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QUANDLE COCYCLES FROM GROUP COCYCLES

YUICHI KABAYA

ABSTRACT. We give a construction of a quandle cocycle from a group cocycle, especially an explicit construction of quandle cocycles of the dihedral quandle R_p from group cocycles of the cyclic group $\mathbb{Z}/p$. The 3-dimensional group cocycle of $\mathbb{Z}/p$ gives a non-trivial quandle 3-cocycle of R_p.

1. INTRODUCTION

A quandle, which was introduced by Joyce [Joy], is an algebraic object whose axioms are motivated by knot theory and conjugation in a group. In [CJKLS], the authors introduced a quandle homology theory, and they defined the quandle cocycle invariants for classical knots and surface knots. The quandle homology is defined as the homology of the chain complex generated by cubes whose edges are labeled by elements of a quandle. On the other hand the group homology is defined as the homology of the chain complex generated by tetrahedra whose edges are labeled by elements of a group. So it is natural to ask a relation between quandle homology and group homology.

In [IK], the authors defined a simplicial version of quandle homology and constructed a homomorphism from the usual quandle homology to the simplicial quandle homology. Applying the construction for $\mathrm{PSL}(2,\mathbb{C})$-representation of the knot complement, we obtained a diagrammatic formula of the hyperbolic volume and the Chern-Simons invariant. The important point of [IK] is to give a triangulation of a knot complement in algebraic fashion by using quandle homology. This construction enable us relate the quandle homology with the topology of a knot complement.

In this note, we apply the work [IK] for finite quandles to construct quandle cocycles from group cocycles. Especially we construct quandle cocycles of the dihedral quandle R_p from group cocycles of the cyclic group $\mathbb{Z}/p$. It will be shown that the 3-dimensional group cocycle gives a non-trivial quandle 3-cocycle of $H^3_Q(R_p;\mathbb{Z}/p)$. Since $\dim H^3_Q(R_p;\mathbb{Z}/p)=1$, our quandle 3-cocycle is a constant multiple of the Mochizuki's 3-cocycle [Moc].

This note is organized as follows. We will review the definition of quandles and their homology theory in Section 2. In Section 3, we recall the definition of the group homology. We will review the construction of [IK] in Section 5 and apply it to construct quandle cocycles of a dihedral quandle. We will propose a general construction in Section 7.

2. QUANDLE AND QUANDLE HOMOLOGY

A quandle is a set X with a binary operation $*$ satisfying the following axioms:

(1) $x*x=x$ for any $x\in X$,
(2) the map $*y : X\rightarrow X$ defined by $x\mapsto x*y$ is a bijection for any $y\in X$,
(3) $(x*y)*z=(x*z)*(y*z)$ for any $x,y,z\in X$.

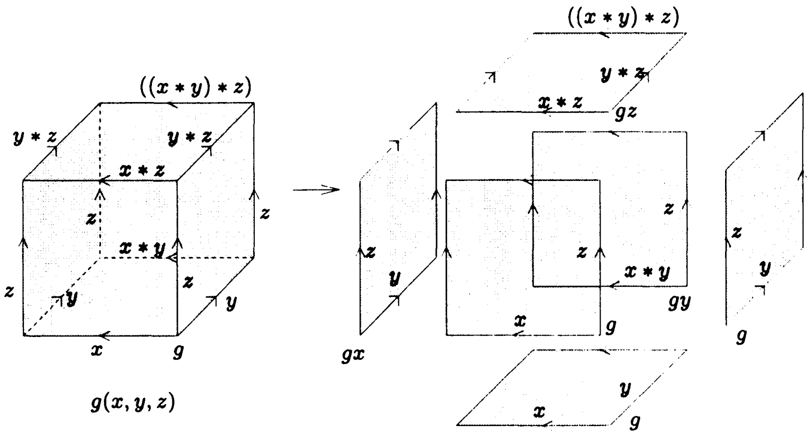


FIGURE 1. $\partial(g(x,y,z)) = -(g(y,z) - gx(y,z)) + (g(x,z) - gy(x*y,z)) - (g(x,y) - gz(x*z, y*z))$. Here $x, y, z \in X$ and $g \in G_X$. Edges are labeled by elements of X and vertices are labeled by elements of G_X.

We denote the inverse of $*y$ by $*^{-1}y$. For a quandle X, we define the associated group G_X by $\langle x \in X | y^{-1}xy = x * y \quad (x, y \in X)\rangle$. A quandle X has a right G_X-action in the following way. Let $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$ be an element of G_X where $x_i \in X$ and $\varepsilon_i = \pm 1$. Define $x * g = (\cdots((x *^{\varepsilon_1} x_1) *^{\varepsilon_2} x_2) \cdots) *^{\varepsilon_n} x_n$. One can easily check that this is a right action of G_X on X. So the free abelian group $\mathbb{Z}[X]$ generated by X is a right $\mathbb{Z}[G_X]$-module.

Let $C_n^R(X)$ be the free (left) $\mathbb{Z}[G_X]$-module generated by X^n. We define the boundary map $C_n^R(X) \to C_{n-1}^R(X)$ by

$$\begin{aligned}\partial(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} (-1)^i ((x_1, \dots, \widehat{x_i}, \dots, x_n) \\ - x_i(x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)).\end{aligned}$$

Figure 1 shows a graphical picture of the boundary map. Let $C_n^D(X)$ be the $\mathbb{Z}[G_X]$-submodule of $C_n^R(X)$ generated by $(x_1, \dots, x_n)$ with $x_i = x_{i+1}$ for some i. Now $C_n^D(X)$ is a subcomplex of $C_n^R(X)$. Let $C_n^Q(X) = C_n^R(X)/C_n^D(X)$. For a right $\mathbb{Z}[G_X]$-module M, we define the *rack homology* of M by the homology of $C_n^R(X; M) = M \otimes_{\mathbb{Z}[G_X]} C_n^R(X)$ and denote it by $H_n^R(X; M)$. We also define the *quandle homology* of M by the homology of $M \otimes_{\mathbb{Z}[G_X]} C_n^Q(X)$ and denote it by $H_n^Q(X; M)$. The homology $H_n^Q(X; \mathbb{Z})$, here $\mathbb{Z}$ is the trivial $\mathbb{Z}[G_X]$-module, is equal to the usual quandle homology $H_n^Q(X)$. Let Y be a set with a right G_X-action. For any abelian group A, the abelian group $A[Y]$ generated by Y over A is a right $\mathbb{Z}[G_X]$-module. The homology group $H_n^Q(X; A[Y])$ is usually denoted by $H_n^Q(X; A)_Y$ ([Kam]).

Let N be a left $\mathbb{Z}[G_X]$-module. We define the *rack cohomology* $H_Q^n(X; N)$ by the cohomology of $C_R^n(X; N) = \mathrm{Hom}_{\mathbb{Z}[G_X]}(C_n^R(X), N)$. The *quandle cohomology* $H_Q^n(X; N)$ is defined in a similar way. For a set Y with a right G_X-action and an abelian group A, we let $\mathrm{Func}(Y, A)$ be the left $\mathbb{Z}[G_X]$-module generated by functions $\phi : Y \to A$, here the action is defined by $(g\phi)(y) = \phi(yg)$ for $y \in Y$ and $g \in G_X$. The cohomology group $H_Q^n(X; \mathrm{Func}(Y, A))$ is usually denoted by $H_Q^n(X; A)_Y$.

3. Group homology

3.1. Let G be a group. Let $C_n(G)$ be the free $\mathbb{Z}[G]$-module generated by $[g_1|\dots|g_n] \in G^n$. Define the boundary map $\partial : C_n(G) \to C_{n-1}(G)$ by

$$\partial([g_1|\dots|g_n]) = g_1[g_2|\dots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_n] + (-1)^n[g_1|\dots|g_{n-1}].$$

Let $C_0(G) \cong \mathbb{Z}[G] \to \mathbb{Z} \to 0$ be the augmentation map. We remark that the chain complex $\{\dots \to C_1(G) \to C_0(G) \to \mathbb{Z} \to 0\}$ is acyclic. So the chain complex $C_*(G)$ gives a free resolution of $\mathbb{Z}$. Let M be a right $\mathbb{Z}[G]$-module. The homology of $M \otimes_{\mathbb{Z}[G]} C_n(G)$ is called the *group homology* of M and denoted by $H_n(G;M)$. In other words, $H_n(G;M) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(M,\mathbb{Z})$.

Let $C'_n(G)$ be the free $\mathbb{Z}$-module generated by $(g_0,\dots,g_n) \in G^{n+1}$. Then $C'_n(G)$ is a left $\mathbb{Z}[G]$-module by $g(g_0,\dots,g_n) = (gg_0,\dots,gg_n)$. Define the boundary operator of $C'_n(G)$ by

$$\partial(g_0,\dots,g_n) = \sum_{i=0}^{n}(-1)^i(g_0,\dots,\widehat{g_i},\dots,g_n).$$

$C_*(G)$ and $C'_*(G)$ are isomorphic as chain complexes. In fact, the following correspondence gives an isomorphism:

$$[g_1|g_2|\dots|g_n] \leftrightarrow (1, g_1, g_1g_2, \dots, g_1\cdots g_n)$$
$$(\quad g_0[g_0^{-1}g_1|g_1^{-1}g_2|\dots|g_{n-1}^{-1}g_n] \leftrightarrow (g_0,\dots,g_n)\quad)$$

The notation using $(g_0,\dots,g_n)$ is called *homogeneous* and the one using $[g_1|\dots|g_n]$ is called *inhomogeneous.*

Factoring out $C_n(G)$ by the degenerate complex, that is generated by $[g_1|\dots|g_n]$ with $g_i = 1$ for some i, we obtain the *normalized* chain complex and its homology group. It is known that the group homology using the normalized chain complex coincides with the unnormalized one. In homogeneous notation, we factor out $C'_n(G)$ by the subcomplex generated by $(g_0,\dots,g_n)$ with $g_i = g_{i+1}$ for some i.

3.2. Let X be a quandle and M be a right $\mathbb{Z}[G_X]$-module. We can construct a map from the rack homology $H_n^R(X;M)$ to the group homology $H_n(G_X;M)$. The following lemma is well-known.

Lemma 3.1. *Let $\dots \to P_1 \to P_0 \to M \to 0$ be a chain complex where P_i are projective (e.g. free). Let $\dots \to C_1 \to C_0 \to N \to 0$ be an acyclic complex. Any homomorphism $M \to N$ can be extended to a chain map from $\{P_*\}$ to $\{C_*\}$. Moreover such a chain map is unique up to chain homotopy.*

So there exists a unique chain map from $C_*^R(X)$ to $C_*(G_X)$ up to homotopy. This map induces $M \otimes_{\mathbb{Z}[G_X]} C_*^R(X) \to M \otimes_{\mathbb{Z}[G_X]} C_*(G_X)$ and then $H_n^R(X;M) \to H_n(G_X;M)$. We can also construct a natural map $H_n^Q(X;M) \to H_n(G_X;M)$. We give an explicit chain map. Let $(x_1,\dots,x_n)$ be a generator of $C_n^R(X)$. We define the map f by

$$f((x_1,\dots,x_n)) = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma)[y_{\sigma,1}|\cdots|y_{\sigma,i}|\cdots|y_{\sigma,n}]$$

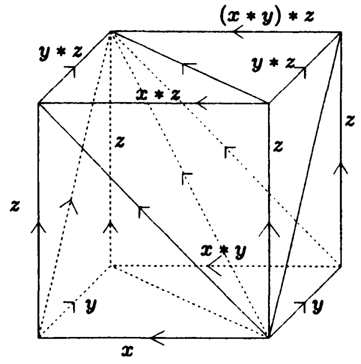


FIGURE 2

where $y_{\sigma,i} \in X$ is defined for a permutation σ and $i \in \{1, \dots, n\}$ as follows. Let $j_1, \dots, j_i < i$ be the maximal set of numbers satisfying $\sigma(i) < \sigma(j_1) < \sigma(j_2) < \cdots < \sigma(j_i)$. Then define

$$y_{\sigma,i} = x_{\sigma(i)} * \left(x_{\sigma(j_1)} x_{\sigma(j_2)} \cdots x_{\sigma(j_i)}\right).$$

The graphical picture of this map is given in Figure 2.

Example 3.2. Let $(x, y, z) \in C_3^R(X)$. Then the chain map $f : C_3^R(X) \to C_3(G_X)$ constructed above is given by

$$\begin{aligned}\partial((x, y, z)) =& [x|y|z] - [x|z|y * z] + [y|z|(x * y) * z] - [y|x * y|z] \\ &+ [z|x * z|y * z] - [z|y * z|(x * y) * z].\end{aligned}$$

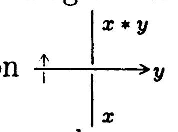
If we use the normalized chain complex for group homology, we obtain a map $f : C_3^Q(X) \to C_3(G_X)$.

Remark 3.3. Fenn, Rourke and Sanderson defined the *rack space* BX. Since $\pi_1(BX)$ is isomorphic to G_X, there exists a unique map, up to homotopy, from BX to the Eilenberg-MacLane space $K(G_X, 1)$ which induces the isomorphism between their fundamental groups. The map we have constructed is essentially same as this map.

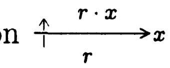
As we have seen, there exists a relation between quandle homology and group homology. We shall give another relation which seems to reflect more geometric feature.

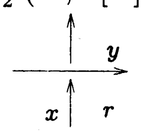
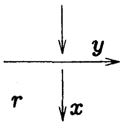
4. SHADOW COLORING AND THE CYCLE INVARIANT

The contents of this section will be used in Section 7.3.

4.1. Let X be a quandle. Let K be an oriented knot in S^3 and D be a diagram of K. An *arc coloring* is a map $\mathcal{A}$: {arcs of D} $\to X$ if it satisfies the relation 

at each crossing, where $x, y \in X$. By the Wirtinger presentation of a knot complement, an arc coloring determines a representation $\pi_1(S^3 \setminus K) \to G_X$. This is obtained by sending each meridian to its color.

Let Y be a set with a right G_X action. A map $\mathcal{D}$: {regions of D} $\to Y$ is called a *region coloring* if it satisfies the relation  for any pair of adjacent regions, where $r \in Y$ and $x \in X$. A pair $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ is called a *shadow coloring.*

We define a cycle $[C(\mathcal{S})]$ of $H_2^Q(X;\mathbb{Z}[Y])$ for a shadow coloring $\mathcal{S}$. Assign $+r \otimes (x,y)$ for a positive crossing colored by  and $-r \otimes (x,y)$ for a negative crossing colored by . Then define

$$C(\mathcal{S}) = \sum_{c:\text{crossing}} \varepsilon_c r_c \otimes (x_c, y_c) \in C_2^Q(X;\mathbb{Z}[Y]),$$

where $\varepsilon_c = \pm 1$. We can easily check that this is a cycle and the homology class $[C(\mathcal{S})]$ is invariant under Reidemeister moves. Moreover it does not depend on the choice of the region coloring if the action of G_X on Y is transitive. So the homology class $[C(\mathcal{S})]$ is an invariant of the arc coloring $\mathcal{A}$.

There are two important sets with right G_X-action, one is $Y = \{*\}$ and the other is $Y = X$. Eisermann showed that the cycle $[C(\mathcal{S})]$ for $Y = \{*\}$ is essentially described by the monodromy of some representation of the knot group along the longitude ([Eis1], [Eis2]). So we study the invariant $[C(\mathcal{S})]$ in the case of $Y = X$.

4.2. Quandle cocycle invariant. Let X be a quandle with $|X| < \infty$. Let A be an abelian group. For any quandle cocycle $f \in H_Q^2(X; \mathrm{Func}(X, A))$,

$$\sum_{\mathcal{S}:\text{shadow colorings}} \langle f, C(\mathcal{S})\rangle \in \mathbb{Z}[A]$$

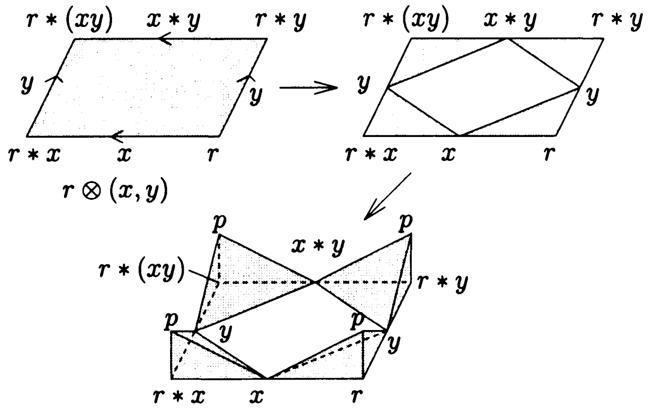
is an invariant of knots. This is called the *quandle cocycle invariant.* When $Y = \{*\}$, Eisermann in [Eis2] showed that $[C(\mathcal{S})]$ is essentially equivalent to the *coloring polynomial,* which is described by the monodromy of some representation of $\pi_1(S^3 \setminus K)$ along the longitude. So we study the case $Y = X$.

5. Simplicial quandle homology $H_n^\Delta(X;\mathbb{Z})$ and the map $H_n^R(X;\mathbb{Z}[X]) \to H_{n+1}^\Delta(X;\mathbb{Z})$

Let X be a quandle. Let $C_n^\Delta(X) = \mathrm{span}_{\mathbb{Z}}\{(x_0, \ldots, x_n) | x_i \in X\}$. We define the boundary operator of $C_n^\Delta(X)$ by

$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

Since X has a right action of G_X, the chain complex $C_n^\Delta(X)$ has a right action of G_X by $(x_0, \ldots, x_n) * g = (x_0 * g, \ldots, x_n * g)$. Let M be a $\mathbb{Z}[G_X]$-module. We denote the homology



$(p, r, x, y) - (p, r * x, x, y)$
$-(p, r * y, x * y, y) + (p, r * (xy), x * y, y)$

FIGURE 3

of $C_n^\Delta(X) \otimes_{\mathbb{Z}[G_X]} M$ by $H_n^\Delta(X; M)$ and call it the *simplicial quandle homology* of X. For any abelian group A, we can also define the cohomology group $H^n_\Delta(X; A)$ in a similar way.

Let I_n be the set consisting of maps $\iota : \{1, 2, \cdots, n\} \longrightarrow \{0, 1\}$. We let $|\iota|$ denote the cardinality of the set $\{i \mid \iota(i) = 1, 1 \le i \le n\}$. For each generator $r \otimes (x_1, x_2, \cdots, x_n)$ of $C_n^R(X; \mathbb{Z}[X])$, here $r, x_1, \ldots, x_n \in X$, we define

$$r(\iota) = r * (x_1^{\iota(1)} x_2^{\iota(2)} \cdots x_n^{\iota(n)}) \in X,$$
$$x(\iota, i) = x_i * (x_{i+1}^{\iota(i+1)} x_{i+2}^{\iota(i+2)} \cdots x_n^{\iota(n)}) \in X,$$

for any $\iota \in I_n$. Fix an element $p \in X$. For each $n \ge 1$, we define a homomorphism

$$\varphi : C_n^R(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^\Delta(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$$

by

$$\varphi(r \otimes (x_1, x_2, \cdots, x_n)) = \sum_{\iota \in I_n} (-1)^{|\iota|} (p, r(\iota), x(\iota, 1), x(\iota, 2), \cdots, x(\iota, n)). \tag{5.1}$$

For example, in the case $n = 2$ (see Figure 3),

$$\varphi(r \otimes (x, y)) = (p, r, x, y) - (p, r * x, x, y) - (p, r * y, x * y, y) + (p, (r * x) * y, x * y, y),$$

and in the case $n = 3$,

$$\begin{aligned} &\varphi(r \otimes (x, y, z)) = \\ &\quad (p, r, x, y, z) - (p, r * x, x, y, z) \\ &\quad - (p, r * y, x * y, y, z) - (p, r * z, x * z, y * z, z) \\ &\quad + (p, (r * x) * y, x * y, y, z) + (p, (r * x) * z, x * z, y * z, z) \\ &\quad + (p, (r * y) * z, (x * y) * z, y * z, z) - (p, ((r * x) * y) * z, (x * y) * z, y * z, z). \end{aligned}$$

Theorem 5.1 (Inoue-Kabaya, [IK]). *The map $\varphi : C_n^R(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^\Delta(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ is a chain map.*

Therefore φ induces a homomorphism $\varphi_* : H_n^R(X;\mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X;\mathbb{Z})$. We remark that the induced map $\varphi_* : H_n^R(X;\mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X;\mathbb{Z})$ does not depend on the choice of $p \in X$.

In general, it is easier to construct cocycles of $H_{n+1}^{\Delta}(X)$ from group cocycles of some group related to X than $H_n^R(X;\mathbb{Z}[X])$. If we have a function f from X^{k+1} to some abelian group A satisfying

(1) $\sum_{i=0}^{k+1}(-1)^i f(x_0,\dots,\widehat{x_i},\dots,x_{k+1}) = 0$,

(2) $f(x_0 * y,\dots,x_k * y) = f(x_0,\dots,x_k)$ for any $y \in X$,

(3) $f(x_0,\dots,x_k) = 0$ if $x_i = x_{i+1}$ for some i,

then f is a cocycle of $H_{\Delta}^k(X;A)$ and $\varphi^* f$ is a cocycle of $H_Q^{k-1}(X;\mathrm{Func}(X,A))$. Moreover $\varphi^* f$ can be regarded as a cocycle in $H_Q^k(X;A)$ by a natural map

$$r \otimes (x_1,\dots,x_{k-1}) \mapsto (r,x_1,\dots x_{k_1}).$$

We will construct functions satisfying these three conditions from group cocycles.

6. Cocycles of dihedral quandles

For any integer $p > 2$, let R_p denote the cyclic group $\mathbb{Z}/p$ with quandle operation defined by $x * y = 2y - x$. The quandle R_p is called the *dihedral quandle*. In this section, we construct quandle cocycles of R_p from group cocycles of $\mathbb{Z}/p$. In the next section, we will propose a general construction of quandle cocycles from group cocycles.

6.1. Group cohomology of cyclic groups. Let G be the cyclic group $\mathbb{Z}/p$ (p is an integer greater than 1). The first cohomology $H^1(G;\mathbb{Z}/p)$ is generated by the 1-cocycle

$$b_1(x) = x$$

and the second cohomology $H^2(G;\mathbb{Z}/p)$ is generated by the 2-cocycle

$$b_2(x,y) = \begin{cases} 1 & \text{if } \bar{x}+\bar{y} \geq p \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{x}$ is an integer $0 \leq \bar{x} < p$ with $\bar{x} \equiv x \mod p$. Moreover any element of $H^*(G;\mathbb{Z}/p)$ can be presented by a cup product of b_1's and b_2's:

$$H^*(G,\mathbb{Z}/p) = \wedge(b_1) \otimes \mathbb{Z}[b_2].$$

6.2. 3-cocycle of R_p. Let f be a k-cocycle of $H^k(G,\mathbb{Z}/p)$. Using homogeneous notation, we obtain a map $f : (R_p)^{k+1} \to \mathbb{Z}/p$ satisfying

(1) $\sum_{i=0}^{k+1}(-1)^i f(x_0,\dots,\widehat{x_i},\dots,x_{k+1}) = 0$,

(3) $f(x_0,\dots,x_k) = 0$ if $x_i = x_{i+1}$ for some i.

Therefore if f also satisfy the condition (2) $f(x_0 * y,\dots,x_k * y) = f(x_0,\dots,x_k)$ for any $y \in R_p$, f gives rise to a quandle k-cocycle of $H_Q^n(R_p;\mathbb{Z}/p)$ by the construction of Section 5. Define $\tilde{f} : (R_p)^{k+1} \to \mathbb{Z}/p$ by

$$\tilde{f}(x_0,\dots,x_k) = f(x_0,\dots,x_k) + f(-x_0,\dots,-x_k).$$

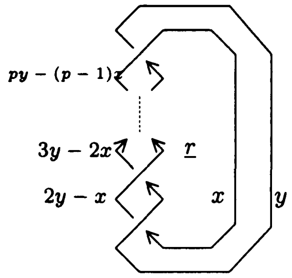


FIGURE 4. A shadow coloring of $(2,p)$-torus knot by R_p. (For any $x, y, r \in R_p$.)

Then $\tilde{f}$ satisfies the conditions (1), (2) and (3). So we obtain a quandle k-cocycle.

We give an explicit presentation of the 3-cocycle coming from $b_1 b_2 \in H^3(G;\mathbb{Z}/p)$. Let

$$d(x,y) = b_2(x,y) - b_2(-x,-y)$$

then d is a 2-cocycle. (We can check that d is cohomologous to $2b_2$.) Then $\widetilde{b_1 b_2}$ is given by $[x|y|z] \mapsto x \cdot d(y,z)$.

Proposition 6.1. *The 3-cocycle coming from $b_1 b_2 \in H^3(G;\mathbb{Z}/p)$ has the following presentation:*

$$(x,y,z) \mapsto 2z(d(y-x,z-y) + d(y-x,y-z)) \quad (x,y,z \in R_p).$$

This is a non-trivial quandle 3-cocycle of R_p.

Proof. In (5.1), since the map φ_* does not depend on the choice of $p \in R_p$, we let $p = 0$. Then we have

$$\begin{aligned}\varphi(x,y,z) &= (0,x,y,z) - (0,x*y,y,z) - (0,x*z,y*z,z) + (0,(x*y)*z,y*z,z)\\ &=(0,x,y,z) - (0,2y-x,y,z) - (0,2z-x,2z-y,z) + (0,2z-2y+x,2z-y,z).\end{aligned}$$

Rewrite in inhomogeneous notation, this is equal to

$$\begin{aligned}&[x|y-x|z-y] - [2y-x|x-y|z-y]\\ &- [2z-x|x-y|y-z] + [2z-2y+x|y-x|y-z].\end{aligned}$$

Evaluating by $\widetilde{b_1 b_2}$, we have

$$\begin{aligned}&xd(y-x,z-y) - (2y-x)d(x-y,z-y)\\ &\quad - (2z-x)d(x-y,y-z) + (2z-2y+x)d(y-x,y-z)\\ =&xd(y-x,z-y) + (2y-x)d(y-x,y-z)\\ &\quad + (2z-x)d(y-x,z-y) + (2z-2y+x)d(y-x,y-z)\\ =&2zd(y-x,z-y) + 2zd(y-x,y-z).\end{aligned}$$

We can check that this cocycle is non-trivial by evaluating at the cycle given by a coloring of the $(2,p)$-torus knot (Figure 4).

□

Since 2 is divisible in $\mathbb{Z}/p$, $(x,y,z) \mapsto z(d(y-x,z-y) + d(y-x,y-z))$ is also a non-trivial quandle 3-cocycle. It is known that $\dim_{\mathbb{F}_p} H^3_Q(R_p;\mathbb{F}_p) = 1$, our cocycle is a constant multiple of the Mochizuki's 3-cocycle [Moc].

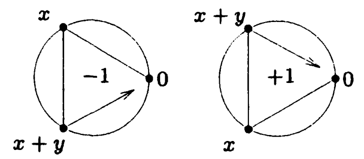


FIGURE 5. The value of $d(x, y)$.

We remark that the cocycle d can be easily understand geometrically. Identify $i \in \mathbb{Z}/p$ with ζ^i where $\zeta = \exp(2\pi\sqrt{-1}/p)$. Then $d(x, y) = -1$ if $(0, x, x + y)$ is counterclockwise, $d(x, y) = +1$ if $(0, x, x + y)$ is clockwise and $d(x, y) = 0$ otherwise (Figure 5). This interpretation and the equation $d(-x, -y) = -d(x, y)$ make various calculations easy.

7. GENERAL CONSTRUCTION

7.1. Let G be a group. Fix an element $h \in G$. Let $\mathrm{Conj}(h) = \{g^{-1}hg | g \in G\}$. Now $\mathrm{Conj}(h)$ has a quandle operation by $x * y = y^{-1}xy$. Let $Z(h) = \{g \in G | gh = hg\}$ be the centralizer of h in G.

Lemma 7.1. *As a set* $\mathrm{Conj}(h) \cong Z(h)\backslash G$ *by*

$$g^{-1}hg \leftrightarrow Z(h)g \text{ (right coset)}$$

From now on we study the quandle structure on $Z(h)\backslash G$ and construct a lift of $\pi : G \to Z(h)\backslash G$. The quandle structure on $\mathrm{Conj}(h)$ induces a quandle operation on $Z(h)\backslash G$:

$$\begin{aligned}(g_1^{-1}hg_1) * (g_2^{-1}hg_2) &= (g_2^{-1}hg_2)^{-1}(g_1^{-1}hg_1)(g_2^{-1}hg_2)\\ &= (g_1g_2^{-1}hg_2)^{-1}h(g_1g_2^{-1}hg_2)\\ &\leftrightarrow Z(h)g_1(g_2^{-1}hg_2)\end{aligned}$$

The quandle operation on $Z(h)\backslash G$ lifts to a quandle operation on G by

$$g_1 \bullet g_2 = h^{-1}g_1(g_2^{-1}hg_2) \quad (g_1, g_2 \in G).$$

We can easily check that $\bullet$ satisfies the quandle axioms and the projection map $\pi : G \to Z(h)\backslash G$ is a quandle homomorphism. Let $s : Z(h)\backslash G \to G$ be a section of π ($\pi \circ s = \mathrm{Id}$). Since $s(x * y)$ and $s(x) \bullet s(y)$ are in the same coset in $Z(h)\backslash G$, there exists an element $c(x, y) \in Z(h)$ satisfying

$$s(x * y) = c(x, y)s(x) \bullet s(y).$$

Lemma 7.2. *If $Z(h)$ is an abelian group, $c : X \times X \to Z(h)$ is a quandle 2-cocycle. If the cocycle c is cohomologous to zero, we can change the section s to satisfy $s(x * y) = s(x) \bullet s(y)$.*

Example 7.3. Let G be the dihedral group $D_{2p} = \langle h, x | h^2 = x^p = hxhx = 1\rangle$ where p is an odd number. Then we have $Z(h) = \{1, h\}$ and $\mathrm{Conj}(h) = \{x^{-i}hx^i | i = 0, 1, \dots, p-1\} = \{hx^{2i} | i = 0, \dots, p-1\}$. We can identify $x^{-i}hx^i \in \mathrm{Conj}(h)$ with $i \in R_p = \{0, 1, 2, \dots, p-1\}$. Define a section $s : Z(h)\backslash G \to G$ by

$$\begin{array}{ccccc}\mathrm{Conj}(h) & \cong & Z(h)\backslash G & \xrightarrow{s} & G\\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid\\ x^{-i}hx^i & \leftrightarrow & Z(h)x^i & \mapsto & hx^i\end{array}$$

Then we have

$$\begin{aligned} s(Z(h)x^i * Z(h)x^j) &= s(Z(h)x^{2j-i}) = hx^{2j-i} \\ &= h^{-1}(hx^i)(x^{-j}hx^j) = s(Z(h)x^i) \bullet s(Z(h)x^j). \end{aligned}$$

Therefore $c(x, y) = 0$ for any $x, y \in R_p$.

Let G be a group. Fix $h \in G$ with $h^l = 1$ $(l > 1)$. We assume that $Z(h)$ is abelian and the 2-cocycle corresponding to $G \to Z(h)\backslash G$ is cohomologous to zero. Let $s : Z(h)\backslash G \to G$ be a section satisfying $s(x * y) = s(x) \bullet s(y)$. Let $f : G^{k+1} \to A$ be a normalized group k-cocycle in homogeneous notation. Then f satisfies

(1) $\sum_{i=0}^{k+1}(-1)^i f(x_0, \dots, \widehat{x_i}, \dots, x_{k+1}) = 0,$
(2) $f(gx_0, \dots, gx_k) = f(x_0, \dots, x_k)$ for any $g \in G$ (left invariance),
(3) $f(x_0, \dots, x_k) = 0$ if $x_i = x_{i+1}$ for some i.

In the following construction, it is convenient to use a right invariant function. So we replace $f(x_0, \dots, x_k)$ by $f(x_0^{-1}, \dots, x_k^{-1})$. Define $\tilde{f} : \mathrm{Conj}(h)^{k+1} \to A$ by

$$\tilde{f}(x_0. \dots, x_k) = \sum_{i=0}^{l-1} f(h^i s(x_0), \dots, h^i s(x_k))$$

for $x_0, \dots, x_k \in \mathrm{Conj}(h)$.

Proposition 7.4. *The function $\tilde{f}$ satisfies the conditions (1), (2) and (3) of Section 5. Therefore $\tilde{f}$ gives rise to a quandle k-cocycle of $H^k_\Delta(\mathrm{Conj}(h); A)$ and $H^k_Q(\mathrm{Conj}(h); A)$.*

Proof. It is clear to satisfy (1) and (3) from the conditions on a normalized group cocycle in homogeneous notation. We only have to check the second property.

$$\begin{aligned} &\tilde{f}(x_0 * y, \dots, x_k * y) \\ &= \sum_{i=0}^{l-1} f(h^i s(x_0 * y), \dots, h^i s(x_k * y)) \\ &= \sum_{i=0}^{l-1} f(h^i s(x_0) \bullet s(y), \dots, h^i s(x_k) \bullet s(y)) \\ &= \sum_{i=0}^{l-1} f(h^{i-1} s(x_0)(s(y)^{-1} h s(y)), \dots, h^{i-1} s(x_k)(s(y)^{-1} h s(y))) \\ &= \sum_{i=0}^{l-1} f(h^{i-1} s(x_0), \dots, h^{i-1} s(x_k)) \quad \text{(right invariance)} \\ &= \tilde{f}(x_0, \dots, x_k) \end{aligned}$$

□

Corollary 7.5. *If $Z(h)$ is abelian and the 2-cocycle corresponding to $G \to Z(h)\backslash G$ is cohomologous to zero, then there is a homomorphism*

$$H^n(G;A) \to H^n_Q(\mathrm{Conj}(h);A)$$

for any abelian group A.

Since there exists a homomorphism from the associated group $G_{\mathrm{Conj}(h)}$ to G, we have a homomorphism

$$H^n(G;A) \to H^n(G_{\mathrm{Conj}(h)};A) \to H^n_Q(\mathrm{Conj}(h);A)$$

from the construction of Section 3.2. I do not know any relation between these homomorphisms.

7.2. We return to the case of R_p discussed in the previous section. Let G be the dihedral group D_{2p}. Consider the short exact sequence

$$0 \to \mathbb{Z}/p \to D_{2p} \to \mathbb{Z}/2 \to 0. \tag{7.1}$$

This induces a map

$$H^*(D_{2p};\mathbb{Z}/p) \to H^*(\mathbb{Z}/p;\mathbb{Z}/p)^{\mathbb{Z}/2}.$$

We can show that this map is an isomorphism. Consider the Hochschild-Serre spectral sequence of (7.1). Since $E_2^{rs} = H^r(\mathbb{Z}/2;H^s(\mathbb{Z}/p;\mathbb{Z}/p)) = 0$ for $r > 0$, we have $E_\infty^{rs} = 0$ for $r > 0$ and $E_\infty^{0s} \cong H^s(\mathbb{Z}/p;\mathbb{Z}/p)^{\mathbb{Z}/2}$. So we have $H^s(D_{2p};\mathbb{Z}/p) \cong E_\infty^{0s} \cong H^s(\mathbb{Z}/p;\mathbb{Z}/p)^{\mathbb{Z}/2}$. Let f be the group 3-cocycle $(x,y,z) \mapsto x \cdot d(y,z)$, which was discussed in the previous section. This is a $\mathbb{Z}/2$-invariant 3-cocycle of $H^3(\mathbb{Z}/p;\mathbb{Z}/p)$, therefore a 3-cocycle of D_{2p}. Applying our construction for this group cocycle, we obtain a quandle 3-cocycle of R_p, which is twice the cocycle constructed in the previous section.

7.3. Considering the dual of our construction, we obtain a group cycle represented by a cyclic branched covering along a knot K in the following way.

Let X be the quandle $\mathrm{Conj}(h)$. Let $\mathcal{S}$ be a shadow coloring of a knot diagram D with arc and region color by X. Then a cycle $C(\mathcal{S})$ was defined in Section 4. Now $\varphi(C(\mathcal{S}))$ is a cycle in $C_3^\Delta(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ but not in $C_3^\Delta(X)$. Define a map $\iota : C_n^\Delta(X) \to C'_n(G)$ by $\iota(x_0,\dots,x_n) \mapsto (s(x_0),\dots,s(x_n))$. Then $\iota(\varphi(C(\mathcal{S}))) \in C'_n(G)$ is still not a cycle in general.

Let $x \in X$ be the color of an arc. Define an arc coloring $\mathcal{A} * x$ by

$$(\mathcal{A} * x)(a) = \mathcal{A}(a) * x, \quad (\mathcal{R} * x)(r) = \mathcal{R}(r) * x \quad \text{(for any arc } a \text{ and region } r).$$

Then $\mathcal{S} * x = (\mathcal{A}(m_i) * x, \mathcal{R} * x)$ is also a shadow coloring. We can show that the sum

$$\iota(\varphi(C(\mathcal{S}))) + \iota(\varphi(C(\mathcal{S} * x))) + \iota(\varphi(C(\mathcal{S} * x^2))) + \cdots + \iota(\varphi(C(\mathcal{S} * x^{l-1})))$$

is a group cycle represented by the l-fold cyclic branched covering along the knot K.

7.4. Relative group homology. Let G be a group and H be a subgroup of G. We define the relative group homology $H_n(G, H; \mathbb{Z})$ by the homology of the mapping cone of the map $C_n(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \to C_n(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. We can compute $H_n(G, H; \mathbb{Z})$ as follows.

Lemma 7.6. *Let K be the kernel of $C_0(H\backslash G) \to \mathbb{Z}$. Let $\cdots \to F_2 \to F_1 \to K \to 0$ be a free resolution of K as $\mathbb{Z}[G]$-module. Then $H_n(G, H; \mathbb{Z}) \cong H_n(F_* \otimes_{\mathbb{Z}[G]} \mathbb{Z})$ for $n \geq 1$.*

The quandle $\mathrm{Conj}(h)$ can be identified with $Z(h)\backslash G$. It is easy to check that the complex $C_*^{\Delta}(Z(h)\backslash G)$ is acyclic and have a $\mathbb{Z}[G]$-module structure. If the following acyclic complex

$$\cdots \to C_2^{\Delta}(Z(h)\backslash G) \to C_1^{\Delta}(Z(h)\backslash G) \to \mathrm{Ker}(C_0^{\Delta}(Z(h)\backslash G) \to \mathbb{Z}) \to 0$$

is a projective resolution, $H_n^{\Delta}(Z(h)\backslash G)$ is isomorphic to the relative group homology $H_n(G, Z(h); \mathbb{Z})$.

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Osaka City University Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, JAPAN

E-mail address: `kabaya@sci.osaka-cu.ac.jp`