数理解析研究所講究録
第 1716 巻 2010 年 64-83

An invitation to V.F.R. Jones' planar algebras

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Abstract

This article is aimed to explain the basic ideas of planar algebras for topological applications in mind. We will start with some familiar diagrammatic algebras and will give some known examples of subfactor planar algebras without any knowledge of subfactor theory. We will briefly see some applications of (subfactor) planar algebras to low dimensional topology.

1 Introduction – diagrammatic algebras –

In the last 25 years, we have witnessed that pictorial expression of some algebras is useful to construct link invariants and 3-manifold invariants. For instance, pictorial expressions of the elements in Temperley-Lieb algebras induce the Jones polynomials for links and those of so-called BMW algebras introduced by J. Murakami in [Mu] and Birman-Wenzl in [BW] independently induce the Kauffman polynomials.

As the introduction of this article, let us recall the definitions of the Temperley-Lieb algebras and the BMW algebras and we will describe the diagrams for the generating elements. In this article, our ground field is always assumed to be the field of complex numbers $\mathbb{C}$.

Example 1 Temperley-Lieb algebras.

Definition 1 Let n be a natural number and δ be a non-zero complex number. The *n-th Temperley-Lieb algebra* $TL_n(\delta)$ is an algebra over $\mathbb{C}$ generated by the unit 1 and $E_1, \ldots, E_{n-1}$ obeying the following relations:

for $i, j = 1, \ldots, n-1$,

$$E_i^2 = \delta E_i, \tag{1.1}$$

$$E_i E_j = E_j E_i \ (|i-j| \geq 2), \tag{1.2}$$

$$E_i E_{i\pm1} E_i = E_{i\pm1} E_i E_{i\pm1}. \tag{1.3}$$

We list a few known facts about $TL_n(\delta)$ for later use:

- It is finite dimensional. The dimension is given by the n-th Catalan number $\dim TL_n(\delta) = \frac{1}{n+1}\binom{2n}{n}$.
- It has a normalized trace τ_n which is compatible with the inclusion $TL_n(\delta) \subset TL_{n+1}(\delta)$. Namely, τ_n is a linear function on $TL_n(\delta)$ with $\tau_n(1) = 1$, $\tau(xy) = \tau(yx)$ for any $x, y \in TL_n(\delta)$ and $\tau_{n+1}(x) = \tau_n(x)$ for any $x \in TL_n(\delta)$.

The generators E_i's have a diagrammatic presentation as in Fig 1.

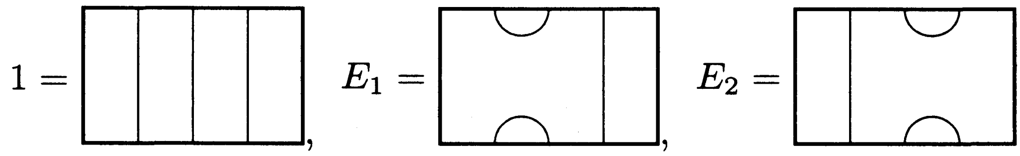


Fig. 1 Diagrams of the unit 1 and the generators E_1, E_2 of $TL_n(\delta)$ $(n = 3)$

In general, we will depict each element of $TL_n(\delta)$ as a diagram in the fixed rectangle which has the n-marked points on both the top and the bottom of the rectangle.

Then, the multiplication $x \cdot y$ of two diagrams x, y in the rectangles corresponding to $x, y \in TL_n(\delta)$ is defined in a similar manner to the way to define the multiplication of the braids. Namely, 'stack y on x'.

However, this is not enough to give an *algebra* of the diagrammatic presentation of Temperley-Lieb algebras and we need the following additional rule.

Rule 1: If we have a loop in the consequence of a multiplication, then we remove a loop and multiply the picture by δ.

We denote this diagrammatically defined algebra obtained from $TL_n(\delta)$ by $\mathcal{TL}_n(\delta)$.

When $n = 3$, by easy computations, we see that basis elements of $\mathcal{TL}_3(\delta)$ have the diagrams shown in Fig. 2.

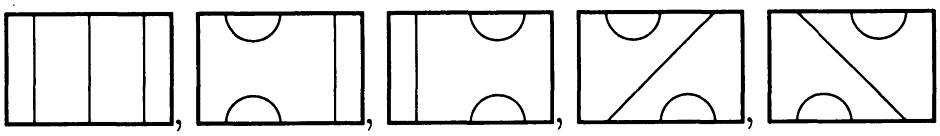


Fig. 2 Diagrams of the basis elements of $\mathcal{TL}_n(\delta)$ $(n = 3)$

Thus, we have an increasing sequence of finite dimensional algebras $\{\mathcal{TL}_n(\delta)\}_{n=1}^{\infty}$. And we will adopt the diagram of the embedded element in $\mathcal{TL}_{n+1}(\delta)$ for $x \in \mathcal{TL}_n(\delta)$ by the following rule.

Rule 2: We use the diagram of a rectangle with one through strand in the right of x for the embedded x. See Fig. 3.

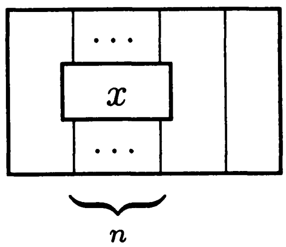


Fig. 3 Diagrams for an embedded element

Then, for each n, we have an obvious isomorphism from $\mathcal{TL}_n(\delta)$ onto $TL_n(\delta)$ by sending the diagrams of generators to corresponding E_i's, which is compatible with the inclusions.

Example 2 BMW algebras.

We will follow the exposition by Birman-Wenzl in [BW].

Definition 2 Let n be a natural number and let $l, m \in \mathbb{C}^{\times}$. We denote by $\mathcal{C}_n(l, m)$ the algebra generated by the unit 1 and G_i, G_i^{-1} $(1 \leq i \leq n-1)$ with the property $G_i G_i^{-1} = G_i^{-1} G_i = 1$ for all i and with the following relations (1)–(8). Firstly, we introduce an element $E_i \in \mathcal{C}_n(l, m)$ in such a way that $G_i + G_i^{-1} = m(1 + E_i)$ for all i. This identity is often called *Kauffman's skein relation*. The relations are:

(1) $G_iG_j = G_jG_i,\ E_iE_j = E_jE_i\ (|i-j| > 1)$
(2) $G_iG_{i\pm 1}G_i = G_{i\pm 1}G_iG_{i\pm 1},\ E_iE_{i\pm 1}E_i = E_i,$
(3) $G_{i\pm 1}G_iE_{i\pm 1} = E_iG_{i\pm 1}G_i = E_iE_{i\pm 1},$
(4) $G_{i\pm 1}E_iG_{i\pm 1} = G_i^{-1}E_{i\pm 1}G_i^{-1},$
(5) $G_{i\pm 1}E_iE_{i\pm 1} = G_i^{-1}E_{i\pm 1},$
(6) $E_{i\pm 1}E_iG_{i\pm 1} = E_{i\pm 1}G_i^{-1},$
(7) $G_iE_i = E_iG_i = l^{-1}E_i,$
(8) $E_iG_{i\pm 1}E_i = lE_i.$

We call the algebra $\mathcal{C}_n(l,m)$ the n-th *BMW algebra*.

We list a few known facts about $\mathcal{C}_n(l,m)$ for later use:

- It is finite dimensional and the dimension is given by

$$\dim \mathcal{C}_n(l,m) = (2n-1)!! = \prod_{j=1}^{n}(2j-1). \tag{1.4}$$

- It has a normalized trace τ_n which is compatible with the inclusion $\mathcal{C}_n(l,m) \subset \mathcal{C}_{n+1}(l,m)$.
- After certain renormalization, the trace τ_n gives rise to the Kauffman polynomial for links with parameters l, m. The links are obtained from the closure of the 'braids' in $\mathcal{C}_n(l,m)$.

It is well known that the generators G_i's and the elements E_i's of the algebra $\mathcal{C}_n(l,m)$ has diagrammatic presentations. The diagrams of E_i's are same as the ones used for $\mathcal{TL}_n(\delta)$ and we also apply **Rule 1** to the diagrams obtained from E_i's. The diagrams of G_i and G_i^{-1} have one crossing for each. See Fig. 4. This diagrammatic presentation of the elements of $\mathcal{C}_n(l,m)$ defines an algebra over $\mathbb{C}$, in a similar way to $\mathcal{TL}_n(\delta)$

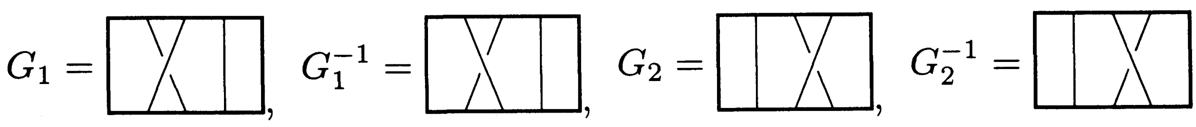


Fig. 4 Diagrams of the generators G_1, G_1^{-1}, G_2 and G_2^{-1} $(n = 3)$

and we denote this diagrammatic algebra by $BMW_n(l,m)$. With **Rule 2**, we will see that $\{BMW_n(l,m)\}_{n=1}^{\infty}$ is an increasing sequence of finite dimensional algebras. Then, it is known that for each n, $BMW_n(l,m)$ is isomorphic to $\mathcal{C}_n(l,m)$ as algebras [MT].

2 Planar tangles

Before we define planar algebras, we will introduce the notion of *planar tangles*, which plays a crucial role in planar algebras.

In the previous section, we depicted elements of $\mathcal{TL}_n(\delta)$ and $BMW_n(l,m)$ as *n-boxes.* Namely, a diagram of each element is presented by a $2n$-marked rectangular box; half of the marked points are located on the top of the rectangle and the rests are located on the bottom.

From now on, we shall change the way to present elements of $\mathcal{TL}_n(\delta)$ and $BMW_n(l,m)$ from n-boxes to "n-disks".

In the sequel of this article, D_0 is always meant to be the unit disk centered at the origin in the complex plane:

$$D_0 = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

We will define planar k-tangles generalizing n-disks following [J2].

Definition 3 A *planar k-tangle T* constitutes of:

- the unit disk D_0 with $2k(\geq 0)$ marked points on the boundary and a finite set of disjoint disks D_i $(1 \leq i \leq b(T))$ inside D_0
- each inner disk D_i has the even number (possibly zero) of marked points on the boundary of the disk, say, $2k_i(\geq 0)$.
- a finite set of disjoint smoothly embedded curves, which are either closed loops or whose boundaries are marked points of D_i $(0 \leq i \leq b(T))$
- each marked point is the boundary of some curve which meets the boundary of the corresponding disk transversally
- each connected component of the complement of the curves in $\mathring{D}_0 - \bigcup_{i=1}^{b(T)} D_i$ is called a region and it is either shaded or unshaded by the following rule:
we shade and unshade the regions inside D_0 so that regions whose

boundaries meet are shaded differently.

- each disk D_i $(0 \leq i \leq b(T))$ has a distinguished interval. We shall mark '$*$' for the distinguished interval.

We think that two planar k-tangles are equal if they are isotopic.

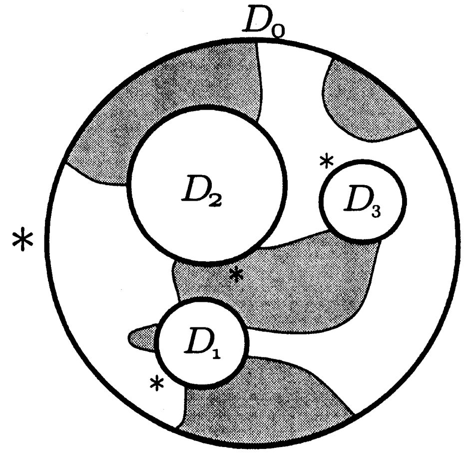


Fig. 5 An example of a planar 3-tangle

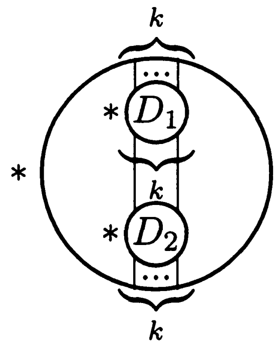
Definition 4 We denote the set of planar k-tangles by $\mathcal{T}_k$. If an element T in $\mathcal{T}_k$ has the distinguished interval (in the boudary of D_0) attached with unshaded (shaded) region, then T is called positive (resp. negative). We denote the subset of positive (negative) planar k-tangles by $\mathcal{T}_{k,+}$ (resp. $\mathcal{T}_{k,-}$). Hence, $\mathcal{T}_k = \mathcal{T}_{k,+} \coprod \mathcal{T}_{k,-}$.

The above Fig. 5 is an example of positive 3-tangles.

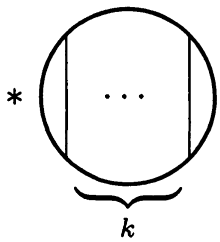
Remark 5 Instead of shading regions, we often use arrows for curves to indicate whether a region is unshaded or shaded. If a region is unshaded, curves consisting of the boundary of it are oriented positively, i.e., in the counter-clockwise direction. If a region is shaded, then the other way around. See [KS1] for more detailed accounts.

We introduce some important planar tangles for later use.

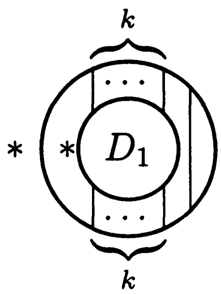
- Multiplication tangle M_k



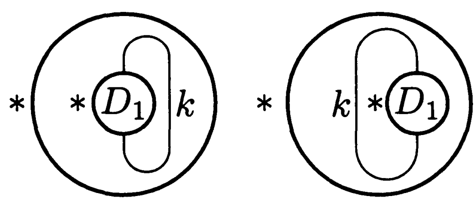
• Identity tangle I_k



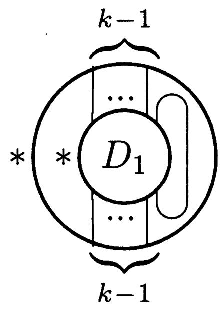
• Inclusion tangle I_k^{k+1}



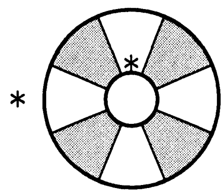
• Right (Left) trace tangle Tr_R (resp. Tr_L), where k stands for k parallel strands



• Partial trace tangle $\mathcal{E}_k^{k-1}$



• Rotation tangle ρ ($*$ for the inner disk D_1 is rotated to the *next unshaded region* in the clockwise direction relative to the $*$ for the outer disk D_0.)



We will explain a natural operation of the *composition* for two planar tangles. Suppose T is a planar k-tangle with $b = b(T)$ internal disks. If one of these internal disks, say D_i, has $2k_i$ marked points, and if S is a planar k_i-tangle, then we have the k-tangle obtained by isotoping S so that the boundary and the distinguished interval with information of the shading of S matches those of D_i. We denote the resulting planar k-tangle by $T \circ_i S$.

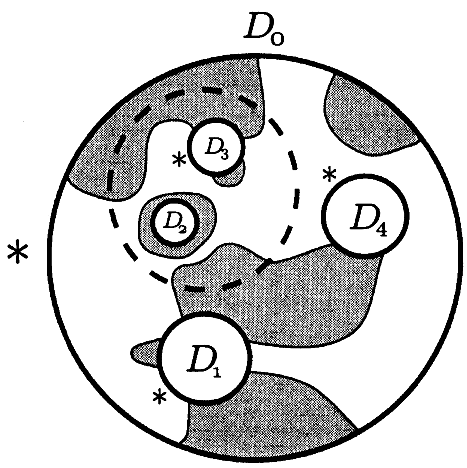


Fig. 6 An example of $T \circ_i S$

In Fig. 6, we give an example of the composition $T \circ_2 S$ with the planar 3-tangle T in Fig. 5. Here S is depicted as the disk surrounded by the dashed circle. (We have used isotopy for S to fit the disk D_2 of T.) Note that the numbers of the disk inside S has been changed from D_1, D_2 of S to D_2, D_3 of $T \circ_2 S$ and that the $*$ indicating the distinguished intervals on the outer disk of S and $*$ for D_2 of T are removed in $T \circ_2 S$.

Example 3 Planar tangles of $\mathcal{TL}_n(\delta)$.

We will use the isotopy from an element of $\mathcal{TL}_n(\delta)$, which is a rectangle with $2n$-marked points (on the top and the bottom of it) connected by non-intersecting strands, to the unit disk D_0 with $2n$-marked points connected by non-intersecting smooth curves. Then, there are $2n$ intervals on the boundary circle of D_0. We shall mark '$*$' for the distinguished interval which is isotoped from the left edge of the rectangle. And we will define the shading in such a way that the region which has the distinguished interval is positive. We will depict basis elements of $\mathcal{TL}_n(\delta)$ in the case $n = 3$ as in Fig. 7.

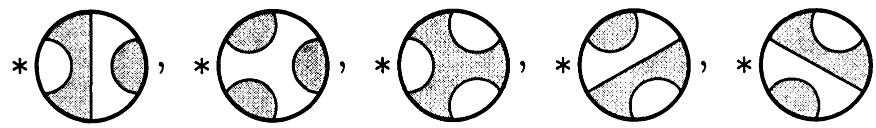


Fig. 7 Basis elements of $\mathcal{TL}_n(\delta)$ as positive planar n-tangles ($n = 3$)

Example 4 Planar tangles of $BMW_n(l, m)$

In a similar manner to define the planar tangles of $\mathcal{TL}_n(\delta)$, we will use the isotopy from a diagram of n-box in $BMW_n(l, m)$ to the n-disk. Then, we have the same pictures of E_i's as those of $\mathcal{TL}_n(\delta)$. However, say, G_1 and G_1^{-1} will be depicted as in Fig. 8 in this procedure. So, these are *not* planar diagrams.

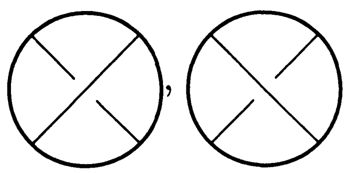


Fig. 8 G_1 and G_1^{-1} as 2-disks

Since we know by (1.4) that $\dim BMW_2(l,m) = 3$, we have a basis $\{1, G_1, G_1^{-1}\}$ of $BMW_2(l,m)$. So, there should be the corresponding elements for planar diagrams if there would be such planar presentations. For this sake, we introduce two planar 2-tangles Q and R as in Fig. 9. They are substitutes of the crossings corresponding to G_1 and G_1^{-1}, respectively. Identifying the elements of planar 2-tangles and the labels for them, we call Q, R the *planar 2-tangles labeled by Q and R.*

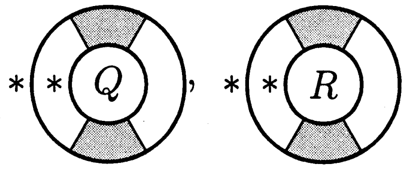


Fig. 9 Planar 2-tangles Q and R

In the case that a planar tangle contains *disks with 4 marked points*, we allow some of these disks to be *labeled* by an element of $L = \{Q, R\}$. Then, we consider the vector space V_n over $\mathbb{C}$ which is the set of linear combinations of *fully labeled* n-tangles. For such a vector space to be isomorphic to $BMW_n(l,m)$, we need to consider the algebra structure on V_n. We will come back and discuss it in the next section.

We will end this section by giving the general definition of *labeled planar tangles.*

Definition 6 Let $L = \coprod_{k \in \mathbb{Z}_{\geq 0}} L_k$ be a disjoint union of an arbitrary collection of a set L_k (possibly empty). A planar k-tangle T is said to be *labeled by L* if T is a planar k-tangle and is allowed to label an element of L_j for each internal disk D with $2j$-marked points only if L_j is not empty. If this is the case, we call L the *label set of planar tangles.*

3 Definitions of planar algebras

3.1 Definition of planar algebras

Definition 7 A (shaded) planar algebra P is a family of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces $\{P_{k,\pm}\}_{k \in \mathbb{Z}_{\geq 0}}$ indexed by the set of non-negative integers $\mathbb{Z}_{\geq 0}$, where $P_{k,\pm}$ denotes the $\pm$ graded space indexed by k. To each planar k-tangle T for $k \geq 0$, if the set of inner disks of the unit disk D_0 of T is not empty, there will be a linear map

$$Z_T : P_{k_1,\epsilon_1} \otimes \cdots \otimes P_{k_b,\epsilon_b} \to P_{k,\epsilon_0} \tag{3.1}$$

where P_{k_i,ϵ_i} is the vector space indexed by half the number of marked boundary points of D_i and $\epsilon_i = +$ if the distinguished interval of D_i is unshaded and $\epsilon_i = -$ if it is shaded.

The map Z_T is subject to the following requirements:

(i) (Naturality)

$$Z_{T\circ_i S} = Z_T \circ (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes Z_S \otimes \mathrm{id} \cdots \otimes \mathrm{id}), \tag{3.2}$$

where Z_S sits at the i-th place.

(ii) (Isotopy invariance) If φ is an orientation preserving diffeomorphism of $\mathbb{C}$ then

$$Z_T = Z_{\varphi(T)}, \tag{3.3}$$

where the sets of internal disks of T and $\varphi(T)$ are identified by φ.

(iii) (Unit preserving) $Z_{I_k} = \mathrm{id}_{P_{k,\epsilon}}$, where I_k is the identity tangle.

Remark 8 (i) We often work with either $\{P_{n,+}\}_n$ or $\{P_{n,-}\}_n$ for $n > 0$. We agree on the convention that P_n will mean $P_{n,+}$.
(ii) The correspondence $Z : T \mapsto Z_T$ may be seen as a "representation of an algebra of planar tangles".

One notices that the multiplication tangle M_n implies a linear map $Z_{M_n} : P_{n,\epsilon} \otimes P_{n,\epsilon} \to P_{n,\epsilon}$. Thus, by using Z_{M_n}, $P_{n,\epsilon}$ has multiplication and becomes an associative algebra over $\mathbb{C}$ with the unit.

Example 5 Temperley-Lieb planar algebras.

Let T be a planar k-tangle which has $b = b(T)$ disks $D_1, \ldots, D_b$ inside the unit disk D_0 and each inner disk D_i has $2k_i$ marked points. Put $P_j = \mathcal{TL}_j(\delta)$ ($j \in \{k_1, \ldots, k_b, k\}$) to simplify the notations. Then, T implies a linear map

$$Z_T : P_{k_1} \otimes \cdots \otimes P_{k_b} \to P_k \tag{3.4}$$

It would be obvious for Z_T to satisfy the conditions (i) - (iii) in the definition of planar algebras for $k > 0$.

When $k = 0$, we need to elaborate a little bit more. The 0-th Temperley-Lieb planar algebra consists of planar 0-tangles by definition and 0-tangles consist of non-intersecting loops. However, by **Rule 1**, each loop is replaced by δ. After all the loops are removed, the rest is an empty diagram inside D_0. According to the shading, we have the unshaded empty diagram and the shaded one denoted by 1_+ and 1_-, respectively. Then, we see that $\mathcal{TL}_{0,\pm}(\delta) = \mathbb{C}1_\pm$. Sending $1_\pm$ to $1 \in \mathbb{C}$ gives an isomorphism from $\mathcal{TL}_{0,\pm}(\delta)$ to $\mathbb{C}$.

Thus, we now know that $\mathcal{TL}(\delta) = \cup_{n=0}^{\infty}\mathcal{TL}_n(\delta)$ is a planar algebra.

Definition 9 Let $\mathcal{TL}(\delta) = \cup_{n=0}^{\infty}\mathcal{TL}_{n,\epsilon_n}(\delta)$, where $\epsilon_n = +$ for $n > 0$. Then, $\mathcal{TL}(\delta)$ with the above correspondence Z is a planar algebra and is called the *Temperley-Lieb planar algebra.* We call an element of $\mathcal{TL}(\delta)$ *Temperley-Lieb element*, or *TL-element* for brevity.

3.2 A planar algebra with generators and relations

Definition 10 Let L be a label set. Define $P_k(L)$ to be the vector space over $\mathbb{C}$ with the basis given by the collection of planar k-tangles all of which inner disks are labeled by L. Each planar k-tangle T induces a linear map Z_T satisfying both (3.1) for $P_j = P_j(L)$ and (3.2). Thus, $P_k(L)$ has an algebra structure given by Z_{M_k}, where M_k is the multiplication tangle. $P = \cup_{n=0}^{\infty}P_n(L)$ is called the *universal planar algebra* with the label set L.

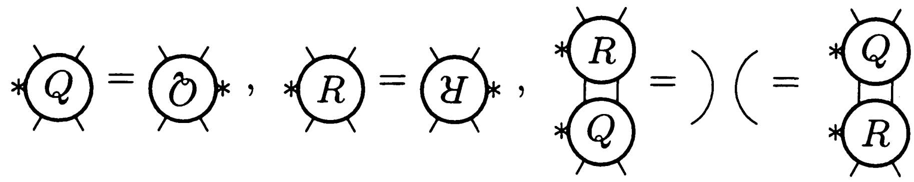
Definition 11 For a universal planar algebra $P = \cup_{n=0}^{\infty}P_n(L)$ with the label set L. Let R be a subset of L. We define $J_k(R)$ to be the smallest vector subspace of $P_k(L)$ containing the set R. Then, $P(L;R) := \cup_{n=0}^{\infty}P_n(L)/J_n(R)$ is called the *planar algebra generated by L with relations R.*

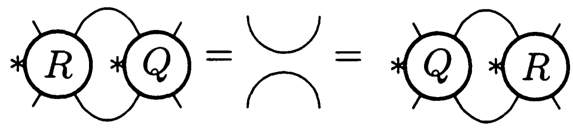
Example 6 BMW planar algebras.

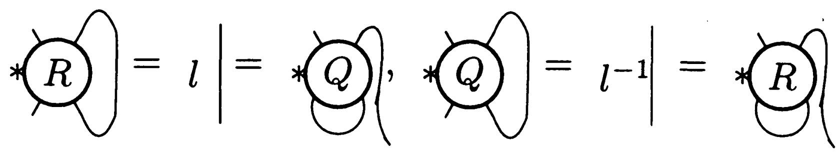
Here is a recipe for V_n in **Example 4** to become a planar algebra due to V.F.R. Jones [J1].

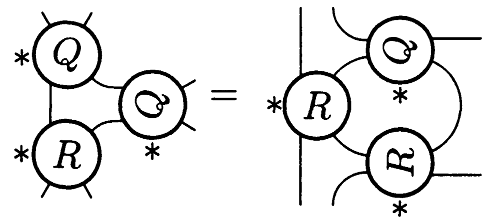
We use the multiplication tangle M_n to define the multiplication on V_n. Hence, V_n in **Example 4** is an associative algebra with the unit.

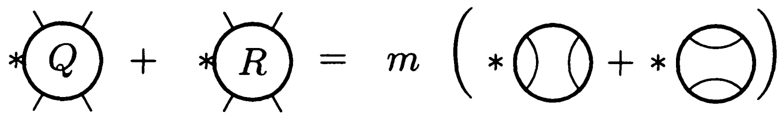
Let $L = L_2 = \{Q, R\}$. Put $P_n(L) = V_n$ in Definition 10, it defines a universal planar algebra $P(L) = \cup_{k=0}^{\infty} P_k(L)$ with the label set L. Then, we define the algebra $\mathcal{BMW}_n(l, m)$ to be $P_k(L)$ modulo the following relations on Q, R:

(i) 

(ii) 

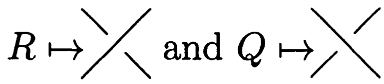
(iii) 

(iv) 

(v) 

Here, we omit the shading since it will be clear from the positions of $*$'s.

We will see that the correspondence

 (3.5)

extends to a homomorphism from $\mathcal{BMW}_n(l, m)$ to $BMW_n(l, m)$ and that it is in fact an isomorphism. See [J1] for more detailed accounts.

When $n = 0$, the 0-th BMW planar algebra consists of planar 0-tangles, which are linear combinations of finitely many links. Since we have assumed the Kauffman skein relation, we see that planar 0-tangles are replaced by the Kauffman polynomials for fixed $l, m \in \mathbb{C}^{\times}$. As in a similar manner to the case of $\mathcal{TL}(\delta)$, we see that $\mathcal{BMW}_{0,\pm}(l, m) = \mathbb{C}1_{\pm}$.

Definition 12 We call $\mathcal{BMW}(l,m) = \cup_{n=0}^{\infty}\mathcal{BMW}_{n,\epsilon_n}(l,m)$, where $\epsilon_n = +$ for $n > 0$, with the correspondence Z the *BMW planar algebra*.

Definition 13 Let P be a planar algebra.

(i) P is called *connected* if $\dim P_{0,\pm} = 1$. If this is the case, we define the isomorphism $Z_{\pm} : P_{0,\pm} \to \mathbb{C}$ by sending positive/negative empty diagrams to $1 \in \mathbb{C}$.

(ii) If P is connected and if $Z_{\mathrm{Tr}_R} = Z_{\mathrm{Tr}_L}$ holds for every planar tangles, then P is called *spherical* and we denote $Z_{\mathrm{Tr}_R} = Z_{\mathrm{Tr}_L}$ by simply Tr. If P is spherical, the map $Z(T) := Z_{\pm} \circ Z_{\mathrm{Tr}}(T)$ gives an isotopy invariant of the closure of a planar tangle T by the trace tangle. We call Z the *partition function* of P.

4 Definitions of subfactor planar algebras

Definition 14 We define the *adjoint tangle* T^* of T by the complex conjugate of T. (Recall T is defined on the complex plane.)

Definition 15 A planar algebra P is a *subfactor planar algebra* if:

(i) P is a connected, spherical planar algebra with $Z(1) = \delta > 0$, where the unit 1 is considered as planar 1-tangle,

(ii) each $P_{k,\pm}$ is a finite dimensional $*$-algebra satisfying the following condition:

$$Z_T(x_1 \otimes \cdots \otimes x_b)^* = Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*), \tag{4.1}$$

for $x_i \in P_{k_i,\epsilon_i}$ $(1 \le i \le b)$,

(iii) for any $x \in P_{k,\pm}$, $\mathrm{tr}_k(x) := \delta^{-k} Z_{\mathrm{Tr}}(x)$ is a positive definite trace on $P_{k,\pm}$ for all $k \ge 1$.

Remark 16 (i) If P is a subfactor planar algebra, then each $P_{k,\pm}$ is a semi-simple $*$-algebra. Namely, $P_{k,\pm}$ is isomorphic to the $*$-algebra of finite direct sums of full matrix algebra over $\mathbb{C}$.
(ii) In the case of a subfactor planar algebra, we will have the canonical vector space isomorphism $P_{n,+} \simeq P_{n,-}$ for positve n [J2, KS1]. For this reason, we sometimes treat only $P_n = P_{n,+}$.
(iii) When we treat the diagrammatic element $\delta^{-1}E_n$ as an element in

P_{n+1}, we set $e_n := \delta^{-1} Z_{E_n}(1) \in P_{n+1}$, and we call it n-th *Jones projection* in P_{n+1}. In fact, this plays a role of the n-th Jones projection in subfactor theory.

Proposition 17 Let $P = \cup_{n=0}^{\infty} P_n$ be a subfactor planar algebra. Then, for any element $x, y \in P_{n+1}$, there exist $x_0, y_0 \in P_n$ such that $x e_n y = x_0 e_n y_0$ holds, where e_n is the n-th Jones projection as in the previous remark (iii).

Proof. It is enough to prove that for any $x \in P_{n+1}$, there exists x_0 such that $x e_n = x_0 e_n$ holds. Put $x_0 := Z_{\mathcal{E}^n_{n+1}}(x e_n)$, then this does the job. (Consider x, x_0, e_n as planar tangles, and apply $\mathcal{E}^n_{n+1}$ to the multiplication of $x e_n$. Readers are encouraged to draw pictures.) □

As a consequence of this proposition, we conclude that $P_n e_n P_n$ is a two sided-ideal in P_{n+1}.

Definition 18 Let $P = \cup_{n=0}^{\infty} P_n$ be a subfactor planar algebra. P is said to have *finite depth* if there exists a positive integer k such that $P_k e_k P_k = P_{k+1}$. We call the smallest such k the *depth* of P.

We have the following very striking result for finite depth subfactor planar algebras ([KT1, KT2]).

Theorem 19 (Kodiyalam-Tupurani) Any finite depth subfactor planar algebra is singly generated with finite relations.

Remark 20 The proof of this result is surprisingly easy based on a simple trick of single generation property in linear algebras. See [KT1, KT2] for details.

5 Examples of finite depth subfactor planar algebras

Let $[n]_q$ be the n-th q-integer defined by $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. For each non-negative integer n, we have the distinguished projection $f^{(n)}$ in $TL_n(\delta)$ defined by the following recursive formula:

$$\begin{cases} f^{(0)} = 1, \\ [n+1]_q f^{(n+1)} = [n+1]_q f^{(n)} + [n]_q f^{(n)} e_n f^{(n)}, \end{cases}$$

where $e_n = \delta^{-1}E_n \in TL_{n+1}(\delta)$. We call $f^{(n)}$ *Jones-Wenzl projections.* In fact, $f^{(n)}$ is the projection in $TL_n(\delta)$ characterized by the following properties:

(i) $f^{(n)} \neq 0$,
(ii) $e_i f^{(n)} = f^{(n)} e_i = 0$ for $i = 1, \dots, n-1$.

As in Section 2, we have the diagrammatic presentation for $f^{(n)}$ and the above identities [MPS1].

Let $\delta = [2]_q = q + q^{-1}$.

5.1 When $\delta < 2$

• A_N-subfactor planar algebra

Let $q = \exp(\pi i/(N+1))$. A_N-subfactor planar algebra $P = \cup_{n=0}^{\infty} P_n$ is the quotient of $TL(\delta)$ by the relation $f^{(N)} = 0$.

• D_{2N}-subfactor planar algebra [MPS1]

Let $q = \exp(\pi i/(4N-2))$. D_{2N}-subfactor planar algebra $P = \cup_{n=0}^{\infty} P_n$ is generated by a single element $S = S^* \in P_{2N-2}$ with the following relations:

(i) $\rho(S) = -S$,
(ii) $\mathcal{E}_{2N-2}^{2N-1}(S) = 0$,
(iii) $S^2 = f^{(2N-2)}$,
(iv) $f^{(4N-3)} = 0$.

Here, we have used the notation of planar tangles exhibited in Section 2, except (iii). In (iii), S^2 stands for S multiplied by itself. So, it should be written with the multiplication tangle M_{2N-2} with the input of two S's.

• E_6- and E_8-subfactor planar algebras

See [B] for these examples.

5.2 Other examples

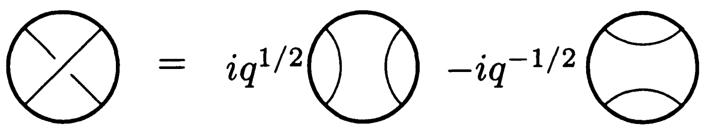
- Asaeda-Haagerup subfactor planar algebra. [P]
- Extended Haagerup subfactor planar algebra. [BMPS]
- Finite dimensional (C^*-) Hopf algebras. [KLS, KS2]

The first two examples are considered to be 'exotic'. Namely, we expect that they would not be constructed from groups, quantum groups or conformal field theories.

6 Applications

Link invariants [MPS2]

We may define a *crossing* (or *braiding*) as the linear combinations of the two planar 2-tangles as in the following figure.



In the case of $\mathcal{TL}(\delta)$ and the A_N-subfactor planar algebra, this gives rises to *colored Jones polynomial* formulated with Jones-Wenzl idempotents [MV]. Namely, each component L_i of a link L is replaced by n_i-parallel strands and we put the Jones-Wenzl idempotent $f^{(n_i)}$ on n_i-cabled link component of L_i.

In the case of the D_{2N}-subfactor planar algebra, by partial braiding property (Theorem 3.2 in [MPS1]), the even number of cabling is allowed to construct the colored Jones polynomial with respect to the new idempotents coming from the D_{2N}-subfactor planar algebra. Practically, on the $2N-2$ cabled link, we put $P = (f^{(2N-2)} + S)/2$ (which is also a projection) instead of Jones-Wenzl projection, then we have a new link invariant $\mathcal{J}_{D_{2N}}$ constructed from the D_{2N}-subfactor planar algebra. The invariant $\mathcal{J}_{D_{2N}}$ induces unexpected relations to some known link polynomials with various specializations. See [MPS2] for the detailed accounts.

Topological Quantum Field Theories in (1+1)-dimension [KPS]

Let us put a planar k-tangle in $\mathbb{R}^3$. Consider the holes instead of the inner disks. Then, planar tangle with this modification gives rise to a 2-sphere with boundary circles, which come from the the boundaries of the inner disks and the outer disk. It is not hard to imagine that this picture will give a TQFT in (1+1)-dimensions. Namely, the objects are closed circles with the decorated arcs, and the morphism are planar tangles seen as decorated 2-spheres with holes which boundaries are the circles, i.e., objects. Thus, we obtain a cobordism category $\mathcal{D}$. In [KPS], the category $\mathcal{D}$ is essentially in one to one correspondence with a subfactor planar algebra. See [KPS] for the detailed accounts.

A topological operad of $\mathbb{P}^1$ [CM]

There is an interesting result on an operadic aspect of planar algebras from real algebraic geometry, although we have mentioned nothing about operads.

Ceyhan and Marcolli have established the following results. Planar algebras, which can be seen as algebras over the planar operad, are algebras over the topological operad of certain moduli spaces of stable maps to $\mathbb{P}^1$ with Lagrangian boundary conditions.

According to this result, we might expect some new results in planar algebras from the viewpoint of the algebras over the topological operad. In fact, they give two general frameworks to construct planar algebras. One is based on sting topology and the other is on Gromov-Witten theory.

Acknowledgment This article is a recollection and a reorganization of some known results on planar algebras. The author does not claim any originality of the contents in this article at all and would like to thank all of the contributors for the subjects appeared in this article.

The author wishes that this article gives a help for the students and researchers in low dimensional topology interested in planar algebras to read further articles in this research area.

Finally, the author would like to thank the organizers to give him an opportunity to include this article in the proceedings of the workshop.

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