{23}+X_{34}]=[X_{42},X_{42}+X_{23}+X_{34}]=0$ and $\varphi\in\mathfrak{F}'_2$, $\varphi(X_{34},X_{42})=\varphi(X_{34},-X_{23}-X_{34})$. By $[X_{23},X_{42}+X_{23}+X_{34}]=[X_{42},X_{42}+X_{23}+X_{34}]=0$ and $\varphi\in\mathfrak{F}'_2$, $\varphi(X_{42},X_{23})=\varphi(-X_{23}-X_{34},X_{23})$. By $[X_{35},X_{52}+X_{23}+X_{35}]=[X_{52},X_{52}+X_{23}+X_{35}]=0$ and $\varphi\in\mathfrak{F}'_2$, $\varphi(X_{35},X_{52})=\varphi(X_{35},-X_{23}-X_{35})$. By $[X_{23},X_{52}+X_{23}+X_{35}]=[X_{52},X_{52}+X_{23}+X_{35}]=0$ and $\varphi\in\mathfrak{F}'_2$, $\varphi(X_{52},X_{23})=\varphi(-X_{23}-X_{35},X_{23})$.

So it follows

$$\begin{aligned}\sum_{i=1}^{12}\sigma_i(P) =&2\Big\{\varphi(X_{34}+X_{35},X_{23})+\varphi(X_{23},-X_{23}-X_{34}-X_{35})\\&+\varphi(-X_{23}-X_{34}-X_{35},X_{34}+X_{35})\Big\}\\&+2\Big\{\varphi(X_{23},X_{34})+\varphi(X_{34},-X_{23}-X_{34})+\varphi(-X_{23}-X_{34},X_{23})\Big\}\\&+2\Big\{\varphi(X_{23},X_{35})+\varphi(X_{35},-X_{23}-X_{35})+\varphi(-X_{23}-X_{35},X_{23})\Big\}.\end{aligned}$$

The elements X_{23}, X_{34} and X_{35} generates completed Lie subalgebra $\mathfrak{F}_3$ of $\mathfrak{P}_5$ which is free of rank 3 and it contains $\sum_{i=1}^{12}\sigma_i(P)$. Let $q:\mathfrak{F}_3\to\mathfrak{F}_2$ be the projection sending $X_{23}\mapsto X$, $X_{34}\mapsto Y$ and $X_{35}\mapsto Y$. Then

$$\begin{aligned}q(\sum_{i=1}^{12}\sigma_i(P)) =&2\Big\{\varphi(2Y,X)+\varphi(X,-X-2Y)+\varphi(-X-2Y,2Y)\Big\}\\&+4\Big\{\varphi(X,Y)+\varphi(Y,-X-Y)+\varphi(-X-Y,X)\Big\}.\end{aligned}$$

By the 2-cycle relation,

$$\begin{aligned}q(\sum_{i=1}^{12}\sigma_i(P)) =&4\Big\{\varphi(X,Y)+\varphi(Y,-X-Y)+\varphi(-X-Y,X)\Big\}\\&-2\Big\{\varphi(X,2Y)+\varphi(2Y,-X-2Y)+\varphi(-X-2Y,X)\Big\}.\end{aligned}$$

Put $R(X,Y)=\varphi(X,Y)+\varphi(Y,-X-Y)+\varphi(-X-Y,X)$. Then $q(\sum_{i=1}^{12}\sigma_i(P))=4R(X,Y)-2R(X,2Y)$. Since $P=0$, it follows $2R(X,Y)=R(X,2Y)$. Expanding this equation in terms of the Hall basis, we see that $R(X,Y)$ must be of the form $\sum_{m=1}^{\infty}a_m(adX)^{m-1}(Y)$ with $a_m\in k$. By the 2-cycle relation, $R(X,Y)=-R(Y,X)$. So $a_1=a_3=a_4=a_5=\cdots=0$. By our assumption $c_{X_0X_1}(\varphi)=0$, a_2 must be 0 either. Therefore $R(X,Y)=0$, which is the 3-cycle relation. This yields the validity of theorem 1.7. □

2. The double shuffle group

This section shows that the pentagon equation (1.3) implies the generalized double shuffle relation (2.3). As a corollary, we obtain an embedding from Drinfel'd's Grothendieck-Teichmüller group GRT_1 to Racinet's double shuffle group DMR_0 ([R]). This realizes the project of Deligne-Terasoma [DT] where a different approach was indicated. Their arguments concerned multiplicative convolutions whereas our methods are based on a bar construction calculus. We also prove that the gamma factorization formula follows from the generalized double shuffle relation. It extends the result in [DT, I] where they show that the GT-relations imply the gamma factorization.

Definition 2.1. *Multiple zeta values* (MZV's for short) $\zeta(k_1,\cdots,k_m)$ are the real numbers defined by the following series

$$\zeta(k_1,\cdots,k_m):=\sum_{0<n_1<\cdots<n_m}\frac{1}{n_1^{k_1}\cdots n_m^{k_m}} \tag{2.1}$$

for m, $k_1,\ldots,k_m\in\mathbf{N}(=\mathbf{Z}_{>0})$.

They converge if and only if the index $(k_1, \cdots, k_m)$ is admissible (i.e. $k_m > 1$). They were studied (allegedly) firstly by Euler [E] for $m = 1, 2$. Several types of relations among MZV's have been discussed. Here we focus on two types of relations, GT-relations and generalized double shuffle relations. Both of them are described in terms of the Drinfel'd associator.

Definition 2.2. The *Drinfel'd associator* $\Phi_{KZ}(X_0, X_1) \in \mathbf{C}\langle\langle X_0, X_1\rangle\rangle$ is the two variables non-commutative formal power series with complex number coefficients introduced in [Dr91] which has the following expression

$$\Phi_{KZ}(X_0, X_1) = 1 + \sum (-1)^m \zeta(k_1, \cdots, k_m) X_0^{k_m - 1} X_1 \cdots X_0^{k_1 - 1} X_1 \\ + (\text{regularized terms}).$$

Its coefficients including regularized terms are explicitly calculated to be linear combinations of MZV's in [F03] proposition 3.2.3 by Le-Murakami's method [LM].

Remark 2.3. The Drinfel'd associator was introduced as the connection matrix of the Knizhnik-Zamolodchikov equation and it was shown in [Dr91] that it is group-like and satisfies the GT-relations (1.1)∼(1.3) with $\mu = \pm 2\pi\sqrt{-1}$, namely $(\pm 2\pi\sqrt{-1}, \Phi_{KZ}) \in M(\mathbf{C})$, by using symmetry of the KZ-system on configuration spaces.

The *generalized double shuffle relation* is a kind of combinatorial relation among MZV's. It consists of the double shuffle relation and the regularization relation. The former is the combination of series shuffle relations and integral shuffle relations. Both of them are product formulae between MZV's. It arises from two ways of expressing MZV's as iterated integrals and as power series. The simplest example of the series shuffle relation is

$$\zeta(n_1)\zeta(n_2) = \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2).$$

It is easily obtained from the expression (2.1) and can be generalized in a similar way to other MZV's. The simplest example of the integral shuffle relation is

$$\zeta(n_1)\zeta(n_2) = \sum_{i=0}^{n_1-1} \binom{n_2 - 1 + i}{i} \zeta(n_1 - i, n_2 + i) + \sum_{j=0}^{n_2-1} \binom{n_1 - j + 1}{j} \zeta(n_2 - j, n_1 + j).$$

This follows from the iterated integral expression for MZV's (note: this also follows from the fact that the Drinfel'd associator Φ_{KZ} is group-like). Using these formulae we can get many relations among the MZV's however the double shuffle relations are not enough to capture all the relations between MZV's. There are two regularization of the MZV's for non-admissible indices: the series regularization, which extends the validity of the series shuffle relation and the integral regularization which extends the validity of the integral shuffle relation. The regularization relation is an algebraic relation among MZV's connecting the above two regularization. The generalized double shuffle relation is expected to capture all relation among MZV's. There are several formulations of the relation (see [IKZ, R]). In [R] it was formulated as (2.3) (see below) for $\varphi = \Phi_{KZ}$.

Let us fix notations and conventions:

Convention 2.4. Let $\pi_Y : k\langle\langle X_0, X_1\rangle\rangle \to k\langle\langle Y_1, Y_2, \ldots\rangle\rangle$ be the k-linear map between non-commutative formal power series rings that sends all the words ending in X_0 to zero and the word $X_0^{n_m-1}X_1 \cdots X_0^{n_1-1}X_1$ $(n_1, \ldots, n_m \in \mathbf{N})$ to $(-1)^m Y_{n_m} \cdots Y_{n_1}$. Define the coproduct Δ_* on $k\langle\langle Y_1, Y_2, \ldots\rangle\rangle$ by $\Delta_* Y_n = \sum_{i=0}^{n} Y_i \otimes Y_{n-i}$ with $Y_0 := 1$. For $\varphi = \sum_{W:\text{word}} c_W(\varphi)W \in k\langle\langle X_0, X_1\rangle\rangle$, define the *series shuffle regularization* $\varphi_* = \varphi_{\text{corr}} \cdot \pi_Y(\varphi)$ with the correction term

$$\varphi_{\text{corr}} = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{X_0^{n-1}X_1}(\varphi) Y_1^n\right). \tag{2.2}$$

For a group-like series $\varphi \in U\mathfrak{F}_2$ the *generalised double shuffle relation* means the equality

$$\Delta_*(\varphi_*) = \varphi_* \widehat{\otimes} \varphi_*. \tag{2.3}$$

Definition 2.5 ([R]). The *double shuffle set* $\underline{DMR}$ (resp. DMR) is the pro-algebraic variety whose set of k-valued points consists of pairs (μ, φ) with $\mu \in k$ (resp. $\mu \in k^\times$) and group-like series φ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = 0$ and $c_{X_0X_1}(\varphi) = \frac{\mu^2}{24}$ which satisfy the generalized double shuffle relation (2.3). [3]

Remark 2.6. We have $(\pm 2\pi\sqrt{-1}, \Phi_{KZ}) \in DMR(\mathbf{C})$ because MZV's satisfy generalized double shuffle relations. Like the case of $M(\mathbf{Q})$ it is proved in [R] that $DMR(\mathbf{Q})$ is non-empty.

Definition 2.7 ([R]). The *double shuffle group* DMR_0 is defined by $\underline{DMR}\backslash DMR$, that is, the pro-algebraic variety whose set of k-valued points consists of group-like series φ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$ which satisfy the generalized double shuffle relation (2.3).

In [R] it is proved that DMR_0 is closed by the multiplication (1.4) as GRT_1.

---

[3] For our convenience, we change some signatures in the original definition ([R] definition 3.2.1.)

Remark 2.8. By the same way to the GRT_1-case, the group DMR_0 is regarded as a subgroup of $\underline{Aut F_2}$.

Theorem 2.9 ([F08]). *Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies Drinfel'd's pentagon equation* (1.3). *Then it also satisfies the generalized double shuffle relation* (2.3).

The following is a direct corollary of our theorem 2.9 since the equations (1.1) and (1.2) for (μ, φ) imply $c_{X_0X_1}(\varphi) = \frac{\mu^2}{24}$.

Corollary 2.10 ([F08]). $\underline{M} \subset \underline{DMR}$. *Particularly $GRT_1 \subset DMR_0$.*

Proof of Theorem 2.9. By [F10] lemma 5, theorem 2.9 is reduced to the following.

Proposition 2.11 ([F08]). *Let φ be a group-like element of $U\mathfrak{F}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = 0$. Suppose that φ satisfies the 5-cycle relation*

$$\varphi(X_{34}, X_{45})\varphi(X_{51}, X_{12})\varphi(X_{23}, X_{34})\varphi(X_{45}, X_{51})\varphi(X_{12}, X_{23}) = 1$$

in the completed universal enveloping algebra $U\mathfrak{P}_5$ of $\mathfrak{P}_5$. Then it also satisfies the generalized double shuffle relation, i.e. $\Delta_*(\varphi_*) = \varphi_* \widehat{\otimes} \varphi_*$.

Proof . Let $\mathcal{M}_{0,4}$ be the moduli space $\{(x_1, \cdots, x_4) \in (\mathbf{P}^1_k)^4 | x_i \neq x_j (i \neq j)\}/PGL_2(k)$ of 4 different points in $\mathbf{P}^1$. It is identified with $\{z \in \mathbf{P}^1 | z \neq 0, 1, \infty\}$ by sending $[(0, z, 1, \infty)]$ to z. Let $\mathcal{M}_{0,5}$ be the moduli space $\{(x_1, \cdots, x_5) \in (\mathbf{P}^1_k)^5 | x_i \neq x_j (i \neq j)\}/PGL_2(k)$ of 5 different points in $\mathbf{P}^1$. It is identified with $\{(x, y) \in \mathbf{G}^2_m | x \neq 1, y \neq 1, xy \neq 1\}$ by sending $[(0, xy, y, 1, \infty)]$ to (x, y).

For $\mathcal{M} = \mathcal{M}_{0,4}/k$ or $\mathcal{M}_{0,5}/k$, we consider the Brown's variant $V(\mathcal{M})$ [Br] of the Chen's reduced bar construction [C]. This is a graded Hopf algebra $V(\mathcal{M}) = \oplus_{m=0}^{\infty} V_m$ $(\subset TV_1 = \oplus_{m=0}^{\infty} V_1^{\otimes m})$ over k. Here $V_0 = k$, $V_1 = H^1_{DR}(\mathcal{M})$ and V_m is the totality of linear combinations (finite sums) $\sum_{I=(i_m,\cdots,i_1)} c_I [\omega_{i_m} | \cdots | \omega_{i_1}] \in V_1^{\otimes m}$ $(c_I \in k,\ \omega_{i_j} \in V_1,$ $[\omega_{i_m} | \cdots | \omega_{i_1}] := \omega_{i_m} \otimes \cdots \otimes \omega_{i_1})$ satisfying the integrability condition

$$\sum_{I=(i_m,\cdots,i_1)} c_I [\omega_{i_m} | \omega_{i_{m-1}} | \cdots | \omega_{i_{j+1}} \wedge \omega_{i_j} | \cdots | \omega_{i_1}] = 0$$

in $V_1^{\otimes m-j-1} \otimes H^2_{DR}(\mathcal{M}) \otimes V_1^{\otimes j-1}$ for all j $(1 \leqslant j < m)$.

For the moment assume that k is a subfield of $\mathbf{C}$. We have an embedding (called a realisation in [Br]§1.2, §3.6) $\rho : V(\mathcal{M}) \hookrightarrow I_o(\mathcal{M})$ as algebra over k which sends $\sum_{I=(i_m,\cdots,i_1)} c_I [\omega_{i_m} | \cdots | \omega_{i_1}]$ $(c_I \in k)$ to $\sum_I c_I \mathrm{It} \int_o \omega_{i_m} \circ \cdots \circ \omega_{i_1}$. Here $\sum_I c_I \mathrm{It} \int_o \omega_{i_m} \circ \cdots \circ \omega_{i_1}$ means the iterated integral defined by

$$\sum_I c_I \int_{0<t_1<\cdots<t_{m-1}<t_m<1} \omega_{i_m}(\gamma(t_m)) \cdot \omega_{i_{m-1}}(\gamma(t_{m-1})) \cdot \cdots \cdot \omega_{i_1}(\gamma(t_1))$$

for all analytic paths $\gamma : (0,1) \to \mathcal{M}(\mathbf{C})$ starting from the tangential basepoint o (defined by $\frac{d}{dz}$ for $\mathcal{M} = \mathcal{M}_{0,4}$ and defined by $\frac{d}{dx}$ and $\frac{d}{dy}$ for $\mathcal{M} = \mathcal{M}_{0,5}$) at the origin in $\mathcal{M}$ (for its treatment see also [De89]§15) and $I_o(\mathcal{M})$ denotes the $\mathcal{O}^{\mathrm{an}}_{\mathcal{M}}$-module generated by all such homotopy invariant iterated integrals with $m \geqslant 1$ and holomorphic 1-forms $\omega_{i_1}, \dots, \omega_{i_m} \in \Omega^1(\mathcal{M})$.

For $\mathbf{a} = (a_1, \cdots, a_k) \in \mathbf{Z}^k_{>0}$, its weight and its depth are defined to be $wt(\mathbf{a}) = a_1 + \cdots + a_k$ and $dp(\mathbf{a}) = k$ respectively. Put $z \in \mathbf{C}$ with $|z| < 1$. Consider the following complex function which is called the *one variable multiple polylogarithm*

$$Li_{\mathbf{a}}(z) := \sum_{0<m_1<\cdots<m_k} \frac{z^{m_k}}{m_1^{a_1} \cdots m_k^{a_k}}.$$

It satisfies the recursive differential equations (cf. [BF, F08]) It gives an iterated integral starting from o, which lies on $I_o(\mathcal{M}_{0,4})$. Actually it corresponds to an element of $V(\mathcal{M}_{0,4})$ denoted by $l_{\mathbf{a}}$.

Similarly for $\mathbf{a} = (a_1, \cdots, a_k) \in \mathbf{Z}^k_{>0}$, $\mathbf{b} = (b_1, \cdots, b_l) \in \mathbf{Z}^l_{>0}$ and $x, y \in \mathbf{C}$ with $|x| < 1$ and $|y| < 1$, consider the following complex function which is called the *two variables multiple polylogarithm*

$$Li_{\mathbf{a},\mathbf{b}}(x,y) := \sum_{\substack{0<m_1<\cdots<m_k \\ <n_1<\cdots<n_l}} \frac{x^{m_k} \quad y^{n_l}}{m_1^{a_1} \cdots m_k^{a_k} n_1^{b_1} \cdots n_l^{b_l}}.$$

It also satisfies the recursive differential equations (cf. [BF]§5). They show that the functions $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{a},\mathbf{b}}(y,x)$, $Li_{\mathbf{a}}(x)$, $Li_{\mathbf{a}}(y)$ and $Li_{\mathbf{a}}(xy)$ give iterated integrals starting from o, which lie on $I_o(\mathcal{M}_{0,5})$. They correspond to elements of $V(\mathcal{M}_{0,5})$ by the map ρ denoted by $l^{x,y}_{\mathbf{a},\mathbf{b}}$, $l^{y,x}_{\mathbf{a},\mathbf{b}}$, $l^x_{\mathbf{a}}$, $l^y_{\mathbf{a}}$ and $l^{xy}_{\mathbf{a}}$ respectively.

The idea of the proof of proposition 2.11 goes as follows: Recall that multiple polylogarithms satisfy the analytic identity, the series shuffle formula in $I_o(\mathcal{M}_{0,5})$

$$Li_{\mathbf{a}}(x) \cdot Li_{\mathbf{b}}(y) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} Li_{\sigma(\mathbf{a},\mathbf{b})}(\sigma(x,y)).$$

Here $Sh^{\leqslant}(k,l) := \cup_{N=1}^{\infty} \{\sigma : \{1, \cdots, k+l\} \to \{1, \cdots, N\} | \sigma \text{ is onto}, \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\}$, $\sigma(\mathbf{a},\mathbf{b}) := ((c_1, \cdots, c_j), (c_{j+1}, \cdots, c_N))$ with $\{j, N\} = \{\sigma(k), \sigma(k+l)\}$,

$$c_i = \begin{cases} a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s,t\} \text{ with } s < t, \\ a_s & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s \leqslant k, \\ b_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s > k, \end{cases}$$

$$\text{and } \sigma(x,y) = \begin{cases} xy & \text{if } \sigma^{-1}(N) = k, k+l, \\ (x,y) & \text{if } \sigma^{-1}(N) = k+l, \\ (y,x) & \text{if } \sigma^{-1}(N) = k. \end{cases}$$

Since ρ is an embedding of algebras, the above analytic identity implies the algebraic identity, the series shuffle formula in $V(\mathcal{M}_{0,5})$

$$l_{\mathbf{a}}^{x} \cdot l_{\mathbf{b}}^{y} = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}^{\sigma(x,y)}. \tag{2.4}$$

Suppose that φ is an element as in proposition 2.11. Evaluation of the equation (2.4) at the group-like element $\varphi_{451}\varphi_{123}$ [4] gives the series shuffle formula

$$l_{\mathbf{a}}(\varphi) \cdot l_{\mathbf{b}}(\varphi) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}(\varphi)$$

for admissible [5] indices $\mathbf{a}$ and $\mathbf{b}$ because of [F08] lemma 4.1. and 4.2.

For non-admissible indices we need a special treatment. The idea is essentially same to the above admissible indices case except that we consider $e^{TX_{51}}\varphi_{451}\varphi_{123}$ (T: a parameter which stands for $\log x$) instead of $\varphi_{451}\varphi_{123}$ (see [F08] in more detail), which completes the proof of theorem 2.9. □

Remark 2.12. Alekseev and Torossian [AT] gave the second proof of the Kashiwara-Vergne (KV) conjecture. It is a conjecture on a property of the Campbell-Baker-Hausdorff formula which was posed in [KV]. Their proof was based on Drinfel'd's theory [Dr91] of the Grothendieck-Teichmüller group. They showed that the set of solutions of the generalized KV-problem admitted a free and transitive action of the (graded) *Kashiwara-Vergne group KRV* (see also [AET] for the definition). It is a subgroup of $\underline{AutF_2}$ and they showed that it contains GRT_1, i,e, there is an embedding $GRT_1 \hookrightarrow KRV$.

They conjectured the following.

Conjecture 2.13 ([AT]§4)**.** The embedding might give an isomorphism between GRT_1 and the degree>1-part $KRV_{>1}$.

One of the main defining equations of KRV is the coboundary Jacobian condition (cf. loc.cit.), which is a lift of the gamma factorization formula (2.5) (see below) to the trace space $\hat{\mathfrak{T}}_2$. The following theorem might be a step to relate KRV with DMR_0.

[4]For simplicity we mean φ_{ijk} for $\varphi(X_{ij}, X_{jk}) \in U\mathfrak{P}_5$.

[5]An index $\mathbf{a} = (a_1, \cdots, a_k)$ is called *admissible* if $a_k > 1$.

Theorem 2.14 ([F08]). *Let φ be a non-commutative formal power series in two variables which is group-like with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = 0$. Suppose that it satisfies the generalized double shuffle relation* (2.3). *Then its meta-abelian quotient* [6] $B_\varphi(x_0, x_1)$ *is gamma-factorisable, i.e. there exists a unique series $\Gamma_\varphi(s)$ in $1 + s^2k[[s]]$ such that*

$$B_\varphi(x_0, x_1) = \frac{\Gamma_\varphi(x_0)\Gamma_\varphi(x_1)}{\Gamma_\varphi(x_0 + x_1)}. \tag{2.5}$$

The gamma element Γ_φ gives the correction term φ_{corr} of the series shuffle regularization (2.2) *by $\varphi_{corr} = \Gamma_\varphi(-Y_1)^{-1}$.*

This theorem was proved in [F08] §5. It extends the result in [DT, I] which shows that for any group-like series satisfying (1.1), (1.2) and (1.3) its meta-abelian quotient is gamma factorisable. We note that it was calculated in [Dr91] that especially $\Gamma_\varphi(s) = \exp\{\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} s^n\} = e^{-\gamma s}\Gamma(1-s)$ for $\varphi = \Phi_{KZ}$ where γ is the Euler constant, $\Gamma(s)$ is the classical gamma function and Φ_{KZ} is the Drinfel'd associator.

3. The motivic Galois group

We recall the motivic Galois group of the category of mixed Tate motives over $\mathbf{Z}$ [DG] in this section. This is related with the Drinfel'd's Grothendieck-Teichmüller group ([Dr91]) in §1 and the Racinet's double shuffle group ([R]) in §2.

Convention 3.1. Let $DM(\mathbf{Q})_{\mathbf{Q}}$ be the triangulated category of *mixed motives* [7] over $\mathbf{Q}$ constructed by Hanamura [Ha], Levine [L2] and Voevodsky [V]. *Tate motives* $\mathbf{Q}(n)$ ($n \in \mathbf{Z}$) are (Tate) objects of the category. Let $DMT(\mathbf{Q})_{\mathbf{Q}}$ be the triangulated sub-category of $DM(\mathbf{Q})_{\mathbf{Q}}$ generated by Tate motives $\mathbf{Q}(n)$ ($n \in \mathbf{Z}$). By the work of Levine [L1] a neutral tannakian $\mathbf{Q}$-category $MT(\mathbf{Q}) = MT(\mathbf{Q})_{\mathbf{Q}}$ of *mixed Tate motives over* $\mathbf{Q}$ can be extracted by taking a heart with respect to a t-structure of $DMT(\mathbf{Q})_{\mathbf{Q}}$.

Each object M of $MT(\mathbf{Q})$ has an increasing filtration of subobjects called *weight filtration*, $W : \cdots \subseteq W_{m-1}M \subseteq W_mM \subseteq W_{m+1}M \subseteq \cdots$, whose intersection is 0 and union is M. The quotient $Gr^W_{2m+1}M :=$

[6]It means $(1 + \varphi_{X_1}X_1)^{ab}$ for the unique expression $\varphi = 1 + \varphi_{X_0}X_0 + \varphi_{X_1}X_1$ ($\varphi_{X_0}, \varphi_{X_1} \in k\langle\langle X_0, X_1\rangle\rangle$) and $(\cdot)^{ab}$ means the image of the abelianization map $k\langle\langle X_0, X_1\rangle\rangle \to k[[x_0, x_1]]$.

[7]As for a nice expository on mixed motives, see [De94]. According to *Wikipedia*, *"the (partly conjectural) theory of motives is an attempt to find a universal way to linearize algebraic varieties, i.e. motives are supposed to provide a cohomology theory which embodies all these particular cohomologies."*

$W_{2m+1}M/W_{2m}M$ is trivial and $Gr^W_{2m}M := W_{2m}M/W_{2m+1}M$ is a direct sum of finite copies of Tate motive $\mathbf{Q}(m)$ for each $m \in \mathbf{Z}$. Morphisms of $MT(\mathbf{Q})$ are strictly compatible with weight filtration. The extension group is related to K-theory as follows

$$Ext^i_{MT(\mathbf{Q})}(\mathbf{Q}(0), \mathbf{Q}(m)) = \begin{cases} K_{2m-i}(\mathbf{Q})_{\mathbf{Q}} & \text{for } i = 1, \\ 0 & \text{for } i > 1. \end{cases}$$

There are realization fiber functors ([L2] and [Hu]) corresponding to usual (Betti, de Rham, étale, etc) cohomology theories.

Definition 3.2. Deligne and Goncharov [DG] defined the full subcategory $MT(\mathbf{Z}) = MT(\mathbf{Z})_{\mathbf{Q}}$ of *unramified mixed Tate motives*, whose objects are mixed Tate motives M (an object of $MT(\mathbf{Q})$) such that for each subquotient E of M which is an extension of $\mathbf{Q}(n)$ by $\mathbf{Q}(n+1)$ for $n \in \mathbf{Z}$, the extension class of E in $Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(n), \mathbf{Q}(n+1)) = Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Q}^\times \otimes \mathbf{Q}$ is equal to $\mathbf{Z}^\times \otimes \mathbf{Q} = \{0\}$.

In this category the following hold:

$$Ext^1_{MT(\mathbf{Z})}(\mathbf{Q}(0), \mathbf{Q}(m)) = \begin{cases} 0 & \text{for } m \leqslant 1, \\ K_{2m-1}(k)_{\mathbf{Q}} & \text{for } m > 1, \end{cases}$$

$$Ext^2_{MT(\mathbf{Z})}(\mathbf{Q}(0), \mathbf{Q}(m)) = 0.$$

Actually $MT(\mathbf{Z})$ forms a neutral tannakian $\mathbf{Q}$-category with the fiber functor $\omega_{\mathrm{can}} : MT(\mathbf{Z}) \to Vect_{\mathbf{Q}}$ ($Vect_{\mathbf{Q}}$: the category of $\mathbf{Q}$-vector spaces) which sends each motive M to $\oplus_n Hom(\mathbf{Q}(n), Gr^W_{-2n}M)$.

Definition 3.3. The *motivic Galois group* of unramified mixed Tate motives $MT(\mathbf{Z})$ is defined to be the pro-algebraic group $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) := \underline{Aut}^{\otimes}(MT(\mathbf{Z}) : \omega_{\mathrm{can}})$.

We have a categorical equivalence $MT(\mathbf{Z}) \simeq \mathrm{Rep}_{\mathbf{Q}}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ where r.h.s means the category of $\mathbf{Q}$-vector space with $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$-action. The action of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ on $\omega_{\mathrm{can}}(\mathbf{Q}(1)) = \mathbf{Q}$ defines a surjection $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \to \mathbf{G}_m$ and its kernel $\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ is the unipotent radical of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$. There is a canonical splitting $\tau : \mathbf{G}_m \to \mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ which gives a negative grading on the Lie algebra $Lie\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ (consult [De89] §8 for the full story). The above computations of Ext-groups follows

Proposition 3.4 ([De89] §8, [DG] §2). *The graded Lie algebra $Lie\mathcal{U}\,Gal^{\mathcal{M}}(\mathbf{Z})$ of the unipotent part $\mathcal{U}\,Gal^{\mathcal{M}}(\mathbf{Z})$ of $Gal^{\mathcal{M}}(\mathbf{Z})$ is a graded free Lie algebra generated by one element in each degree $-m$ ($m \geqslant 3$: odd).*

In [DG] §4 they constructed the *motivic fundamental group* $\pi_1^{\mathcal{M}}(X : \overrightarrow{01})$ with $X = \mathbf{P}^1\backslash\{0,1,\infty\}$, which is an ind-object of $MT(\mathbf{Z})$. This is an affine group $MT(\mathbf{Z})$-scheme (cf. [DG]). Since all the structure morphism of $\pi_1^{\mathcal{M}}(X : \overrightarrow{01})$ belong to the set of morphisms of $MT(\mathbf{Z})$ and $\omega_{\mathrm{can}}(\pi_1^{\mathcal{M}}(X : \overrightarrow{01})) = \underline{F_2}$ where $\underline{F_2}$ is the free pro-unipotent algebraic group of rank 2, we have

$$\varphi : \mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \to \underline{Aut}\underline{F_2}.$$

On this map φ the following is one of the basic problems.

Problem 3.5. Is φ injective?

Remark 3.6. This might be said a problem which asks a validity of a unipotent variant of the so-called 'Belyĭ's theorem' in [Be] in the pro-finite setting. Equivalently this asks if the motivic fundamental group $\pi_1^{\mathcal{M}}(X : \overrightarrow{01})$ is a 'generator' of the tannakian category $MT(\mathbf{Z})$. It is related with various conjectures in several realizations (cf. [F07] note 3.10); Zagier conjecture (partially proved by Terasoma [T] and Deligne-Gonchaov [DG]) in Hodge realization, Deligne-Ihara conjecture (partially proved by Hain-Matsumoto [HM]) in étale realization and Furusho-Yamashita conjecture (partially proved by Yamashita [Y]) in crystalline realization.

By using geometric interpretation of pentagon and hexagons equations, the defining equations of the Grothendieck-Teichmüller group GRT_1, we could show that the unipotent part of the motivic Galois group $\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ is mapped into GRT_1 (clearly explained in [A, F07]):

Proposition 3.7. *$\varphi(\mathcal{U}Gal^{\mathcal{M}}(\mathbf{Z})) \subset GRT_1$.*

In [Ko] Kontsevich raised a mysterious speculation which connects motivic Galois groups and deformation quantizations. His speculation was based on several conjectures and one of which was the following.

Conjecture 3.8. The map φ might induce the isomorphism $\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \simeq GRT_1$.

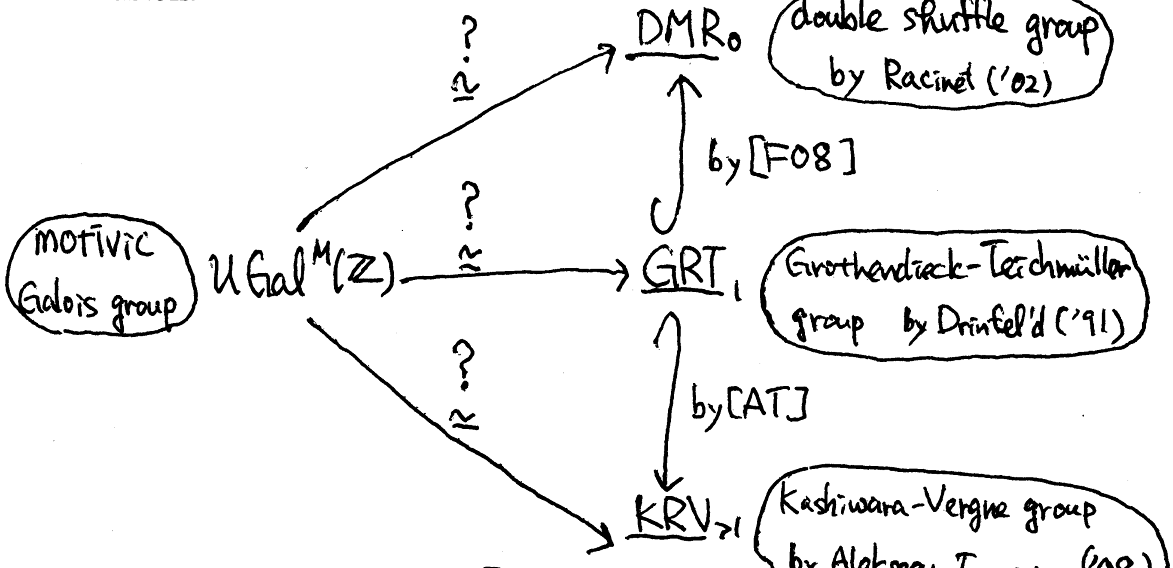
It is known that the double shuffle group DMR_0 also contains the motivic Galois image:

Proposition 3.9. *$\varphi(\mathcal{U}Gal^{\mathcal{M}}(\mathbf{Z})) \subset DMR_0$.*

This follows from the result in [Go] and another proof is given in [F07]. As an analogue of conjecture 3.8, the following conjecture is posed (cf. [R] and see also [A].)

Conjecture 3.10. The map φ might induce the isomorphism $\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \simeq DMR_0$.

The validities of conjecture 3.8 and conjecture 3.10 would imply that GRT_1 might be isomorphic to DMR_0. Due to corollary 2.10 we have the injection $GRT_1 \hookrightarrow DMR_0$ however we do not know the opposite inclusion.



REFERENCES

[A] André, Y., Une introduction aux motifs (motifs purs, motifs mixtes, périodes), Panoramas et Synthèses, 17, Société Mathématique de France, Paris, 2004.

[AT] Alekseev, A. and Torossian, C., The Kashiwara-Vergne Conjecture and Drinfeld's associators, preprint arXiv:0802.4300.

[AET] ———, Enriquez, B. and ———, Drinfeld associators, braid groups and explicit solutions of the Kashiwara-Vergne equations, preprint arXiv:0903.4067.

[Ba] Bar-Natan, D., On associators and the Grothendieck-Teichmuller group. I, Selecta Math. (N.S.) 4 (1998), no. 2, 183–212.

[Be] Belyĭ, G. V., Galois extensions of a maximal cyclotomic field, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 2, 267–276, 479.

[BF] Besser, A. and Furusho, H.; The double shuffle relations for p-adic multiple zeta values, AMS Contemporary Math, Vol 416, (2006), 9-29.

[Br] Brown, F., Multiple zeta values and periods of moduli spaces $\mathfrak{M}_{0,n}$, arXiv:math/0606419.

[C] Chen, K. T., Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977), no. 5, 831–879.

[De89] Deligne, P., Le groupe fondamental de la droite projective moins trois points, Galois groups over Q (Berkeley, CA, 1987), 79–297, Math. S. Res. Inst. Publ., 16, Springer, New York-Berlin, 1989.

[De94] ———, À quoi servent les motifs?, In Motives (Seattle, WA, 1991), volume 55 of Proc. Sympos. Pure Math., pages 143-161. Amer. Math. Soc., Providence, RI, 1994.

[DG] ——— and Goncharov, A.; Groupes fondamentaux motiviques de Tate mixte, Ann. Sci. Ecole Norm. Sup. (4) 38 (2005), no. 1, 1-56.

[DT] ——— and Terasoma, T., Harmonic shuffle relation for associators, preprint available from www2.lifl.fr/mzv2005/DOC/Terasoma/lille_terasoma.pdf

[Dr86] Drinfel'd, V. G., Quantum groups, Proceedings of the International Congress of Mathematicians (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[Dr90] ———, Quasi-Hopf algebras, Leningrad Math. J. 1 (1990), no. 6, 1419–1457.

[Dr91] ———, On quasitriangular quasi-Hopf algebras and a group closely connected with Gal($\overline{Q}/Q$), Leningrad Math. J. 2 (1991), no. 4, 829–860.

[E] Euler, L., Meditationes circa singulare serierum genus, Novi Commentarii academiae scientiarum Petropolitanae 20, 1776, pp. 140-186 and Opera Omnia: Series 1, Volume 15, pp. 217 - 267 (also available from www.math.dartmouth.edu/ euler/).

[F03] Furusho, H., The multiple zeta value algebra and the stable derivation algebra, Publ. Res. Inst. Math. Sci. Vol 39. no 4. (2003). 695-720.

[F07] ———, p-adic multiple zeta values II – tannakian interpretations, Amer. J. Math, Vol 129, No 4, (2007),1105-1144.

[F08] ———, Double shuffle relation for associators, preprint arXiv:0808.0319v2.

[F10] ———, Pentagon and hexagon equations, Annals of Mathematics, Vol. 171 (2010), No. 1, 545-556.

[Go] Goncharov, A. B., Periods and mixed motives, arXiv:math/0202154, preprint in 2002.

[Gr] Grothendieck, A., Esquisse d'un programme, 1983 (available on pp. 243–283. London Math. Soc. LNS 242, Geometric Galois actions, 1, 5–48, Cambridge Univ).

[HM] Hain, R. and Matsumoto, M., Weighted Completion of Galois Groups and Some Conjectures of Deligne, Compositio Math. 139 (2003), no.2, 119–167.

[Ha] Hanamura, M.;Mixed motives and algebraic cycles. I, Math. Res. Lett. 2 (1995), no. 6, 811–821; II, Invent. Math. 158 (2004), no. 1, 105–179; III, Math. Res. Lett. 6 (1999), no. 1, 61–82.

[Hu] Huber, A.; Realization of Voevodsky's motives, J.Alg.Geom. 9. 2000, 755-799.

[IKZ] Ihara, K., Kaneko, M. and Zagier, D., Derivation and double shuffle relations for multiple zeta values, Compos. Math. 142 (2006), no. 2, 307–338.

[I] Ihara, Y., On beta and gamma functions associated with the Grothendieck-Teichmuller groups, Aspects of Galois theory, 144–179, London Math. Soc. LNS 256, Cambridge Univ. Press, Cambridge, 1999.

[JS] Joyal, A. and Street, R., Braided tensor categories, Adv. Math. 102 (1993), no. 1, 20–78.

[KV] Kashiwara, M. and Vergne, M., The Campbell-Hausdorff formula and invariant hyperfunctions, Invent. Math. 47 (1978), no. 3, 249–272.

[Ka] Kassel, C., Quantum groups, Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.

[Ko] Kontsevich, M., Operads and motives in deformation quantization, Lett. Math. Phys. 48 (1999), no. 1, 35–72.

[LM] Le, T.T.Q. and Murakami, J., Kontsevich's integral for the Kauffman polynomial, Nagoya Math. J. 142 (1996), 39–65.

[L1] Levine, M.; Tate motives and the vanishing conjectures for algebraic K-theory, in: Algebraic K-theory and algebraic topology (Lake Louis, 1991), 167-188, NATO Adv. Sci, but. Ser.C Math. Phys. 407, Kluwer, 1993.

[L2] ———; Mixed Motives, Math surveys and Monographs 57, AMS, 1998.

[R] Racinet, G.; Doubles melanges des polylogarithmes multiples aux racines de l'unite, Publ. Math. Inst. Hautes Etudes Sci. No. 95 (2002), 185–231.

[T] Terasoma, T., Mixed Tate motives and multiple zeta values, Invent. Math. 149 (2002), no. 2, 339–369.

[V] Voevodsky, V.; Triangulated categories of motives over a field, in: Cycles, transfer and motivic homology theories, 188-238, Ann of Math Studies 143, Princeton University Press, 1993.

[Y] Yamashita, G.; Bounds for the dimensions of the p-adic multiple L-value spaces, preprint.

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, CHIKUSA-KU, FURO-CHO, NAGOYA, 464-8602, JAPAN

E-mail address: `furusho@math.nagoya-u.ac.jp`