

AN EXTENSION OF BURAU REPRESENTATION OF THE BRAID GROUPS

HIROSHI MATSUDA

Artin [3] introduced the braid group

$$B_n = \left\{ \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\}.$$

Alexander [1] found a connection between the braid group B_n and links in S^3 . Markov [10] introduced (three) “Markov moves” on closed braids, and announced “Markov Theorem.” (Weinberg [13] showed that one of three Markov moves was unnecessary.) This theorem says that studying closed braids modulo Markov moves, “Markov equivalence classes”, is equivalent to studying ambient isotopy classes of links in S^3 . Birman [6] gave a first complete proof of Markov Theorem.

Around the same time as the announcement of Markov theorem, Burau investigated a connection between a representation of B_n and Alexander polynomial of links. We introduce Burau representation of B_n in §1.

1. BURAU REPRESENTATION

Burau [7] defined a representation, Burau representation, $\varphi_n: B_n \rightarrow M(n; \mathbb{Z}[t, t^{-1}])$ of the braid group B_n . The image $\varphi_n(\sigma_i)$ of a generator σ_i of B_n is represented by the matrix

$$\left(\begin{array}{c|cc|c} I_{i-1} & O & O & O \\ \hline O & 1-t & t & O \\ O & 1 & 0 & O \\ \hline O & O & O & I_{n-(i+1)} \end{array} \right).$$

This representation φ_n is reducible, and is reduced to an irreducible representation $\varphi'_n: B_n \rightarrow M(n-1; \mathbb{Z}[t, t^{-1}])$. The images of generators σ_1, σ_{n-1} and σ_i ($2 \leq i \leq n-2$) of B_n by φ'_n are represented by the following matrices:

$$\varphi'_n(\sigma_1) = \left(\begin{array}{cc|c} -t & 1 & O \\ 0 & 1 & O \\ \hline O & O & I_{n-3} \end{array} \right) \quad \varphi'_n(\sigma_{n-1}) = \left(\begin{array}{c|cc} I_{i-3} & O & O \\ \hline O & 1 & 0 \\ O & t & -t \end{array} \right)$$

$$\varphi'_n(\sigma_i) = \left(\begin{array}{c|ccc|c} I_{i-2} & O & O & O & O \\ \hline O & 1 & 0 & 0 & O \\ O & t & -t & 1 & O \\ O & 0 & 0 & 1 & O \\ \hline O & O & O & O & I_{n-(i+2)} \end{array} \right).$$

Remark 1. The representation φ_n is faithful when $n \leq 3$ [9], and $n \geq 5$ [12], [8], [5]. It is not known whether $\varphi_4: B_4 \rightarrow M(4; \mathbb{Z}[t, t^{-1}])$ is faithful.

Burau obtained a knot invariant, Alexander polynomial, by measuring how far “1” departs from being an eigenvalue of $\varphi'_n(\beta)$.

Theorem 2. [7] *Let β denote a word in B_n . Let K denote a link in S^3 that is a closed n -braid corresponding to β . Then $\frac{\det(\varphi'_n(\beta) - I_{n-1})}{\det(\varphi'_n(\sigma_1\sigma_2 \cdots \sigma_{n-1}) - I_{n-1})}$ is equal to Alexander polynomial of K , $\Delta_K(t)$, up to multiplications by t .*

2. EXTENSION OF BURAU REPRESENTATION

In this section, we use 2×2 matrices instead of $\mathbb{Z}[t, t^{-1}]$ in Burau representation, that is, we study a mapping $\psi_n: B_n \rightarrow M(n; 2 \times 2 \text{ matrices})$.

Let Λ denote a set of elementary functions with variables $a, b, c, d, e, f, g, h, p, q, r, s, t, u, v, w$. Let

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, L = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, N = \begin{pmatrix} t & u \\ v & w \end{pmatrix}$$

denote elements in $GL(2; \Lambda)$. We assume that the 4×4 matrix $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ is invertible in $M(4; \Lambda)$. We define a mapping $\psi_n: B_n \rightarrow M(n; GL(2; \Lambda))$ by

$$\psi_n(\sigma_i) = \left(\begin{array}{c|cc|c} I_{2(i-1)} & O & O & O \\ \hline O & K & L & O \\ O & M & N & O \\ \hline O & O & O & I_{2(n-(i+1))} \end{array} \right).$$

This mapping ψ_n is a homomorphism when the following conditions are satisfied;

$$M = K^{-1}L^{-1}K(I_2 - K),$$

$$N = I_2 - K^{-1}L^{-1}KL,$$

$$h = \frac{1}{acf + b(bc + d - ad)g} \{cf(a(e - de) + c(be + f)) + (bde + (a - d)(-a - bc - d + ad)f)g - b^2g^2\}, \text{ and}$$

$$e = \frac{1}{2bc(-bc + ad)} \{a^2cf + 2bc^2f - acdf + 2b^2cg - ab^2cg - abdg + a^2bdg + b^2cdg + bd^2g - abd^2g + (acf + b(bc + d - ad)g)\sqrt{4bc + (a - d)^2}\}.$$

Therefore we obtain a representation of B_n , $\psi_n: B_n \rightarrow M(n; GL(2; \Lambda))$. This representation ψ_n is reducible, and is reduced to a representation

$$\psi'_n: B_n \rightarrow M(n - 1; GL(2; \Lambda)).$$

Remark 3. The representation ψ_n has something to do with a “biquandle”.

See [4], for example.

In a similar method as Burau obtained Alexander polynomial from Burau representation, we obtain a knot invariant from the representation ψ'_n .

Theorem 4. [11] *Let β denote a word in B_n . Let K denote a link in S^3 that is a closed n -braid corresponding to β . Then $\frac{\det(\psi'_n(\beta) - I_{2(n-1)})}{\det(\psi'_n(\sigma_1\sigma_2 \cdots \sigma_{n-1}) - I_{2(n-1)})}$ is a knot invariant, up to multiplications by $(a - 1)(d - 1) - bc$.*

3. EXAMPLE AND PROBLEM

In this section, we denote $\Delta_2(\beta) = \frac{\det(\psi'_n(\beta) - I_{2(n-1)})}{\det(\psi'_n(\sigma_1\sigma_2\cdots\sigma_{n-1}) - I_{2(n-1)})}$, and $D = (a-1)(d-1) - bc$, $T = (a-1) + (d-1)$.

Example 5. We calculate $\Delta_2(\beta)$ for some numbers of β .

- (1) Suppose $\beta = \sigma_1 \in B_2$. Then K is a trivial knot, and $\Delta_2(\sigma_1) = 1$.
- (2) Suppose $\beta = 1 \in B_2$. Then K is a 2-component trivial link, and $\Delta_2(1) = 0$.
- (3) Suppose $\beta = \sigma_1^2 \in B_2$. Then K is a positive Hopf link, and $\Delta_2(\sigma_1^2) = D + T + 1$.
- (4) Suppose $\beta = \sigma_1^{-2} \in B_2$. Then K is a negative Hopf link, and $\Delta_2(\sigma_1^{-2}) = \frac{1}{D^2}\Delta_2(\sigma_1^2)$. This is equal to $\Delta_2(\sigma_1^2)$, up to multiplications by D^2 .
- (5) Suppose $\beta = \sigma_1^3 \in B_2$. Then K is a right-handed trefoil knot, and $\Delta_2(\sigma_1^3) = D^2 + DT + T^2 - D + T + 1$.
- (6) Suppose $\beta = \sigma_1^{-3} \in B_2$. Then K is a left-handed trefoil knot, and $\Delta_2(\sigma_1^{-3}) = \frac{1}{D^3}\Delta_2(\sigma_1^3)$. This is equal to $\Delta_2(\sigma_1^3)$, up to multiplications by D^3 .

Observing calculations in Example 5, we pose the following conjecture.

Conjecture 6. *The knot invariant $\Delta_2(\beta)$ is an element in $\mathbb{Z}[D, D^{-1}, T, T^{-1}]$, up to multiplications by D , for every $\beta \in B_n$.*

In order to extend our extension of Burau representation, we pose the following problem.

Problem 7. *Choose your favorite algebra Ω with unit 1. (For example, Ω might be a “quandle algebra” or $GL(n; \Lambda)$ for some algebra Λ .) Let $\kappa, \lambda, \mu, \nu$ denote elements*

in Ω such that $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$ is invertible in $M(2; \Omega)$. We define a mapping

$$\phi_n: B_n \rightarrow M(n; \Omega) \text{ by } \phi_n(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & O & O & O \\ \hline O & \kappa & \lambda & O \\ O & \mu & \nu & O \\ \hline O & O & O & I_{n-(i+1)} \end{array} \right).$$

- (1) *Select $\kappa, \lambda, \mu, \nu \in \Omega$ so that $\phi_n: B_n \rightarrow M(n; \Omega)$ is a homomorphism. (We refer to [2] for one answer.)*

(2) Let β denote a word in B_n . Let K denote a link in S^3 that is a closed n -braid corresponding to β . Construct an invariant of “Markov equivalence classes” of β , that is, an invariant of K by evaluating the determinant, the trace, etc., of a matrix obtained from $\phi_n(\beta)$.

REFERENCES

- [1] J. W. Alexander, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. U.S.A., **9** (1923), 93–95.
- [2] M. Arik, F. Aydin, E. Hizel, J. Kornfilt, A. Yildiz, *Braid group related algebras, thier representations and generalized hydrogenlike spectra*, J. Math. Phys. **35** (1994), 3074–3088.
- [3] E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg, **4** (1925), 47-72;
Theory of braids, Ann. of Math. (2) **48** (1947), 101–126.
- [4] A. Bartholomew, R. Fenn, *Biquandles of small size and some invariants of virtual and welded knots*, arXiv:1004.1320.
- [5] S. Bigelow, *The Burau representation is not faithful for $n = 5$* , Geometry & Topology **3** (1999), 397–404.
- [6] J. S. Birman, *Braids, links and mapping class groups*, Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974; Errata: Canad. J. Math. **34** (1982), 1396–1397.
- [7] W. Burau, *Über Zopfgruppen und gleichsinnig verdrillte Verkettungen*. Abh. Math. Sem. Hamburg **11** (1936), 179–186.
- [8] D. D. Long, M. Paton, *The Burau representation is not faithful for $n \geq 6$* , Topology **32** (1993), 439–447.
- [9] W. Magnus, A. Peluso, *On a theorem of V. I. Arnol’d*, Comm. Pure Appl. Math. **22** (1969), 683–692.
- [10] A. A. Markov, *Über die freie aquivalenz der geschlossenen Zöpfe*, Recueil de la Soc. Math. de Moscou **43** (1936), 73–78.
- [11] H. Matsuda, *An extension of Burau representation, and a deformation of Alexander polynomial*, preprint.
- [12] J. A. Moody, *The Burau representation of the braid group B_n is unfaithful for large n* , Bull. Amer. Math. Soc. **25** (1991), 379–384.
- [13] N. Weinberg, *Sur l’équivalence libre des tresses fermées*, Comptes Rendus (Doklady) de l’Academie des Sciences de l’URSS **23** (1939), 215–216.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY,
HIROSHIMA 739-8526, JAPAN