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Jacobi's theta-hypergeometric formula in 2 variables: some applications of the modular forms on the hyperball

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1 Introduction

一変数保型函数論とくにテータ函数論において、以下に述べる Jacobi の公式、倍角公式、算術幾何平均公式、の三者が超幾何函数と関連しながら一つの理論の萌芽を内蔵しているかのように見える。これらを、多変数化する試みは散発的な試みは見られるが、これまで十分説得力のある結果は得られていなかった。理由は、楕円モジュラー函数に比肩する幾何学的精密さを持った多変数保型函数が与えられていなかったからである。Picard の原論文 [P] に依拠して筆者が得た保型函数 [S] は 1989 年の K. Matsumoto [Mat] の保型函数とともに、この意味で十分な精密さを持った 2 変数保型函数である。その実際の応用例を本稿で幾つか示したい。

2 Classical model : Jacobi-Gauss formulas

Start from the family of elliptic curves in the Legendre normal form:

$$y^2 = x(x-1)(x-\lambda),\ \lambda \in \boldsymbol{P}^1 - \{0,1,\infty\}.$$

Put

$$\tau(\lambda) = \frac{\int_1^\infty \frac{dx}{y}}{\int_{-\infty}^0 \frac{dx}{y}}.$$

(to avoid the ambiguity we suppose $0 < \lambda < 1, \tau \in \boldsymbol{H}$ for the moment). Recall that we have

Theorem 2.1.

$$\lambda(\tau) = \frac{\vartheta_{01}^4(\tau)}{\vartheta_{00}^4(\tau)},\ \tau \in \boldsymbol{H} \tag{2.1}$$

, here we use

$$\vartheta_{jk}(\tau) = \sum_{n\in\mathbb{Z}} \exp[\pi i(n+\frac{j}{2})^2\tau + 2\pi i(n+\frac{j}{2})\frac{k}{2}]\ (i,j \in \{0,1\}^2).$$

Especially

$$\begin{cases} \vartheta_{00}(\tau) = 1 + 2\tilde{q} + 2\tilde{q}^4 + \cdots + 2\tilde{q}^{n^2} + \cdots, \\ \vartheta_{01}(\tau) = 1 - 2\tilde{q} + 2\tilde{q}^4 + \cdots + (-1)^n 2\tilde{q}^{n^2} + \cdots,\ \tilde{q} = e^{\pi i\tau}. \end{cases} \tag{2.2}$$

According to Jacobi we have

Theorem 2.2. (Jacobi, Complete works I, p.235) *Under the relation (2.1) we have*

$$\vartheta_{00}^2(\tau) = F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda\right) = \frac{1}{\pi}\int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-(1-\lambda))}}. \tag{2.3}$$

Here

$$F(a,b,c;\lambda) := \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!}\lambda^n \quad \textit{for } |\lambda| < 1,$$

with

$$(a,n) = \begin{cases} a(a+1)\cdots(a+n-1) & \textit{for } n > 0, \\ 1 & \textit{for } n = 0. \end{cases}$$

Remark 2.1. *At first, we have the equality (2.3) for a pure imaginary τ. Then we can make analytic continuations on $\boldsymbol{H}$ both sides. So we obtain (2.3) on the whole $\boldsymbol{H}$.*

2.1 From the Jacobi formula to Gauss AGM

The following duplication formula (due to Gauss 1818) plays an important role:

Theorem 2.3. (duplication formula)

$$\begin{cases} \vartheta_{00}^2(2\tau) = \frac{1}{2}\left(\vartheta_{00}^2(\tau) + \vartheta_{01}^2(\tau)\right), \\ \vartheta_{01}^2(2\tau) = \vartheta_{00}(\tau)\vartheta_{01}(\tau). \end{cases} \tag{2.4}$$

For an initial data (a,b) with $0 < a, b$, set $\psi(a,b) = \left(\frac{a+b}{2}, \sqrt{ab}\right)$, and set $(a_n, b_n) = \psi^n(a,b)$. We have a common limit $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. We define Gauss' arithmetic geometric mean

$$M(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Theorem 2.4. (Gauss AGM theorem)
We have

$$\frac{1}{M(1,x)} = F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-x^2\right).$$

By putting $x = \vartheta_{01}^2(\tau)/\vartheta_{00}^2(\tau)$ we can derive the Gauss AGM theorem from the above Jacobi formula. In fact, we have

$$M(\vartheta_{00}^2(\tau), \vartheta_{01}^2(\tau)) = \lim_{n\to\infty} \frac{1}{2}\left(\vartheta_{00}^2(2^n\tau) + \vartheta_{01}^2(2^n\tau)\right) = \lim_{\tau\to i\infty} \vartheta_{00}^2(\tau) = 1.$$

So we have $\vartheta_{00}^2(\tau)\, M(1,x) = 1$.

This proof shows the close relation among Jacobi's formula, the Duplication formula and Gauss' AGM formula.

3 Jacobi-Gauss formulas for the Picard modular case

3.1 Picard modular forms

We express the Picard curve with the projective parameters:

$$C(\xi) : y^3 = x(x-\xi_0)(x-\xi_1)(x-\xi_2), \tag{3.1}$$

where

$$\xi \in \Lambda = \{[\xi_0 : \xi_1 : \xi_2] \in \mathbb{P}^2(\mathbb{C}) : \xi_0\xi_1\xi_2(\xi_0 - \xi_1)(\xi_1 - \xi_2)(\xi_2 - \xi_0) \neq 0\}.$$

It is a curve of genus three, The Jacobian variety $Jac(C(\xi))$ of $C(\xi)$ has a generalized complex multiplication by $\sqrt{-3}$ of type $(2,1)$. In fact we have a basis of holomorphic differentials

$$\varphi = \varphi_1 = \frac{dz}{w}, \quad \varphi_2 = \frac{dz}{w^2}, \quad \varphi_3 = \frac{zdz}{w^2}.$$

Putting $\lambda_1 = \xi_1/\xi_0, \lambda_2 = \xi_2/\xi_0$, we assume $0 < \lambda_1 < \lambda_2 < 1$. Under this condition we choose the following symplectic basis of $H_1(C, \mathbb{Z})$ already used in [S]. Here we put cut lines starting from branch points in the lower half z-plane to get simply connected sheets. The real line(resp. dotted line, chained line) indicates an arc on the first sheet (resp. second sheet, third sheet).

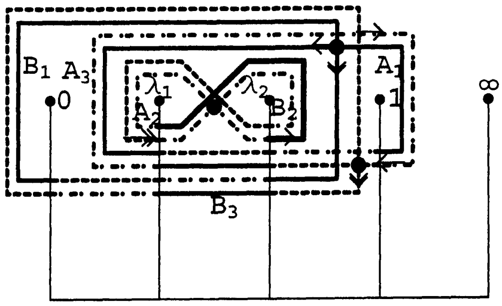


Figure 1. homology basis

Setting $\rho(z,w) = (z, \omega w)$, we have

$$B_3 = \rho(B_1), \quad A_3 = -\rho^2(A_1), \quad B_2 = -\rho^2(A_2),$$

here ω stands for $\exp[2\pi i/3]$. We have $A_iB_j = \delta_{ij}$. Put

$$\eta_0 = \int_{A_1} \varphi, \quad \eta_1 = -\int_{B_3} \varphi, \quad \eta_2 = \int_{A_2} \varphi. \tag{3.2}$$

By the analytic continuation, they are multivalued analytic functions on the domain $\Lambda = \{\lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2) \neq 0\} \subset \boldsymbol{C}^2$. It holds

$$\begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \int_{A_1} \varphi_1 \\ -\int_{B_3} \varphi_1 \\ \int_{A_2} \varphi_1 \end{pmatrix} = \begin{pmatrix} -\omega^2 \int_{A_3} \varphi_i \\ -\omega^2 \int_{B_1} \varphi_i \\ -\omega^2 \int_{B_2} \varphi_i \end{pmatrix}, \quad \begin{pmatrix} \int_{A_1} \varphi_i \\ -\int_{B_3} \varphi_i \\ \int_{A_2} \varphi_i \end{pmatrix} = \begin{pmatrix} -\omega \int_{A_3} \varphi_i \\ -\omega \int_{B_1} \varphi_i \\ -\omega \int_{B_2} \varphi_i \end{pmatrix}, \quad (i = 2,3). \tag{3.3}$$

Set

$$\Omega_1 = \left(\int_{A_j} \varphi_i\right), \quad \Omega_2 = \left(\int_{B_j} \varphi_i\right), \quad (1 \le i,j \le 3).$$

The normalized period matrix of $C(\xi)$ is given by $\Omega = \Omega_1^{-1}\Omega_2$. By the relations of periods (3.3) together with the symmetricity ${}^t\Omega = \Omega$, we can rewrite

$$\Omega = \Omega_1^{-1}\Omega_2 = \begin{pmatrix} \frac{u^2+2\omega^2 v}{1-\omega} & \omega^2 u & \frac{\omega u^2 - \omega^2 v}{1-\omega} \\ \omega^2 u & -\omega^2 & u \\ \frac{\omega u^2 - \omega^2 v}{1-\omega} & u & \frac{\omega^2 u^2 + 2\omega^2 v}{1-\omega} \end{pmatrix}, \tag{3.4}$$

here we put $u = \frac{\eta_2}{\eta_0}$, $v = \frac{\eta_1}{\eta_0}$. So we set $\Omega = \Omega(u,v)$. The Riemann period relation Im $\Omega > 0$ induces the inequality $2\mathrm{Re}\,(v) + |u|^2 < 0$. We set

$$\mathcal{D} = \{\eta = [\eta_0 : \eta_1 : \eta_2] \in \boldsymbol{P}^2 : \eta H {}^t\bar{\eta} < 0\} = \{(u,v) \in \mathbb{C}^2 : 2\mathrm{Re}\,(v) + |u|^2 < 0\},$$

here we put $H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We define our period map $\Phi : \Lambda \to \mathcal{D}$ by

$$\Phi(\lambda_1, \lambda_2) = [\eta_0, \eta_1, \eta_2].$$

Set the Picard modular group

$$\Gamma = U(2,1; \boldsymbol{Z}[\omega]) = \{g \in \mathrm{GL}_3(\boldsymbol{Z}[\omega]) : {}^t\bar{g} H g = H\}.$$

The element $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in \Gamma$ acts on $\mathcal{D}$ by

$$g(u,v) = \left(\frac{p_3 + q_3 v + r_3 u}{p_1 + q_1 v + r_1 u}, \frac{p_2 + q_2 v + r_2 u}{p_1 + q_1 v + r_1 u} \right). \tag{3.5}$$

Set $\Gamma(\sqrt{-3}) = \{g \in \Gamma : g \equiv I_3 \mod \sqrt{-3}\}$. We have $\Gamma/\Gamma(\sqrt{-3}) \cong S_4$.

The Riemann theta constant is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega) = \sum_{n \in \boldsymbol{Z}^3} \exp[\pi i (n+a) \Omega {}^t(n+a) + 2\pi i (n+a) {}^t b],$$

here $a, b \in \boldsymbol{Q}^3$ (row vectors) and $\Omega \in \mathfrak{S}_3$.

We use the following Riemann theta constants and their Fourier expansions (see [S], p.327):

$$\vartheta_k(u,v) = \vartheta \begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix} (\Omega(u,v)) = \sum_{\mu \in \boldsymbol{Z}[\omega]} \omega^{2k\mathrm{tr}(\mu)} H(\mu u) q^{\mathrm{N}(\mu)} \tag{3.6}$$

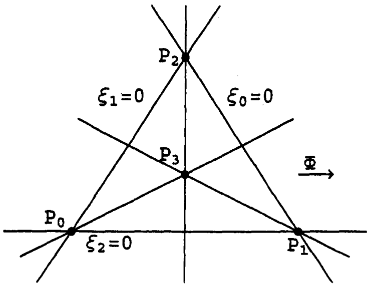
with an index $k \in \boldsymbol{Z}$, where $\mathrm{tr}(\mu) = \mu + \bar{\mu}$, $\mathrm{N}(\mu) = \mu\bar{\mu}$ and

$$H(u) = \exp[\frac{\pi}{\sqrt{3}} u^2] \vartheta \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix} (u, -\omega^2), \quad q = \exp[\frac{2\pi}{\sqrt{3}} v].$$

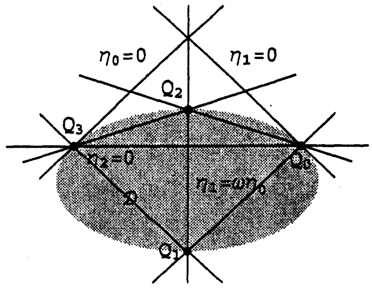
Apparently it holds $\vartheta_k(u,v) = \vartheta_{k+3}(u,v)$, so k runs over $\{0,1,2\} = \boldsymbol{Z}/3\boldsymbol{Z}$.

The following properties are already established.

Fact 3.1. *(Originally by Picard 1881, [S] p.349, 1988) The period map Φ induces a biholomorphic isomorphism from ξ-space $\mathbb{P}^2(\mathbb{C})$ to the Satake compactification $\overline{\mathcal{D}/\Gamma(\sqrt{-3})}$ of $\mathcal{D}/\Gamma(\sqrt{-3})$. This compactification is obtained by atatching 4 boundary points corresponding to 4 points $[\xi_0, \xi_1, \xi_2] = [0,0,1], [0,1,0], [1,0,0], [1,1,1]$. We have an action of the S_4 that is composed of projective linear transformations which causes a permutation of above 4 points on $\boldsymbol{P}^2$.*



$\xi-$ *space*



$\eta-$ *space*

Fact 3.2. *The following theorem is due to M. Namba [Nam], 1981.*

Theorem. *Let $C(\lambda)$ and $C(\lambda')$ be two Picard curves. They are isomorphic as Riemann surfaces if and only if we have an automorphism f of $\boldsymbol{C}$ such that $f(\{0,1,\lambda_1,\lambda_2\}) = \{0,1,\lambda_1',\lambda_2'\}$.*

So $\boldsymbol{P}^2/S_4 \cong (\mathscr{D}/\Gamma(\sqrt{-3}))/S_4 = (\mathscr{D}/\Gamma)^\circ$ is the moduli space of our family of Picard curves, here $^\circ$ means the one point compactification.

Fact 3.3.

Theorem 3.1. (Theta representation of Φ^{-1}, [S] p.327)

$$(\lambda_1,\lambda_2) = \left(\frac{\vartheta_1(u,v)^3}{\vartheta_0(u,v)^3}, \frac{\vartheta_2(u,v)^3}{\vartheta_0(u,v)^3}\right). \tag{3.7}$$

Fact 3.4. *([S] p.329) The projective group $\overline{\Gamma(\sqrt{-3})} = \Gamma(\sqrt{-3})/\{1,\omega,\omega^2\}$ is generated by*

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ \omega-\omega^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ \omega-1 & 1 & \omega-1 \\ 1-\omega^2 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 1 & \omega-\omega^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & \omega-1 & \omega-1 \\ 0 & 1 & 0 \\ 0 & 1-\omega^2 & 1 \end{pmatrix}.$$

This is the projective monodromy group of the multivalued map $\Phi : \Lambda \longrightarrow \mathscr{D}$.

Fact 3.5. *([S] p.346) We have the automorphic property:*

$$\vartheta_k(g(u,v))^3 = (p_1 + q_1 v + r_1 u)^3\, \vartheta_k(u,v)^3 \tag{3.8}$$

for $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in \Gamma(\sqrt{-3})$. *The sytem $\{\vartheta_k(g(u,v))^3\}_{k=0,1,2}$ is a basis of the vector space of automorphic forms with the property (3.8).*

Fact 3.6. *The system of periods $\{\eta_0,\eta_1,\eta_2\}$ is a basis of the space of solutions for the Appell hypergeometric differential equation $E_1(a,b,b',c)$ with $(a,b,b',c) = (\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$:*

$$E_1(a,b,b',c) : \begin{cases} r\,(1-x)\,x + p\,(c-(1+a+b)\,x) - b\,q\,y + s\,(1-x)\,y - a\,b\,z = 0 \\ \\ -(b'\,p\,x) + s\,x\,(1-y) + t\,(1-y)\,y + q\,(c-(1+a+b')\,y) - a\,b'\,z = 0, \end{cases} \tag{3.9}$$

with $r = z_{xx}, s = z_{xy}, t = z_{yy}, p = z_x, q = z_y$. It has singularities along $\boldsymbol{P}^2 - \Lambda$. $\overline{\Gamma(\sqrt{-3})}$ is the projective monodromy group of $E_1(\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$ also.

3.2 A Jacobi type formula in two variables

Under the relation

$$(\lambda_1, \lambda_2) = \left(\frac{\vartheta_1(u,v)^3}{\vartheta_0(u,v)^3}, \frac{\vartheta_2(u,v)^3}{\vartheta_0(u,v)^3}\right) \tag{3.10}$$

stated in Fact 3.1 , we have the following:

Theorem 3.2. (A Jacobi type formula in two variables, [M-S] 2010)

$$\vartheta_0(u,v) = C_0\, F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2), \tag{3.11}$$

$$C_0 = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (-\omega^2),$$

here $F_1(a,b,b',c;\lambda_1,\lambda_2)$ indicates the Appell hypergeometric function

$$F_1(a,b,b',c;\lambda_1,\lambda_2) = \sum_{m,n\geq 0} \frac{(a,m+n)(b,m)(b',n)}{(c,m+n)m!n!} \lambda_1^m \lambda_2^n. \tag{3.12}$$

Remark 3.1. *By using the power series expansion of F_1, we have the equality (3.11) for an arbitrary point in a neighborhood of the set $\{(u,v) \in \mathscr{D} : u = 0, v < 0\}$ in $\mathscr{D}$. By making the analytic continuation of the both sides we have the equality on the whole domain $\mathscr{D}$.*

Theorem 3.3. ([M-S] 2010) *We have*

$$\vartheta_i(u,v)^3 = C_0^3\, \lambda_i \left(F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2)\right)^3, \quad (i = 1,2). \tag{3.13}$$

Remark 3.2. *According to some classical literature (also in* [M-T-Y]*), it holds*

$$C_0 = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (-\omega^2) = \frac{3^{3/8}}{2\pi} \exp(\frac{5\pi\sqrt{-1}}{72}) \Gamma(\frac{1}{3})^{3/2}. \tag{3.14}$$

3.3 Application to a three terms AGM theorem

In [K-S1], a new three terms arithmetic geometric mean $M_3(a,b,c)$ is introduced. For three positive numbers a,b,c, set a new triple (a',b',c') with

$$\begin{cases} a' = \frac{1}{3}(a+b+c), \\ b'^3 + c'^3 = \frac{1}{3}(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2), \\ b'^3 - c'^3 = \frac{1}{3\sqrt{-3}}(a-b)(b-c)(c-a). \end{cases}$$

Define our AGM process by

$$(a',b',c') = \psi(a,b,c).$$

We can take a nice choice of the cubic roots for b', c' so that $\psi^2(a,b,c)$ becomes to be a triple of positive numbers again. Thus, we get a unique positive number

$$M_3(a,b,c) := \lim_{n\to\infty} \psi^n(a,b,c).$$

For the proof of the convergence of $\psi^n(a,b,C)$ see [K-S1] Theorem 2.1. As a consequence of Theorem 3.3 i.e. (3.11) we obtain a new proof of the three terms AGM theorem in [K-S1] (p.134 Theorem 2.2). For it we use

Theorem 3.4. (Isogeny formula, [K-S1] 2009)

$$\begin{cases} \vartheta_0(\sqrt{-3}u, 3v) = \frac{1}{3}(\vartheta_0 + \vartheta_1 + \vartheta_2), \\ \vartheta_1^3(\sqrt{-3}u, 3v) + \vartheta_2^3(\sqrt{-3}u, 3v) = \frac{1}{3}(\vartheta_0^2\vartheta_1 + \vartheta_1^2\vartheta_2 + \vartheta_2^2\vartheta_0 + \vartheta_0\vartheta_1^2 + \vartheta_1\vartheta_2^2 + \vartheta_2\vartheta_0^2), \\ \vartheta_1^3(\sqrt{-3}u, 3v) - \vartheta_2^3(\sqrt{-3}u, 3v) = \frac{1}{3\sqrt{-3}}(\vartheta_0 - \vartheta_1)(\vartheta_1 - \vartheta_2)(\vartheta_2 - \vartheta_0), \end{cases}$$

By using this isogeny formula, we obtain

Theorem 3.5. (AGM formula in two variables, [K-S1] 2009)

$$\frac{1}{M_3(1, x, y)} = F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1 - x^3, 1 - y^3), \quad (|x| < 1, |y| < 1). \tag{3.15}$$

We can prove it from Theorem 3.2 by the same method as the classical case.

Example 3.1. *We have* $F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; \frac{1}{2}, \frac{1}{2}) = \frac{\sqrt{\pi}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}$. *Put* $x = y = 1/\sqrt[3]{2}$. *By three times procedure* ψ *for them, we get a forty digits approximation*

$$1/M_3(1, 1/\sqrt[3]{2}, 1/\sqrt[3]{2}) = 1.159595266963928365769992051570020881945\cdots$$

of $\frac{\sqrt{\pi}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}$.

4 Application to Shimura curve calculation

4.1 General consideration

A Shimura space corresponding to a quaternion algebra $B = \left(\frac{a,b}{F}\right)$ is a coarse moduli space embedded in the coarse moduli space $\mathfrak{S}_g$ of principally polarized abelian varieties $\{A\}$ of a certain dimension g with an endmorphism structure $B \subset \mathrm{End}_0 A$. We can extract the Shimura space with its modular group action as a pair of the upper half plane $\boldsymbol{H}$ and a discrete group $G \subset SL(2, \boldsymbol{R})$. Here note that the Shimura curve $\boldsymbol{H}/G$ is compact.

In the article of A. Kurihara we find how we can realize the Shimura curve with discriminant 6 in the projective space generated by the modular forms on $\boldsymbol{H}$ with respect to G. (It is explained originally this description is given by Y. Ihara.)

We restrict our attention only for the case $B = \left(\frac{-3,2}{\boldsymbol{Q}}\right)$. Even in this case, we may allowed to ask the following questions:

(1) Is it possible to get more explicit descrition of the Shimura curve ? We are wishing to have a explicit algebraic curves corresponding to the point on the Shimura curve. In this case what does it mean our coordinates, in other words what is our embient space ?

(2) Is it possible to get more precise and more simple description of the modular group G ?

(3) Can we describe the period differential equation for our algebraic curves ?

(4) May I find a direct relation between the Gauss hypergeometric differential equation and our modular group G?

(5) Is it possible to give an explicit "Fourier expansion" for our modular form defined on $\boldsymbol{H}$ with respect to the modular group G?

We answer for these questions. We use the Picard modular forms for $k = \boldsymbol{Q}(\sqrt{-3})$ as our main tool.

4.2 Ball quotient geometry

Set $k = \boldsymbol{Q}(\sqrt{-3})$, and let $\mathcal{O}_k$ be the ring of algebraic intergers of k. Set $H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and we denote $\langle \eta, \eta' \rangle_H = \eta H {}^t\overline{\eta'}$ for $\eta, \eta' \in \boldsymbol{C}^3$, here $\bar{\cdot}$ indicates the complex conjugate. Let us recall $\mathscr{D} = \{\eta = [\eta_0, \eta_1, \eta_2] \in \boldsymbol{P}^2(\boldsymbol{C}) : \eta H {}^t\overline{\eta} < 0\}$ is the image of our period map Φ. For a generic point $\eta \in \mathscr{D}$, it corresponds a Picard curve $C(\lambda)$. We set $A(\lambda) = Jac(C(\lambda))$.

Let $c \in \mathcal{O}_k^3$ be a fixed vector with $\|c\|_H = \langle c, c \rangle_H > 0$. We call

$$D_c = \{\eta \in \mathscr{D} : \langle \eta, c \rangle = 0\}$$

a k-disc in $\mathscr{D}$. According to the study by R. P. Holzapfel and M. Petkova, we have the following:

Proposition 4.1. *Let D_c be a k-disc. Then for a generic point $\eta \in D_c$, the corresponding Jacobi variety $A(\lambda)$ is isogenous to a product type abelian variety $E_0 \times A'(\lambda)$, with $E_0 = \boldsymbol{C}/(\boldsymbol{Z} + \omega \boldsymbol{Z})$ and a two dimensional abelian variety $A'(\lambda)$.*

Proposition 4.2. *The k-disc D_c is a Shimura curve for the quaternion algebra $B = \left(\frac{-3, \langle c,c \rangle_H}{\boldsymbol{Q}}\right)$ in the sense that we have*

$$B \subset \mathrm{End}_0(A'(\lambda)).$$

4.3 Results

Set $\mathbf{c} = (1, 1, 0)$. Its H-norm $\mathbf{c}H\bar{\mathbf{c}}$ is 2. We have $D_{\mathbf{c}} = \{(u, v) \in \mathscr{D} : v = -1\} = \mathbf{c}^{\perp} \cap \mathscr{D}$. According to Petkova and Holzapfel ([Pet]) we know that $\mathscr{D}_{\mathbf{c}}$ is the one dimensional Shimura space for

$$B = \left(\frac{-3, \|\mathbf{c}\|_H}{\boldsymbol{Q}}\right) = \left(\frac{-3, 2}{\boldsymbol{Q}}\right)$$

with $\mathrm{Disc}(B) = 6$. (We have

$$\mathrm{Disc}(B) = \prod_{(-3,2)_p = -1} p = 2 \cdot 3 = 6.$$

)

So this is the case discussed in the Clay Lecture by Voight and also studied in the thesis of Maria Petkova.

Theorem 4.1. ([P-S] 2010) *Set a complex line $L_{\mathbf{c}} = \{\lambda_1 + \lambda_2 = 1\}$ in (λ_1, λ_2) space $\boldsymbol{P}^2$. Then we have*

$$\Theta(D_{\mathbf{c}}) = L_{\mathbf{c}}.$$

Namely our Shimura curve is realized as a hyperplane section in the vector space space $\langle \vartheta_0^3, \vartheta_1^3, \vartheta_2^3 \rangle$ of Picard modular forms.

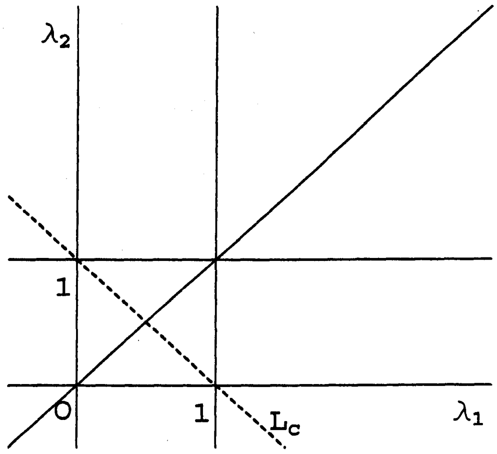


Figure of the Shimura curve L_c

Proposition 4.3. *For the Picard curves* $C(\lambda): y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)$ *with* $\lambda_1 + \lambda_2 = 1$, *we have a decomposition*

$$J(C(\lambda)) \cong E_0 \times A'(\lambda) \quad (\textit{up to isogeny})$$

where $E_0 = \boldsymbol{C}/(\omega\boldsymbol{Z} + \boldsymbol{Z})$ *and* $A'(\lambda)$ *is a 2-dimentional abelian variety. And we have*

$$\mathrm{End}_0(A'(\lambda)) \supset B.$$

Namely $C(\lambda)$ *with* $\lambda_1 + \lambda_2 = 1$ *is the corresponding curve for our Shimura variety.*

Proposition 4.4. *Put* $\lambda_1 = \frac{1}{2}(1+s), \lambda_2 = \frac{1}{2}(1-s)$. *On the line* L_c *the Appell differential equation* $E(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$ *reduces to*

$$\begin{aligned}&[27\, s^2\left(-1+s^2\right)^2\left(3+s^2\right) z_{sss} + 18\, s\left(3 - 38\, s^2 + 27\, s^4 + 8\, s^6\right) z_{ss} \\ &\quad +6\left(-9 - 60\, s^2 + 127\, s^4 + 22\, s^6\right) z_s + 8\, s^3\left(9+s^2\right)]\, z = 0.\end{aligned} \tag{4.1}$$

Remark 4.1. *This is a Fuchsian differential equation of rank 3. Looking at (4.1) we know that it has new singularities* $u = \pm\sqrt{-3}$ *other than expected singularities* $L_c \cap (\boldsymbol{P}^2 - \Lambda) = \{0, \pm 1, \infty\}$. *We have the Riemann scheme of (4.1):*

$$\left\{\begin{matrix} 0 & 1 & -1 & \sqrt{-3} & -\sqrt{-3} & \infty \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 2 & \frac{1}{3} & \frac{1}{3} & 1 & 1 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 3 & 3 & \frac{4}{3} \end{matrix}\right\}.$$

So $u = \pm\sqrt{-3}$ *are apparent singularities.*

Let

$$F_1(\lambda_1, \lambda_2) = F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; \lambda_1, \lambda_2) = 1 + \sum_{m+n>0} \frac{(\frac{1}{3}, m+n)(\frac{1}{3}, m)(\frac{1}{3}, n)}{(1, m+n)m!n!} \lambda_1^m \lambda_2^n$$

be the Appell hypergeometric series that is a solution of $E(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$.

Theorem 4.2. ([P-S] 2010) *Let* $f(s) = F_1(\frac{1}{2}(1+s), \frac{1}{2}(1-s))$ *be the restriction of* F_1 *on* L_c. *Then* $f(s)$ *is an even function of* s. *So we put* $f(s) = g(t)$ *with* $t = s^2$. *In this situation* $g(t)$ *satisfies the following Gauss hypergeometric differential equation:*

$$g''(t) + \frac{9t-5}{6t(t-1)}g'(t) + \frac{1}{18t(t-1)}g(t) = 0. \tag{4.2}$$

Remark 4.2. (1) *This theorem shows that the system (4.1) contains the subsystem (4.2) with rank 2. And it corresponds to the fact that we have the linear relation* $\eta_0 + \eta_1 = 0$ *in the period domain* $\mathscr{D}$.

(2) *The Riemann scheme of (4.2) is*

$$\begin{Bmatrix} 0 & 1 & \infty \\ 0 & 0 & \frac{1}{6}. \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{Bmatrix}$$

So (4.2) is the Gauss hypergeometric differential equation $E(\frac{1}{6}, \frac{1}{3}, \frac{2}{3})$, *and its monodromy group is the triangle group* $\Delta(3,6,6)$. *We can find it in the list of arithmetic co-compact triangle groups by K. Takeuchi [Tak].*

Definition 4.1. *Set* $B = \left(\frac{a,b}{\boldsymbol{Q}}\right)$ *be an indefinite quaternion algebra over* $\boldsymbol{Q}$. *We say* S_B *is a Shimura curve for* B, *if it is a moduli space of the isomorphism classes of principally polarized Abelian surfaces with the condition* $B' \subset \mathrm{End}_0(A)$ *under some identification* $B \cong B'$ *as quaternion algebras over* $\boldsymbol{Q}$.

Theorem 4.3. ([P-S] 2010) *Set* $M = \Gamma_{|D_c} = \{g \in \Gamma : g(D_c) = D_c\}/\{g \in \Gamma : g_{|D_c} = id_{D_c}\}$. *Under the identification induced from the isomorphism* $\Theta : \overline{\mathscr{D}/\Gamma(\sqrt{-3})} \to \boldsymbol{P}^2$, *we have the representation of the Shimura curve:*

$$S_B = L_c/\langle\sigma\rangle \cong D_c/M \cong \boldsymbol{H}/\Delta(3,6,6),$$

where σ *stands for the involution* $(\lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1)$.

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