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A survey: 4-fold symmetric quandle invariants of 3-manifolds

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Abstract

This report is a survey of two papers [N2, HN]. The first paper [N2] introduces 4-fold symmetric quandles. For a finite 4-fold symmetric quandle, we construct the 4-fold symmetric homotopy invariant of 3-manifolds. We classify 4-fold symmetric quandles herein, investigate their properties and explicitly determine the inner automorphism groups. We calculate the container of the 4-fold symmetric homotopy invariant. We also discuss 4-fold symmetric quandle cocycle invariants and coloring polynomials.

The second paper [HN] gives a topological interpretation of 4-fold symmetric quandle invariants. We demonstrate a close relation between a certain coloring and a homomorphism from the fundamental group of a 3-manifold. Further, we show that our 4-fold symmetric quandle homotopy invariants are at least as strong as Dijkgraaf-Witten invariants. Also, we reformulate the Chern-Simons invariant of $SL(2;\mathbb{C})$ as a symmetric quandle cocycle invariant via the extended Bloch group. As an application, for any odd m, the quandle homotopy invariant of the dihedral quandle R_m of links is equivalent to the Dijkgraaf-Witten invariant of $\mathbb{Z}/m\mathbb{Z}$ of the double branched covering spaces, which is a generalization of [H2].

1 Introduction

We review some invariants of links and of 3-manifolds using quandles. A quandle is a set with a certain binary operation like a group. Quandles are adapted to the oriented link theory. For *un*oriented links a symmetric quandle introduced by Kamada [Kam] is suitable. Given a quandle X, Fenn, Rourke and Sanderson [FRS1] defined the Rack space. Further, for oriented links, they proposed a quandle homotopy invariant valued in the group ring $\mathbb{Z}[\pi_2(BX)]$, where the space BX is a certain modification of the Rack space. The second author calculated $\pi_2(BX)$ for some quandles [N1]. On the other hand, quandle cocycle invariants of oriented links introduced by [CJKLS] are computable and practical. However, thier invariants are derived from the above homotopy invariant [FR]. In another direction, the first author [H] reformulated certain Dijkgraaf-Witten invariants of 3-manifolds [DW] as quandle cocycle invariants. To see this, she made use of the fact that any 3-manifold can be presented by some 4-fold irregular branched covering of S^3 along some link.

Our papers [N2, HN] generalize her reconstruction using symmetric quandles. Our aim is to construct an invariant of 3-manifolds using a certain quandle, and further to research the invariant. It is known [Apo, BP] that isotopy classes of 3-manifolds are in 1-1 correspondence with the set of links with "simple" monodromy representations onto $\mathfrak{S}_4$ modulo some link moves (see Figure 2). Roughly speaking, we define a 4-fold symmetric

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quandle and an invariant of 3-manifolds to be unchangeable under these link moves. Although the idea behind the definitions seems very naive, we show some interesting phenomena and results of the quandle and the invariant.

This report is organized as follows. In §2 we prepare some notation. In §3, we define a 4-fold symmetric quandle and introduce a 4-fold symmetric quandle homotopy invariant. In §5, we classify 4-fold symmetric quandles. In §6, we note the inner automorphism group. In §7, we give a topological interpretation of 4-fold symmetric quandle homotopy invariants. In §8, we compare the 4-fold symmetric quandle homotopy invariant with the Dijkgraaf-Witten invariant. In §9, we discuss 4-fold symmetric quandle cocycles invariants. In §10, we present some examples of 4-fold symmetric quandle cocycles.

We note a relation between these sections our papers. For more detail of §2, 3, 5, 6, 9, 10, see the paper [N2]. On the other hand, see [HN] for §7, 8, 10 and 12.

2 Review: symmetric quandle and labeled diagram

We review symmetric quandles and X_ρ-colorings introduced by Kamada [Kam]. A *symmetric quandle* is a triple of a set X, a binary operation $*$ on X and an involution $\rho : X \to X$ satisfying that for any $x, y, z \in X$, $x*x = x$, $(x*y)*z = (x*z)*(y*z)$, $\rho(x*y) = \rho(x) * y$, $(x * y) * \rho(y) = x$ (See also [KO]). For example, $\mathcal{S} := \{(ij) \in \mathfrak{S}_4\}$ with $x * y := y^{-1}xy$ and $\rho(x) = x$ is a symmetric quandle.

Let D be an unoriented link diagram on $\mathbb{R}^2$. For a symmetric quandle (X, ρ), an *X_ρ-coloring* of D is a map C : {the two orientations on arcs of D} $\to X$ satisfying

(X1) For the two orientations α_1, α_2 of the same arc as shown in Figure 1, the colors satisfy $C(\alpha_1) = \rho\big(C(\alpha_2)\big)$. (Hence, we will later draw the only one color of the two).
(X2) At each crossing such as the right hand side of Figure 1, the three orientations satisfy $C(\gamma) = C(\alpha) * C(\beta)$.

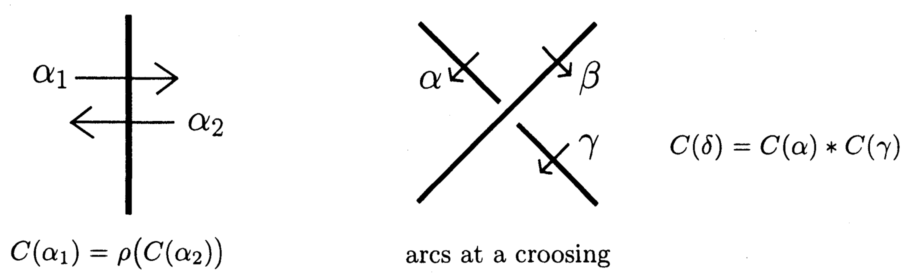


Figure 1: The condition of a symmetric coloring on semi-arcs and at each crossings

The conditions (X1)(X2) are well-defined by using the axioms of (X, ρ). Let $\mathrm{Col}_{X,\rho}(D) :=$ {X_ρ-colorings of D }. It is known [KO, Proposition 6.2] that for two diagrams D_1 and D_2 related by Reidemeister moves, there exists a bijection between $\mathrm{Col}_{X,\rho}(D_1)$ and $\mathrm{Col}_{X,\rho}(D_2)$.

We will interpret 3-manifolds [3] as $\mathcal{S}_{\rm id}$-colorings. It is well-known that any 3-manifold M can be obtained by a 4-fold irregular branched covering space of a link $L \subset S^3$ with its monodoromy $\phi : \pi_1(S^3 \backslash L) \to \mathfrak{S}_4$. Remark that ϕ is so-called "simple", i.e., ϕ is surjective and sends each meridian of L to a transposition in $\mathfrak{S}_4$. Let D_ϕ be a link diagram of L with the monodoromy ϕ, which we call *labeled diagram.* Then by Wirtinger presentation, we regard D_ϕ as an $\mathcal{S}_{\rm id}$-coloring.

It is known that MI and MII moves of labeled diagrams, shown in Figure 2, do not change the topological type of the covering space. Conversely, Apostolakis [Apo], Bobtcheva and Piergallini [BP] showed

Theorem 2.1. *([Apo]. A special case of [BP, Theorem 3]) Two 4-fold simple branched coverings of links $\subset S^3$ represent the same 3-manifold if and only if their associated labeled diagrams can be related by a finite sequence of MI, MII and Reidemeister moves on $\mathbb{R}^2$.*

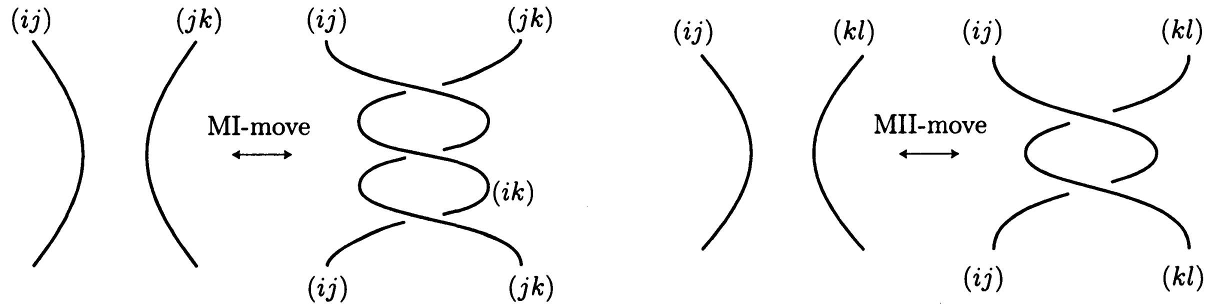


Figure 2: MI, II moves of labeled diagrams

Throughout this survey, the symbols $1 \leq i, j, k, l \leq 4$ mean distinct indices.

3 Definition: 4-fold symmetric quandle homotopy invariant

Hence, roughly speaking, if we can find a certain quandle whose colorings are unchangeable under the MI and MII moves, we obtain an invariant of 3-manifolds. Then we introduce such quandle as follows.

Definition 3.1. A *4-fold symmetric quandle* is a triple (X, p_X, ρ) satisfying
(F1) (X, ρ) is a symmetric quandle.
(F2) The map $p_X : X \to \mathcal{S}$ is a symmetric quandle epimorphism. For $(ij) \in \mathcal{S}$, let us denote the preimage $p_X^{-1}(ij) \subset X$ by X_{ij} later.
(F3) For any $x_{ij} \in X_{ij}$ and $y_{jk} \in X_{jk}$, it satisfies $x_{ij} * y_{jk} = \rho(y_{jk}) * x_{ij}$.
(F4) For any $z_{ij} \in X_{ij}$ and $w_{kl} \in X_{kl}$, it satisfies $z_{ij} * w_{kl} = z_{ij}$.

For a 4-fold symmetric quandle (X, p_X, ρ), notice that the epimorphism $p_X : X \to \mathcal{S}$ induces $(p_X)_* : \mathrm{Col}_{X,\rho}(D) \to \mathrm{Col}_{\mathcal{S},\rm id}(D)$. For a labeled diagram $D_\phi \in \mathrm{Col}_{\mathcal{S},\rm id}(D)$, we

[3]In this survey, 3-manifolds are assumed to be C^∞-smooth, connected, oriented and compact with no boundary.

denote the preimage $(p_X)_*^{-1}(D_\phi)$ by $\mathrm{Col}_{X,\rho}(D_\phi)$. An element of $\mathrm{Col}_{X,\rho}(D_\phi)$ is called an *X_ρ-coloring of D_ϕ*. The following proposition indicates that the axioms $(F3)$, $(F4)$ above correspond to MI, MII-moves, respectively.

Proposition 3.2. *Let (X, p_X, ρ) be a 4-fold symmetric quandle. If two labeled diagrams D_ϕ and $D'_{\phi'}$ are related by a finite sequence of MI, MII and Reidemeister moves on $\mathbb{R}^2$, then there is a bijection* $\mathrm{Col}_{X,\rho}(D_\phi) \xleftrightarrow{1:1} \mathrm{Col}_{X,\rho}(D'_{\phi'})$.

Proof. If $D_\phi \xleftrightarrow{\mathrm{MI}} D'_{\phi'}$, the required bijection follows from Figure 3 using the axiom $(F4)$.

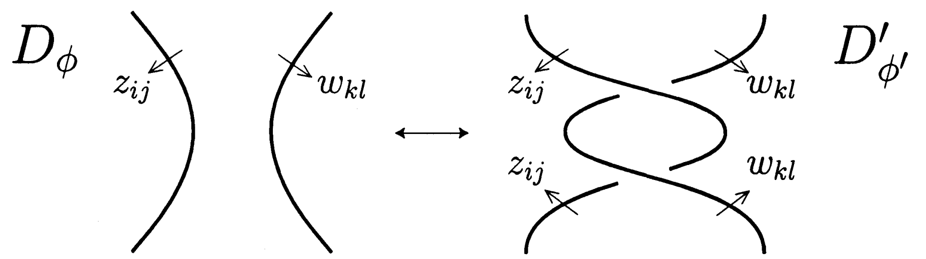


Figure 3: $X_{\widetilde{\rho}}$-colorings of D_ϕ and $D'_{\phi'}$ related by a single MII move

Similarly, if $D_\phi \xleftrightarrow{\mathrm{MII}} D'_{\phi'}$, the purpose results from Figure 4 and the axiom $(F3)$. $\square$

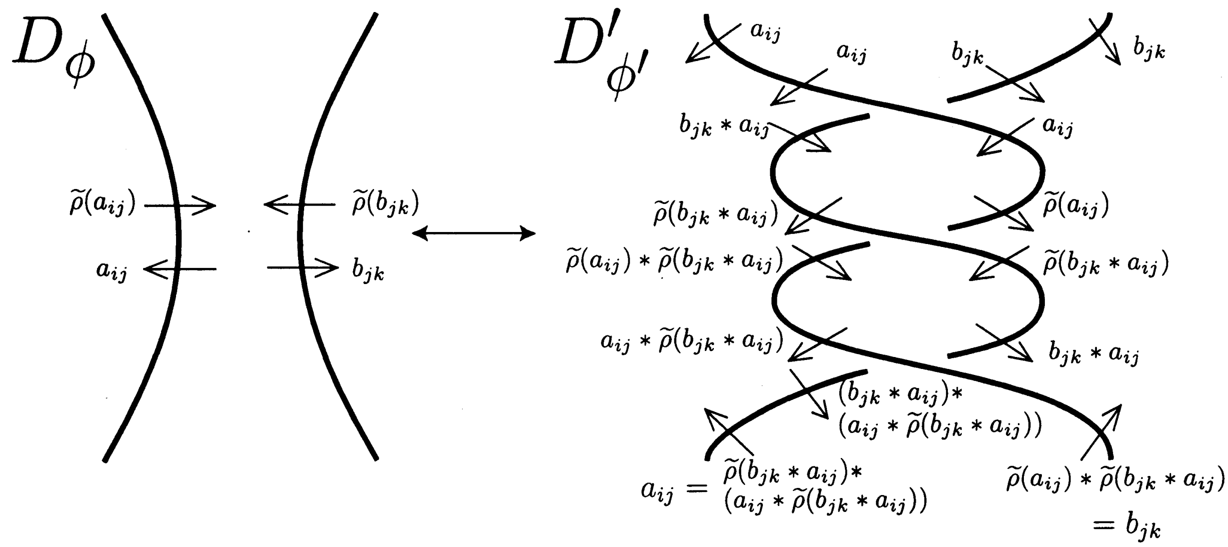


Figure 4: $X_{\widetilde{\rho}}$-colorings of D_ϕ and $D'_{\phi'}$ related by a single MI move

In addition, we will equip the invariant $\mathrm{Col}_{X,\rho}(D_\phi)$ with a grading using an Abel group $\Pi^{4\mathrm{f}}_{2,\widetilde{\rho}}(X)$ as follows. $\Pi^{4\mathrm{f}}_{2,\rho}(X)$ is a modification of Fenn, Rourke and Sanderson [FRS1] denoted by $\mathcal{D}(n, BX)$. $\Pi^{4\mathrm{f}}_{2,\rho}(X)$ is defined to be the set of all X_ρ-colorings of all diagrams in $\mathbb{R}^2$ subject to Reidemeister-I,II,III moves and *symmetric concordance relations* as shown in Figure 5 and to all X_ρ-colorings of all trefoils and of all Hopf links illustrated in Figure

6, where indicies i, j, k, l run over all distinct natural numbers of ≤ 4 and $x_{ij}, y_{jk}, z_{ij}, w_{kl}$ run over X_{ij}, X_{jk}, X_{ij}, X_{kl}, respectively. The set $\Pi^{4\mathrm{f}}_{2,\rho}(X)$ has a multiplication given by disjoint union which turns $\Pi^{4\mathrm{f}}_{2,\rho}(X)$ into an Abel group. From the definition of $\Pi^{4\mathrm{f}}_{2,\rho}(X)$ we have a canonical map:

$$\Xi^{4\mathrm{f}}_X(D_\phi; \bullet) : \mathrm{Col}_{X_\rho}(D_\phi) \longrightarrow \Pi^{4\mathrm{f}}_{2,\rho}(X), \tag{1}$$

that is, $\Xi^{4\mathrm{f}}_X(D; \bullet)$ maps an X_ρ-coloring C to the canonical class $[C] \in \Pi^{4\mathrm{f}}_{2,\rho}(X)$.

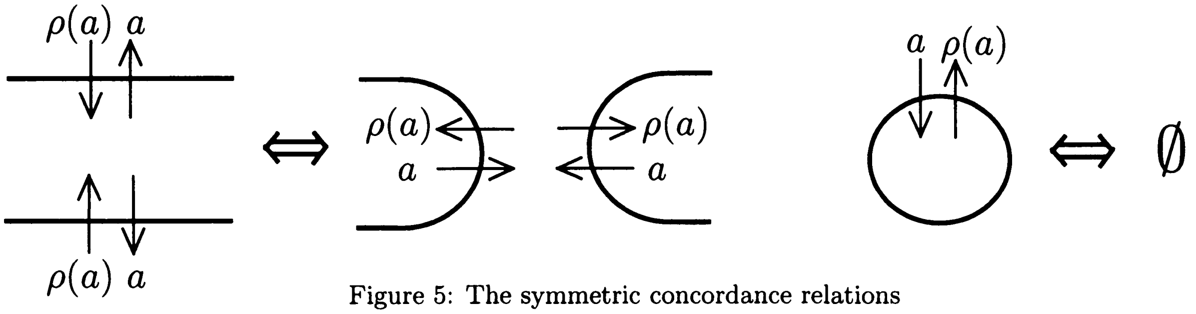


Figure 5: The symmetric concordance relations

Figure 6: X_ρ-colorings of trefoil and Hopf link

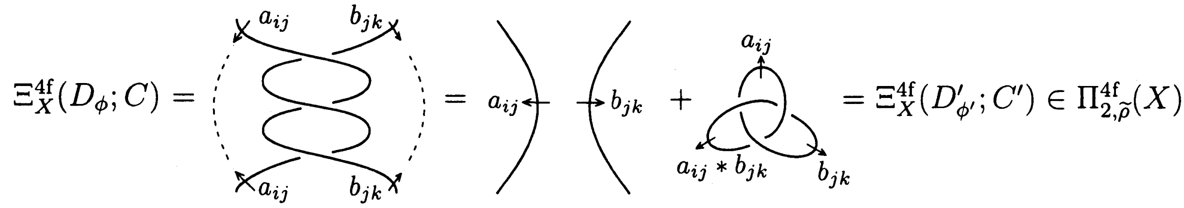
Definition 3.3. Let X be a finite 4-fold symmetric quandle. Let D_ϕ be a labeled diagram. Then a 4-*fold symmetric quandle homotopy invariant* of D_ϕ is the expression

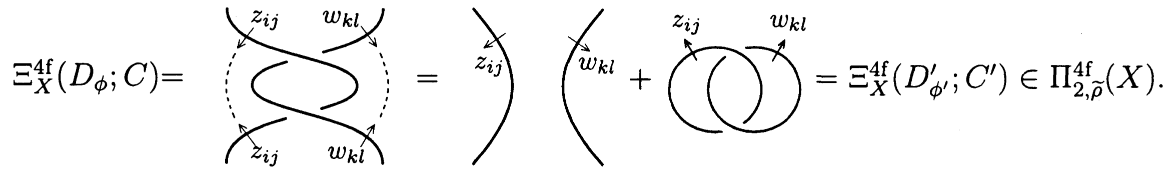
$$\Xi^{4\mathrm{f}}_X(D_\phi) := \sum_{C \in \mathrm{Col}_{X,\rho}(D_\phi)} \Xi^{4\mathrm{f}}_X(D_\phi; C) \in \mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(X)].$$

Theorem 3.4. *Let D_ϕ and $D'_{\phi'}$ be labeled diagrams related by a finite sequences of MI, MII and Reidemeister moves. For a finite* 4*-fold symmetric quandle X, $\Xi^{4\mathrm{f}}_X(D_\phi) = \Xi^{4\mathrm{f}}_X(D'_{\phi'}) \in \mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(X)]$. In particular, for a 3-manifold M presented by D_ϕ, the 4-fold symmetric quandle homotopy invariant $\Xi^{4\mathrm{f}}_X(D_\phi) \in \mathbb{Z}[\Pi^{4\mathrm{f}}_{2,\rho}(X)]$ is an invariant of M.*

Therefore we often denote the invariant of a 3-manifold M by $\Xi^{4\mathrm{f}}_X(M)$.

Proof. It is suffices to show the invariance under MI, MII-moves, which follows from





where we use concordance relations along the dashed lines in the second equalities. □

4 Some Questions about 4-fold symmetric quandle

Although we have obtained an invariant of 3-manifolds, the definitions of the 4-fold symmetric quandle (homotopy invariant) seem teleological and abstract. Particularly, it is a problem to explicitly determine what the container $\Pi^{4f}_{2,\widetilde{\rho}}(X)$ is. So we pose some questions:

- How broad is concretely the class of 4-fold symmetric quandles? (see §5)
- How large is the container of our invariant? (see §6)
- Is our invariant related to other invariants? (see §8)
- How do we compute the 4-fold symmetric quandle homotopy invariants? (see §9)
- Do 4-fold symmetric quandle homotopy invariants have an application? (see §12)

From now on, we will answer these questions in turn.

5 Classification of 4-fold symmetric quandles

We consider a pair of a group G and its central element $c \in G$ such that $c^2 = e$. Such a pair is called *cored group*. Given a cored group (G, c), we give an example of 4-fold symmetric quandles. Further, we classify 4-fold symmetric quandles (Theorem 5.2).

Example 5.1. Fix a cored group (G, c). Putting $T_{12} := \{(i,j) \in \mathbb{Z}^2 | 1 \leq i, j \leq 4, \ \ i \neq j\}$, we define $\widetilde{G}_c$ to be a quotient set $G \times T_{12}/\sim$, where the equivalent $\sim$ on $G \times T_{12}$ is defined by $(g, i, j) \sim (g^{-1}c, j, i)$, for any $(i,j) \in T_{12}$ and $g \in G$. Further, we equip $\widetilde{G}_c$ with an operation $* : \widetilde{G}_c \times \widetilde{G}_c \to \widetilde{G}_c$ defined by Table 1 below. Define $\rho : \widetilde{G}_c \to \widetilde{G}_c$ by $\rho(g, i, j) = (g \cdot c, i, j)$. Further, we remark a projection $p_{\widetilde{G}_c} : \widetilde{G}_c \to \mathcal{S}$ which sends (g, i, j) to $(ij) \in \mathcal{S}$. Then the triple $(\widetilde{G}_c, p_{\widetilde{G}_c}, \rho)$ is a 4-fold symmetric quandle.

$(g,\ t)$	$(g',\ t')$	$(g,\ t) * (g',\ t')$
(g,i,j)	(g',i,j)	$(g'g^{-1}g',i,j)$
(g,i,j)	(g',j,k)	(gg',i,k)
(g,i,j)	(g',k,l)	(g,i,j)

Table 1: The binary operation $*$ in $\widetilde{G}_c$. In each line i,j,k,l are all distinct. $t,t' \in T_{12}$.

Theorem 5.2. *Let (X, p_X, ρ) be a 4-fold symmetric quandle. Then there is a cored group (G,c) related to X by a 4-fold symmetric quandle isomorphism $\widetilde{G}_c \cong X$.*

Moreover, we show the following corollary (see [N2] for notation):

Corollary 5.3. *The functor $\mathcal{T}$ which takes a cored group (G,c) to $\widetilde{G}_c$ gives a category equivalence between the category of cored groups and a category of (based) 4-fold symmetric quandles. Moreover, the restriction of the functor to the category of groups* **Grp** *induces the category equivalence between* **Grp** *and a category of (based) 4-fold symmetric quandles with $\rho = \mathrm{id}_X$.*

Then the results can be summarized as follows

$$
\begin{array}{ccccccc}
\begin{pmatrix}\text{4-fold symmetric}\\ \text{quandles of } \rho=\mathrm{id}\end{pmatrix} & \subset & \begin{pmatrix}\text{4-fold symmetric}\\ \text{quandles}\end{pmatrix} & \subset & \begin{pmatrix}\text{symmetric}\\ \text{quandles}\end{pmatrix} & \subset & \begin{pmatrix}\text{quandles}\end{pmatrix} \\
\cong & & \cong & & & & \\
\begin{pmatrix}\text{groups}\end{pmatrix} & \subset & \begin{pmatrix}\text{cored groups}\end{pmatrix} & & & &
\end{array}
$$

By the classification of Theorem 5.2, we mainly deal with quandles of the form $\widetilde{G}_c$.
Lastly, we comment some properties of $\widetilde{G}_c$.

Proposition 5.4. *For any $x,y \in \widetilde{G}_c$, there exist $a,b \in \widetilde{G}_c$ s.t. $(x*a)*b = y$. In particular, $\widetilde{G}_c$ is connected.*

Proposition 5.5. *The quandle $\widetilde{G}_c$ is of type 4, i.e., $^{\forall}x,y \in \widetilde{G}_c$, $(((x*y)*y)*y)*y = x$. Further, $\widetilde{G}_c$ is of type 2, if and only if $c = e$.*

6 Inner automorphism group $\mathrm{Inn}(\widetilde{G}_c)$

Given a symmetric quandle (X,ρ), for any $z \in X$, $(\bullet * z) : X \to X$ is bijective by the axioms of (X,ρ). Then we denote by $\mathrm{Inn}(X)$ a subgroup of $\mathfrak{S}_{|X|}$ generated by the right actions $(\bullet * z)$. It is known [Joy] that any connected quandle X is determined by the inner automorphism group $\mathrm{Inn}(X)$.

For our 4-fold symmetric quandle $\widetilde{G}_c$ of a finite cored group (G,c), we conclude

Theorem 6.1. *Let (G,c) be a finite cored group, and let $Z(G)$ be the center of G. Then* $\mathrm{Inn}(\widetilde{G}_c)$ *is isomorphic to a quotient group $I_{G,c}/Z_{G,c}$, where*

$$I_{G,c} = \{(x,y,z,w;\sigma) \in G^4 \rtimes \mathfrak{S}_4 \mid c^{\frac{\mathrm{sgn}(\sigma)-1}{2}} xyzw \in [G,G] \},$$
$$Z_{G,c} = \{(z,z,z,z;e) \in G^4 \rtimes \mathfrak{S}_4 \mid z^4 \in [G,G] \ , z \in Z(G) \}. \tag{2}$$

This theorem have some corollaries: we estimate the container of our invariant:

Corollary 6.2. *Given a finite cored group (G,c), $\Pi^{4\mathrm{f}}_{2,\widetilde{\rho}}(\widetilde{G}_c)$ is a finite Abel group whose elements are annihilated by $2^{12} \cdot 3^4 \cdot |G|^{12} \cdot |[G,G]|^4$.*

Corollary 6.3. *When $(G,c) = (\mathbb{Z}/2\mathbb{Z}, 0)$, $\Pi^{4\mathrm{f}}_{2,\widetilde{\rho}}(\widetilde{G}_c) \cong \mathbb{Z}/2\mathbb{Z}$ whose generator is presented by the real projective space $\mathbb{R}P^3$.*

To prove these corollaries, we use some results in [N1]; we view a perspective that $\Pi^{4\mathrm{f}}_{2,\rho}(\widetilde{G}_c)$ is a quotient of a homotopy group $\pi_2(BX)$.

Further, we give another corollary of "second quandle homology groups" $H_2^Q(X;\mathbb{Z})$(see [CJKLS] for the definition): following the covering theory of Eisermann [Eis2], for a quandle X of type 2, $H_2^Q(X;\mathbb{Z})$ is computable from the presentation of $\mathrm{Inn}(X)$. Therefore we obtain

Corollary 6.4. *Given a finite group G, the second quandle homology $H_2^Q(\widetilde{G}_e;\mathbb{Z})$ is given by $\mathrm{Ab}\big(T_{G,e}/Z_{G,e}\big)$. Here $Z_{G,e}$ is given in (2), and*

$$T_{G,e} = \big\{(x,x,z,w;\sigma) \in G^4 \rtimes \mathfrak{S}_4 \mid x^2 zw \in [G,G], \ \sigma \in \{e, \ (12)(34)\}\big\}, \tag{3}$$

Consequently, if we know $[G,G]$ and $Z(G)$, we can calculate $H_2^Q(\widetilde{G}_e;\mathbb{Z})$. For instance,

Example 6.5. Let $G = \mathbb{Z}/m\mathbb{Z}$. We decompose $m = 2^k \cdot n$, where n is odd.

$$H_2^Q(\widetilde{G}_e;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & (m: \ \text{odd}), \\ \mathbb{Z}/2^{k-1}n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}^2, & (k = 2 \cdot n, \ \text{or} \ \ 4 \cdot n), \\ \mathbb{Z}/2^{k-1}n\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, & (m = 2^k \cdot n, \ k > 2). \end{cases}$$

Example 6.6. Let G be a perfect group: $G = [G,G]$. Then $H_2^Q(\widetilde{G}_e;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 6.7. Let G be a quaternion group Q_8 of order 8. Then $H_2^Q(\widetilde{G}_e;\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^5$.

7 $\widetilde{G}_c$-colorings, fundamental quandle and class of 3-manifold

7.1 $\widetilde{G}_c$-colorings of a 3-manifold

We give a topological interpretation of $\widetilde{G}_c$-colorings of a labeled diagram.

Theorem 7.1. *Let (G, c) be a cored group, and D_ϕ a labeled diagram which presents a 3-manifold M. Then there is a canonical bijection*

$$\mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi) \simeq G^3 \times \mathrm{Hom}_{grp}(\pi_1(M), G). \tag{4}$$

This is a slight generalization of [H, Proposition 3.5]. Namely, restricting to the case $c = e$, the statement above is reduced to be the same with the proposition.

As a result, for a finite cored group (G, c), the cardinally of $\widetilde{G}_c$-colorings is a classical invariant, and does not depend on the choice of central element $c \in G$. Hence, for a search of a new invariant, our next step is to study the group $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ (see §9, 10,11).

Incidentally, we give a topological interpretation of colorings of core quandles. For a group G, the *core quandle* Q_G is a set G with a symmetric quandle operation of $g * h = hg^{-1}h$ and $\rho = \mathrm{id}_G$.

Corollary 7.2. *Let D be a link diagram of a link L, Q_G a core quandle of a group G, and M_L the double branched covering space of L. Then the set of the colorings $\mathrm{Col}_{Q_G,\mathrm{id}}(D)$ is in a 1:1 correspondence with $G \times \mathrm{Hom}(\pi_1(M_L), G)$.*

Proof. By definitions a subquandle $\{(g, 1, 2) \in \widetilde{G}_c \mid g \in G\}$ is isomorphic to Q_G. Moreover, we can regard D as a labeled diagram whose all arcs are labeled by $(12) \in \mathcal{S}$ shown as Figure 7. Hence $G^2 \times \mathrm{Col}_{Q_G,\mathrm{id}}(D) \simeq \mathrm{Col}_{\widetilde{G}_e,\mathrm{id}}(D_\phi) \overset{\mathrm{Thm.7.1}}{\simeq} G^3 \times \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M_L), G)$. □

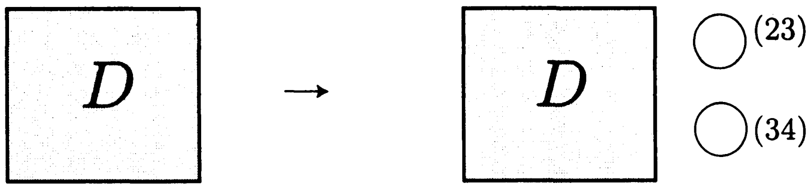


Figure 7: A labeled diagram D_ϕ from a link diagram D.

7.2 A fundamental symmetric quandle of a 3-manifold

We introduce a fundamental quandle and a fundamental class of a 3-manifold. For this, given a link $L \subset S^3$, we recall the symmetric link quandle $SQ(L)$ introduced by Kamada [Kam], which is, roughly speaking, the conjugacy class of $\pi_1(S^3 \setminus L)$ including meridians of L. Kamada showed a canonical bijection $\mathrm{Col}_{X,\rho}(D) \simeq \mathrm{Hom}_{\mathrm{sQnd}}(SQ(L), X)$. When $X = \mathcal{S}$, we can regard a labeled diagram D_ϕ as the associated quandle epimorphism $\phi : SQ(L) \to \mathcal{S}$. We consider the following relations on $SQ(L)$:

$$R_L^{3,\phi} := |x_{ij} * y_{jk} = \rho(y_{jk}) * x_{ij} \quad (x_{ij} \in \phi^{-1}(ij),\ y_{jk} \in \phi^{-1}(jk)).\rangle$$

$$R_L^{4,\phi} := |z_{ij} * w_{kl} = z_{ij} \quad (z_{ij} \in \phi^{-1}(ij),\ w_{kl} \in \phi^{-1}(kl))\rangle$$

Then, we consider the quotient symmetric quandle $SQ(L)/\langle R_L^{3,\phi}, R_L^{4,\phi}\rangle$. It goes without saying that this quandle is a 4-fold symmetric quandle.

Corollary 7.3. *For a 3-manifold M presented by a labeled diagram D_ϕ, $\widetilde{G(M)}_{c(M)} \cong SQ(L)/\langle R_L^{3,\phi}, R_L^{4,\phi}\rangle$ as a quandle isomorphism. Here the cored group $\big(G(M), c(M)\big) = (\pi_1(M) \oplus \mathbb{Z}/2\mathbb{Z}, (e,1))$.*

This immediately follows from Yoneda's embedding. Anyway, we call the quandle $SQ(L)/\langle R_L^{3,\phi}, R_L^{4,\phi}\rangle$ *a fundamental symmetric quandle* of M. We denote it by $SQ(M)$. Let us focus on a class of the natural transformations: by Yoneda's lemma, we have a bijection

$$\mathrm{Nat}(\mathrm{Hom}_{\mathrm{4sQnd}}(SQ(M), \widetilde{\bullet}), \Pi_{2,\rho}^{\mathrm{4f}}(\widetilde{\bullet})) \simeq \Pi_{2,\rho}^{\mathrm{4f}}(SQ(M)),$$

which sends $\Xi_{\widetilde{\bullet}}^{\mathrm{4f}}(D_\phi; \dagger)$ to $\Xi_{SQ(M)}^{\mathrm{4f}}(D_\phi; \mathrm{id}_{SQ(M)})$, where $\mathrm{id}_{SQ(M)}$ is the identity map of $SQ(M)$. We call $\Xi_{SQ(M)}^{\mathrm{4f}}(D_\phi; \mathrm{id}_{SQ(M)})$ *a fundamental class* of M. By the naturality, we thus reformulate the 4-fold symmetric quandle homotopy invariant by

$$\Xi_{\widetilde{G}_c}^{\mathrm{4f}}(M) = \sum_{F \in \mathrm{Hom}_{\mathrm{4sQnd}}(SQ(M), \widetilde{G}_c)} F_*\big(\Xi_{SQ(M)}^{\mathrm{4f}}(D_\phi; \mathrm{id}_{SQ(M)})\big) \in \mathbb{Z}[\Pi_{2,\rho}^{\mathrm{4f}}(\widetilde{G}_c)]. \tag{5}$$

In summary, the study of the 4-fold symmetric quandle homotopy invariant of M is a research of $\Pi_{2,\rho}^{\mathrm{4f}}(SQ(M))$ and of the fundamental class with using the relativity toward other 4-fold symmetric quandles $\widetilde{G}_c$.

8 Toward Dijkgraaf-Witten invariant

In [H], the second author reformulated some Dijkgraaf-Witten invariant [DW] as a cocycle invariant of $\widetilde{G}_e$. However, her work needs a certain condition of G. For example, the reformulation does not hold for $G = \mathbb{Z}/6\mathbb{Z}$. To settle the condition, in [HN], we discuss oriented bordism groups of G, and show that any Dijkgraaf-Witten invariant is derived from the 4-fold symmetric homotopy invariant.

8.1 Preliminaries: Bordism Dijkgraaf-Witten invariant

Let (G, c) be a cored group and let $n \in \mathbb{Z}$ be ≥ 3. In this subsection, we make a modification of Dijkgraaf-Witten invariant in the view of an oriented bordism group of (G, c). We consider a pair of an n-manifold M without boundary and a homomorphism $\pi_1(M) \to G$. Then a set $\Omega_n(G, c)$ is defined to be the quotient of such pairs $(M,\ \pi_1(M) \to G)$ subject to the following (G, c)-bordant equivalence. Such a pair $(M,\ f : \pi_1(M) \to G)$ is (G, c)-*bordant*, if there exists an $(n+1)$-manifold W, two homomorphisms $\bar{f} : \pi_1(W) \oplus \mathbb{Z}/2\mathbb{Z} \to G$ and $\tilde{f} : \pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$ such that $\bar{f}(e, 1) = c \in G$, the boundary is $\partial W = M$, and $f = \bar{f} \circ \big((i_M)_* \oplus \tilde{f}\big)$, where $i_M : M \to W$ is a natural inclusion. Further, $\Omega_n(G, c)$ has an Abel group structure by connected sum, that is,

$$(M_1,\ f_1 : \pi_1(M_1) \to G) \cdot (M_2,\ f_2 : \pi_1(M_2) \to G) := (M_1 \sharp M_2,\ f_1 * f_2 : \pi_1(M_1 \sharp M_2) \to G),$$

where $f_1 * f_2$ is the free product of f_1 and f_2. The inverse element of $(M,\ f : \pi_1(M) \to G)$ is $(-M,\ f : \pi_1(M) \to G)$, where $-M$ stands for M with the opposite orientation.

Then *bordism Dijkgraaf-Witten invariant* of a closed n-manifold N is defined by

$$DW_{\Omega}^{G_c}(N) := \sum_{f \in \mathrm{Hom}_{\mathrm{grp}}(\pi_1(N),G)} [(N, f : \pi_1(N) \to G)] \in \mathbb{Z}[\Omega_n(G,c)]. \tag{6}$$

Remark 8.1. When $c = e$, it easily can be verified that the group $\Omega_n(G,e)$ coincides with the usual oriented bordism group of the Eilenberg-MacLane space $K(G;1)$, using the obstruction theory and $\pi_i(K(G;1)) \cong 0$ $(i \geq 2)$ (cf. [Ati]). Moreover, if $n = 3$ and $c = e$, we can see $\Omega_3(G,e) \cong \Omega_3(K(G;1)) \cong H_3(K(G;1);\mathbb{Z})$ by Atiyah-Hirzebruch spectral sequence. Then, $DW_{\Omega}^{G_c}(M)$ is equivalent to the original Dijkgraaf-Witten invariant [DW].

8.2 From $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$ to the oriented bordism group $\Omega_3(G,c)$

Returning into our quandle homotopy invariant, our goal is to obtain an epimorphism $\Phi_{\Pi\Omega} : \Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c) \to \Omega_3(G,c)$, which implies that our 4-fold symmetric quandle homotopy invariant is at least as strong as the bordism Dijkgraaf-Witten invariant (Theorem 8.3). For this, the following is a key lemma:

Lemma 8.2. *Assume that two $\widetilde{G}_c$-colorings $C_1 \in \mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi)$ and $C_2 \in \mathrm{Col}_{\widetilde{G}_c,\rho}(D'_{\phi'})$ are related by either Reidemeister moves, MI, MII moves or symmetric concordance relations. Let C_i present a 3-manifold M_i with $\pi_1(M_i) \to G$ for $i = 1,2$. Then their connected sum $\big(-M_1\sharp M_2,\ \pi_1(M_1\sharp M_2) \to G\big)$ is (G,c)-concordant.*

The proof is reduced to a construction of a 4-manifold W which bounds $-M_1\sharp M_2$. Roughly, such W is obtained from a 4-fold branched covering of a saddle which bounds the symmetric concordance relation in Figure 5.

Let us explain Theorem 8.3. Put a composite map $\mathrm{Col}_{\widetilde{G}_c,\rho}(D_\phi) \simeq G^3 \times \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M),G)$ $\xrightarrow{\mathrm{proj}} \mathrm{Hom}_{\mathrm{grp}}(\pi_1(M),G)$, where the first map is the bijection in Theorem 7.1. Moreover, recall the definition of $\Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c)$. Then, by running over all $\widetilde{G}_c$-coloring of all labeled diagram and all homomorphism $f : \pi_1(M) \to G$ of all 3-manifolds, by Lemma 8.2, the composite maps induce a map

$$\Phi_{\Pi\Omega} : \Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c) \longrightarrow \Omega_3(G,c). \tag{7}$$

By a certain presentation of the connected sum of labeled diagrams, the map is an epimorphism by construction. In conclusion, when G is finite, we see

Theorem 8.3. *Let (G,c) be a finite cored group. There exists an epimorphism $\Phi_{\Pi\Omega} : \Pi_{2,\rho}^{4\mathrm{f}}(\widetilde{G}_c) \longrightarrow \Omega_3(G,c)$. Moreover, the bordism Dijkgraaf-Witten invariant is derived from the 4-fold symmetric quandle homotopy invariant by the formula*

$$|G|^3 \cdot DW_{\Omega}^{G_c}(M) = \Phi_{\Pi\Omega}\big(\Xi_{\widetilde{G}_c}^{4\mathrm{f}}(M)\big) \in \mathbb{Z}[\Omega_3(G,c)].$$

Conversely, we pose a problem.

Problem 8.4. Are 4-fold symmetric quandle homotopy invariants of (G, c) stronger than Dijkgraaf-Witten invariants?

We suggest negative approaches to answer the question. Hence, if we expect the equivalence of the two invariants, it suffices to show that the map (7) is isomorphic. Further, this would come down to a problem whether any 4-manifold with boundaries is a 4-fold simple branched covering branched over a locally flat surface in a 4-ball or not. For reference, we remark the result of Iori and Piergallini [IP], which says that any closed PL 4-manifold is a **5-fold** simple branched covering of S^4 branched over a locally flat surface in S^4.

9 4-fold symmetric quandle cocycle invariant

However, it is difficult to directly calculate the 4-fold symmetric homotopy invariants valued in $\Pi_{2,\rho}^{4\mathrm{f}}(X)$, since so is the computation of $\Pi_{2,\rho}^{4\mathrm{f}}(X)$. For the reduction of the invariant to a computable invariant, we introduce 4-fold symmetric quandle cocycles, modifying symmetric quandle cocycles introduced by Kamada and Oshiro [Kam, KO]. Inspired by them, we will define the 4-fold symmetric quandle cocycle invariant of 3-manifolds. Further, we show that the symmetric cocycle invariants are derived from 4-fold symmetric homotopy invariants (Proposition 9.3).

Let us define the 4-fold symmetric quandle cocycle. For a 4-fold symmetric quandle (X, ρ), an (X, ρ)-*set* is a set Λ equipped with a map $* : \Lambda \times X \longrightarrow \Lambda$ satisfying $(\lambda * x) * x' = (\lambda * x') * (x * x')$ and $(\lambda * x) * \rho(x) = \lambda$ for any $\lambda \in \Lambda$ and $x, x' \in X$. For an Abel group A and an (X, ρ)-set Λ, a map $\theta : \Lambda \times X \times X \longrightarrow A$ is called a *4-fold symmetric quandle 2-cocycle*, if it satisfies the following five conditions:

(C1) $^\forall(\lambda, x, y, z) \in \Lambda \times X^3$,

$$\theta(\lambda, y, z)^{-1} \cdot \theta(\lambda * x, y, z) \cdot \theta(\lambda, x, z) = \theta(\lambda * y, x * y, z) \cdot \theta(\lambda, x, y) \cdot \theta(\lambda * z, x * z, y * z)^{-1}.$$

(C2) $^\forall(\lambda, x) \in \Lambda \times X, \ \ \theta(\lambda, x, x) = 1_A.$

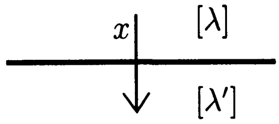
(C3) $^\forall(\lambda, x, y) \in \Lambda \times X^2, \ \ \theta(\lambda, x, y) = \theta(\lambda * x, \rho(x), y)^{-1}, \quad \theta(\lambda, x, y) = \theta(\lambda * y, x * y, \rho(y))^{-1}.$

(C4) $^\forall\lambda \in \Lambda, \ \ x_{ij} \in X_{ij}, \, y_{jk} \in X_{jk}, \ \ \theta(\lambda, x_{ij}, y_{jk}) \cdot \theta(\lambda, y_{jk}, x_{ij} * y_{jk}) \cdot \theta(\lambda, x_{ij} * y_{jk}, x_{ij}) = 1_A.$

(C5) $^\forall\lambda \in \Lambda, \ \ z_{ij} \in X_{ij}, w_{kl} \in X_{kl}, \, \theta(\lambda, z_{ij}, w_{kl}) \cdot \theta(\lambda, w_{kl}, z_{ij}) = 1_A.$

Remark 9.1. For a symmetric quandle (X, ρ), if the map $\theta : \Lambda \times X^2 \to A$ satisfies (C1)$\sim$ (C3), then θ is a symmetric quandle 2-cocycle introduced by Kamada and Oshiro [KO]. However, in general, it is difficult to find a presentation of a symmetric quandle 2-cocycle.

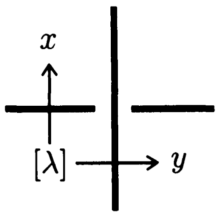
We prepare X_Λ-colorings. Let D_ϕ be a labeled diagram. An *X_Λ-coloring of D_ϕ* is defined to be an X_ρ-coloring of D_ϕ with an assignment of elements of Λ to each complementary regions of D such that, for each regions separated by the arc, the colors satisfies the following figure.



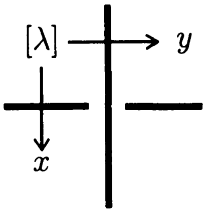
$\boxed{\lambda * x = \lambda'. \ (\lambda, \lambda' \in \Lambda)}$

Fix $\lambda_0 \in \Lambda$. An X_Λ-coloring of D_ϕ is *at* λ_0, if this satisfies that the unbounded region contain the infinity point is assigned by λ_0. Denote by $\mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0}$ a set of all X_Λ-coloring of D_ϕ at λ_0. We can obtain a bijection between $\mathrm{Col}_{X,\rho}(D_\phi)$ and $\mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0}$ (see [KO, Proposition 6.1]).

For a 4-fold symmetric quandle 2-cocycle θ, we will provide X_Λ-colorings of D at λ_0 with a grading by A. Let C be an X_Λ-coloring of D at λ_0. For a crossing v of C, there are four complementary regions of D around v. Choose one of the four regions. If the region is assigned by $\lambda \in \Lambda$, then the *weight of* v is defined to be $\theta(\lambda, x, y)^\epsilon \in A$, where x, y and the sign $\epsilon \in \{+1, \ -1\}$ are determined by the orientations shown as Figure 8.



$\boxed{\theta(\lambda, x, y)^{+1}}$



$\boxed{\theta(\lambda, x, y)^{-1}}$

Figure 8: Weight of a crossing v

It is known [KO, Lemma 6.2] that the weight of any crossing does not depend on the choice of four complementary regions and their orientations by (C1)(C2)(C3). Now we give $\Phi_\theta(D; C)_{\lambda_0} \in A$ by the sum of the weights of all crossing of D. Then the sum can be considered as a map

$$\Phi_\theta(D_\phi; \bullet)_{\lambda_0} : \ \mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0} \longrightarrow A. \tag{8}$$

Definition 9.2. Let X be a finite 4-fold symmetric quandle, let Λ be an (X, ρ)-set, and let D_ϕ be a labeled diagram. Fix $\lambda_0 \in \Lambda$. For a 4-fold symmetric quandle 2-cocycle θ, the *4-fold symmetric quandle cocycle invariant* of D_ϕ [4] is

$$\Phi_\theta(D_\phi)_{\lambda_0} = \sum_{C \in \mathrm{Col}_{X_\Lambda}(D_\phi)_{\lambda_0}} \Phi_\theta(D_\phi; C)_{\lambda_0} \in \mathbb{Z}[A].$$

[4] If X transitively acts on Λ, the value $\Phi_\theta(D_\phi)_{\lambda_0}$ does not depend of the choice of λ_0. To be precise, if λ_0, λ_0' are related by $\lambda_0' = (\cdots(\lambda_0 * x_1)\cdots * x_{n-1}) * x_n$ for some $x_1, \ldots, x_n \in X$, then $\Phi_\theta(D_\phi)_{\lambda_0} = \Phi_\theta(D_\phi)_{\lambda_0'}$.

This is a topological invariant of 3-manifolds, and is derived from the 4-fold quandle homotopy invariant as follows:

Proposition 9.3. *Let (X, p_X, ρ) be a finite 4-fold symmetric quandle, and Λ an $(X, \widetilde{\rho})$-set. We fix a 4-fold symmetric quandle 2-cocycle $\theta \in \mathrm{Map}(\Lambda \times X \times X, A)$. Then there exists a homomorphism $\mathcal{H}_\theta : \Pi_{2,\widetilde{\rho}}^{4\mathrm{f}}(X) \to A$ satisfying that for any labeled diagram D_ϕ,*

$$\mathcal{H}_\theta\big(\Xi_X^{4\mathrm{f}}(D_\phi)\big) = \Phi_\theta(D_\phi)_{\lambda_0} \in \mathbb{Z}[A]. \tag{9}$$

In particular, $\Phi_\theta(D_\phi)_{\lambda_0}$ is a topological invariant of the 3-manifold M presented by D_ϕ.

Notice that the axioms $(C4)$ (resp. $(C5)$) means that weights of the $\widetilde{G}_c$-colorings of trefoils (resp. of Hopf link) are zero. This $\mathcal{H}_\theta$ is obtained from the maps (8) by running over all $\widetilde{G}_c$-colorings of all labeled diagrams.

Remark 9.4. By Theorem 6.1, if $A \otimes_{\mathbb{Z}} \mathbb{Z}/6|G|\mathbb{Z} \cong 0$, say $A = \mathbb{Q}$, then the 4-fold symmetric quandle cocycle invariant of $\widetilde{G}_c$ is trivial.

We give two examples of 4-fold symmetric quandle invariants. In §10, we first discuss some 4-fold symmetric quandle invariants in the case where Λ is a single point. The second example is a reconstruction of the Chern-Simons invariant (see §10), which follows the work of the first author [H].

10 4-fold symmetric cocycles with the trivial coefficient

In this section, we assume that Λ is a single point and $c = e \in G$. We show that every 4-fold symmetric quandle cocycle invariant of such Λ can be computable without knowing the presentation of the 4-fold symmetric cocycle (Theorem 10.1).

We briefly review the coloring polynomial of [Eis1]. Let (X, x_0) be a quandle of type 2 with a point. Assume that the action of $\mathrm{Inn}(X)$ on X is transitive. We let $Z(x_0) \subset \mathrm{Inn}(X)$ be the stabilizer subgroup of $X \curvearrowleft \mathrm{Inn}(X)$. Let K be a knot, m_K a meridian of K, and l_K a longitude of K. Eisermann introduced the following invariant of knots:

$$\mathcal{P}_X^{x_0}(K) := \sum_{\gamma \in \mathrm{Hom}_{\mathrm{grp}}^{m_K, x_0}(\pi_1(S^3 \setminus K), \mathrm{Inn}(X))} \gamma(l_K) \in \mathbb{Z}[\mathrm{Inn}(X)], \tag{10}$$

where $\mathrm{Hom}_{\mathrm{grp}}^{m_K, x_0}(\pi_1(S^3 \setminus K), \mathrm{Inn}(X))$ stands for a set of the homomorphisms which sends m_K to $(\bullet * x_0) \in \mathrm{Inn}(X)$. It is shown that $\mathcal{P}_X^{x_0}(K)$ is the universal invariant among the original quandle cocycle invariant of knots. Also, note that l_K lies in $[\pi_1(S^3 \setminus K), \pi_1(S^3 \setminus K)]$ and commutes with m_K. Hence, we may regard $\gamma(l_K) \in Z(x_0) \cap [\mathrm{Inn}(X), \mathrm{Inn}(X)]$.

Next, we consider our 4-fold symmetric quandle $\widetilde{G}_e$. For short, we denote $(e, (1, 2)) \in \widetilde{G}_e$ by e_{12}. When G is finite, by Theorem 6.1, the above container $Z(e_{12}) \cap [\mathrm{Inn}(\widetilde{G}_e), \mathrm{Inn}(\widetilde{G}_e)]$ is isomorphic to $T_{G,e}/Z_{G,e}$ in (10), where $T_{G,e}$ is given in (3).

Recall that M is presented by a 3-fold branched covering of a knot K with the monodromy $\phi : \pi_1(S^3 \setminus K) \to \mathfrak{S}_4$. For applying the coloring polynomials to labeled diagrams, we consider the $\mathbb{Z}/2\mathbb{Z}$-Abelinization $H_G := \mathrm{Ab}(T_{G,e}/Z_{G,e})/2\mathrm{Ab}(T_{G,e}/Z_{G,e})$, and let $\pi_{H_G} : T_{G,e}/Z_{G,e} \to H_G$ be the projection. Projecting (10) on H_G, we define

$$\mathcal{P}^{e_{12}}_{\widetilde{G}_e}(D_\phi) := \sum_{\gamma \in \mathrm{Hom}^{m_K, e_{12}}_{\mathrm{grp},\phi}(\pi_1(S^3\setminus K), \mathrm{Inn}(\widetilde{G}_e))} \pi_H\big(\gamma(l_K)\big) \in \mathbb{Z}[H_G], \tag{11}$$

where $\mathrm{Hom}^{m_K, e_{12}}_{\mathrm{grp},\phi}(\pi_1(S^3 \setminus K), \mathrm{Inn}(\widetilde{G}_e))$ stands for the preimage of ϕ via the natural projection $\mathrm{Hom}^{m_K, e_{12}}_{\mathrm{grp}}(\pi_1(S^3 \setminus K), \mathrm{Inn}(\widetilde{G}_e)) \to \mathrm{Hom}_{\mathrm{grp}}(\pi_1(S^3 \setminus K), \mathfrak{S}_4)$.

Theorem 10.1. *Let $\widetilde{G}_e$ and H_G be as above. Let a 3-manifold M be presented by a 3-fold branched covering of a knot K with the monodromy $\phi : \pi_1(S^3 \setminus K) \to \mathfrak{S}_4$. Then there exists a 4-fold symmetric 2-cocycle $\theta_{2\mathbb{Z}}$, such that the 4-fold symmetric cocycle invariant $\Phi_\theta(M) = |G|^3 \cdot \mathcal{P}^{e_{12}}_{\widetilde{G}_e}(D_\phi) \in \mathbb{Z}[H_{2\mathbb{Z}}]$. In particular, the polynomial (11) is an invariant of M. Furthermore, any 4-fold symmetric cocycle invariant of $\widetilde{G}_e$ is derived from $\mathcal{P}^{e_{12}}_{\widetilde{G}_e}(D_\phi)$.*

In general, it is difficult to explicitly find a presentation of a quandle 2-cocycle. However, Theorem 10.1 say that, when the coefficient is trivial, the 4-fold quandle cocycle invariant can be computable without quandle 2-cocycle. Although we have obtained an easy calculation of the 4-fold symmetric quandle cocycle invariant, unfortunately the authors have not been able to find examples of a non-trivial invariant.

Problem 10.2. Find an example of a non-trivial 4-fold symmetric quandle cocycle invariant which is stronger than Dijkgraaf-Witten invariant.

11 The Chern-Simons invariant as a cocycle invariant

In this Section, we reformulate the Chern-Simons invariant of closed 3-manifolds as a 4-fold symmetric quandle cocycle invariant.

11.1 Review: 4-fold symmetric 2-cocycle from normalized group 3-cocycle

We review some 4-fold symmetric quandle 2-cocycles introduced in [H] obtained from normalized group 3-cocycles. For a cored group (G, c), we define a map $* : G^4 \times \widetilde{G}_c \to G^4$ by

$$(s_1, s_2, s_3, s_4)*(g, 1, 2) = (cgs_2, g^{-1}s_1, s_3, s_4), \quad (s_1, s_2, s_3, s_4)*(g, 1, 3) = (cgs_3, s_2, g^{-1}s_1, s_4),$$

$$(s_1, s_2, s_3, s_4)*(g, 1, 4) = (cgs_4, s_2, s_3, g^{-1}s_1), \quad (s_1, s_2, s_3, s_4)*(g, 2, 3) = (s_1, cgs_3, g^{-1}s_2, s_4),$$

$$(s_1, s_2, s_3, s_4)*(g, 2, 4) = (s_1, cgs_4, s_3, g^{-1}s_2), \quad (s_1, s_2, s_3, s_4)*(g, 3, 4) = (s_1, s_2, cgs_4, g^{-1}s_3),$$

where $g \in G$ and $(s_1, s_2, s_3, s_4) \in G^4$. Then G^4 is a $(\widetilde{G}_c, \rho)$-set via the operation $*$.

A map $\theta : G^3 \to A$ is a *(strong) normalized 3-cocycle*, if for any $x, y, z, w \in G$, it satisfies

$$\theta(y,z,w)\cdot\theta(xy,z,w)^{-1}\cdot\theta(x,yz,w)\cdot\theta(x,y,zw)^{-1}\cdot\theta(x,y,z) = 1_A$$

$$\theta(e,x,y) = \theta(x,e,y) = \theta(x,y,e) = \theta(x,x^{-1},y) = \theta(x,y^{-1},y) = 1_A.$$

For a normalized 3-cocycle θ, we define a function $\mathcal{X}_\theta : G^4 \times \widetilde{G}_e \times \widetilde{G}_e \to A$ as follows:

$$\begin{aligned}
&\mathcal{X}_\theta\big((s_1,s_2,s_3,s_4),(g,i,j),(g',i,j)\big)\\
&= \theta(g,g^{-1}g',g'^{-1}gs_j)\cdot\theta(g',g'^{-1}g,g^{-1}s_i)\cdot\theta(g'g^{-1}g',g'^{-1}g,s_j)\cdot\theta(g',g^{-1}g',g'^{-1}s_i),\\
&\mathcal{X}_\theta\big((s_1,s_2,s_3,s_4),(g,i,j),(g',j,k)\big) = \theta(g'^{-1},g^{-1},s_i)^{-1}\cdot\theta(g'^{-1},g^{-1},gs_j),\\
&\mathcal{X}_\theta\big((s_1,s_2,s_3,s_4),(g,i,j),(g',k,l)\big) = 1.
\end{aligned}$$

The function $\mathcal{X}_\theta$ is introduced in [H, Section 4.2], and the first author showed

Theorem 11.1. *([H, Proposition 4.1. and Theorem 4.2.]) For a normalized 3-cocycle θ, the resulting map $\mathcal{X}_\theta : G^4 \times \widetilde{G}_e \times \widetilde{G}_e \to A$ is a 4-fold symmetric quandle 2-cocycle. Moreover, under the bijection $\mathrm{Col}_{\widetilde{G}_e,\rho}(D_\phi) \simeq G^3 \times \mathrm{Hom}(\pi_1(M), G)$ in Theorem 7.1, for $C_f \in \mathrm{Col}_{\widetilde{G}_e,\rho}(D_\phi)$ corresponding with $f \in \mathrm{Hom}(\pi_1(M), G)$, the 4-fold cocycle invariant $\Phi_{\mathcal{X}_\theta}(D_\phi; C_f) = \langle [M], f^*(\theta)\rangle \in A$. Here $[M] \in H_3(M; A)$ is the fundamental class of M.*

This implies that Dijkgaaf-Witten invariant of normalized 3-cocycles can be reformulated as a 4-fold symmetric quandle cocycle invariant (see [H] for detail).

11.2 Chern-Simons invariant

Let $G = SL(2;\mathbb{C})$. The *Cheeger-Chern-Simons class* is a map $\widehat{C}_2 : G^3 \to \mathbb{C}/4\pi^2\mathbb{Z}$ introduced by [CS]. See [DG], for the explicit presentation of $\widehat{C}_2$ using the extended Bloch group [Neu]. It is known that $\widehat{C}_2$ can be represented by an element of the group cohomology $H^3(G; \mathbb{C}/4\pi^2\mathbb{Z})$. *Chern-Simons invariant* of $f : \pi_1(M) \to SL(2;\mathbb{C})$ is defined by $\langle [M], f^*(\widehat{C}_2)\rangle \in \mathbb{C}/4\pi^2\mathbb{Z}$.

Lemma 11.2. $6\cdot\widehat{C}_2$ *is a normalized* 3*-cocycle.*

Therefore, combing this with Theorem 11.1, we immediately conclude

Theorem 11.3. *Let $G = SL(2;\mathbb{C})$. Let $\widehat{C}_2$ be as above. Let $\mathcal{X}_{6\widehat{C}_2} : G^4 \times \widetilde{G}_e \times \widetilde{G}_e \to \mathbb{C}/4\pi^2\mathbb{Z}$ be the resulting 4-fold symmetric quandle 2-cocycle given by Theorem 11.1. For $f \in \mathrm{Hom}(\pi_1(M), G)$, we put the associated $\widetilde{G}_e$-coloring $C_f \in \mathrm{Col}_{\widetilde{G}_e,\rho}(D_\phi)$ by Theorem 7.1. Then the 4-fold cocycle invariant coincides with the Chern-Simons invariant multiplicated by 6: $\Phi_{\mathcal{X}_{6\widehat{C}_2}}(D; C_f) = 6\langle [M], f^*(\widehat{C}_2)\rangle \in \mathbb{C}/4\pi^2\mathbb{Z}$.*

Remark 11.4. Notice an inclusion $\mathbb{Z}/m\mathbb{Z} \cong H_3(\mathbb{Z}/m\mathbb{Z};\mathbb{Z}) \hookrightarrow H_3(SL(2;\mathbb{C});\mathbb{Z})$. If we know the values of Dijkgraaf-Witten invariants of $G = \mathbb{Z}/6^a\mathbb{Z}$ for all $a \in \mathbb{N}$, then we can easily make a recovery of the Chern-Simons invariant from the multiplication by 6.

We emphasize an advantage of Theorem 11.3. Following the description of [Neu, Z], for the computation of the Chern-Simons invariant we have to choose a (flattened) triangulation of M. However, in general, a triangulation of M are composed of many simplicies, which make the computation the Chern-Simons invariant complicated.

On the other hand, Theorem 11.3 says that if we know a labeled diagram of M and a $\widetilde{G}_e$-coloring corresponding with $\pi_1(M) \to G$, the formulation is to make the Chern-Simons invariant computable without using triangulation of M.

In general, for any 3-manifold M, it is not easy to find a labeled diagram of M. However, if we find a labeled diagram of M, it is easy to find a $\widetilde{G}_c$-coloring C_f corresponding with $f : \pi_1(M) \to G$ by Theorem 7.1. We expect a good computer program for the calculation of the Chern-Simons invariant of f from labeled diagrams. It goes without saying that a double branched covering of a link is precisely presented by a labeled diagram similar to Figure 7. So, the Chern-Simon invariant of the double branched covering would be easily computable.

12 An application: a generalization of [H2]

We give an application obtained from the 4-fold symmetric quandle homotopy invariant.

Let m be an odd number. To begin with, let us roughly recall a quandle homotopy invariant of a dihedral quandle. A *dihedral quandle* R_m of order m is $\mathbb{Z}/m\mathbb{Z}$ with a quandle operation given by $x * y = 2y - x$. Note that the dihedral quandle R_m is isomorphic to a subquandle $\{(g,(1,2)) \in \widetilde{G}_e \mid g \in G\} \subset \widetilde{G}_e$, where $G = \mathbb{Z}/m\mathbb{Z}$. Further, for an oriented link $L \subset S^3$, the second author studied "the quandle homotopy invariant" of R_m denoted by $\Xi_{R_m}(L) \in \mathbb{Z}[\pi_2(BR_m)]$. (see [N1] for more detail). He showed that if m is prime, then the invariant is equivalent to "the quandle cocycle invariant" of "Mochizuki 3-cocycle [Moc]" (see, e.g., [Iwa] for the definition).

We give a topological interpretation of the invariant $\Xi_{R_m}(L)$ as follows.

Corollary 12.1. *Let m, G and $L \subset S^3$ be as above. Let M_L denote the double branched covering space of L.*
(i) We obtain an isomorphism $\pi_2(BR_m) \to \Omega_3(G)$ using the map (7). In particular, since $\Omega_3(G,e) \cong H_3(G,\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$ (Remark 8.1), $\pi_2(BR_m) \cong \mathbb{Z}/m\mathbb{Z}$.
(ii) Further, the quandle homotopy invariant $\Xi_{R_m}(L)$ is equal to a scalar multiple of the Dijkgraaf-Witten invariant $DW_\Omega^{G_c}(M_L)$ given in (6). Namely,

$$\Xi_{R_m}(L) \doteq m \cdot DW_\Omega^{G_c}(M_L) \in \mathbb{Z}[\Omega_3(G,e)] \cong \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}].$$

Remark 12.2. From the perspective of the quandle cocycle invariant of links, the first author [H2] showed the similar equivalence between the Mochizuki 3-cocycle invariant of L and the Dijkgraaf-Witten invariant of M_L. However, the her work needs a certain condition of odd m.

Note that Corollary 12.1 drops the condition. Further, since the quandle homotopy invariant is the universal among quandle cocycle invariants, Corollary 12.1 is a generalization of [H2].

Recall that the quandle homotopy (cocycle) invariants are defined by combinatorial methods. However, we give the quandle homotopy (coycle) invariant of R_m a topological meaning. In general, it is a problem for the future how topological interpretation the quandle cocycle invariant of any quandle has.

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